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Łukasiewicz fuzzy positive implicative ideals in BCK-algebras

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Abstract

In BCK-algebras, the notion of Łukasiewicz fuzzy positive implicative ideal is introduced, and several properties are investigated. The relationship between Łukasiewicz fuzzy ideal and Łukasiewicz fuzzy positive implicative ideal is discussed, and characterizations of a Łukasiewicz fuzzy positive implicative ideal are considered. Conditions for a Łukasiewicz fuzzy ideal to be a Łukasiewicz fuzzy positive implicative ideal are provided, and conditions for the $\in$-set, q-set and O-set to be positive implicative ideals are explored.

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1 Introduction

A BCI/BCK-algebra was introduced by K. Iséki and it is an important class of logical algebras (see [2] and [3]). Since then, it has been extensively investigated by several researchers. In particular, ideal of BCK/BCI-algebra based on crossing cubic structure was studied by Jun and Song (see [7]). Jan Łukasiewicz (1878-1956) was a Polish scientist, logician, philosopher, and mathematician. He was the author of three-valued logic, the first non-classical logic on the basis of which modal logic, probabilistic logic and fuzzy logic were created. In mathematics and philosophy, Łukasiewicz logic is a non-classical, many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. It was later generalized to n-valued (for all finite n) as well as infinitely-many-valued variants, both propositional and first-order. Infinite-valued Łukasiewicz logic is a real-valued logic in which sentences from sentential calculus may be assigned a truth value of not only zero or one but also any real number. Jun dealt with so called a Łukasiewicz fuzzy set which is a fuzzy set based on Łukasiewicz t-norm, and applied it to BCK-algebras and BCI-algebras (see [4, 5]).

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In this paper, we address the concept of Łukasiewicz fuzzy positive implicative ideal in BCK-algebras and investigate several properties. We consider characterization of a Łukasiewicz fuzzy positive implicative ideal. We discuss the relationship between a Łukasiewicz fuzzy ideal and a Łukasiewicz fuzzy positive implicative ideal. We give a condition for a Łukasiewicz fuzzy ideal to be a Łukasiewicz fuzzy positive implicative ideal. We provide conditions for the $\in$-set, q-set and O-set to be positive implicative ideals.

2 Preliminaries

2.1 Basic concepts about BCI/BCK-algebras

This section provides the definitions and default results required for this manuscript. For more information about BCK-algebras and BCI-algebras, see the books [1, 8].

Let T be a set containing a special element "0" and a binary operation "$*$". If it satisfies the conditions below:

(I_1) $(\forall r, u, d \in T)$ $(((r * u) * (r * d)) * (d * u) = 0)$,

(I_2) $(\forall r, u \in T)$ $((r * (r * u)) * u = 0)$,

(I_3) $(\forall r \in T)$ $(r * r = 0)$,

(I_4) $(\forall r, u \in T)$ $(r * u = 0,\ u * r = 0 \ \Rightarrow\ r = u)$,

then we say that T is a *BCI-algebra*. If a BCI-algebra T has the additional condition

(K) $(\forall r \in T)$ $(0 * r = 0)$,

then it is called a *BCK-algebra.*

The order relation "$\leq$" in a BCI/BCK-algebra T is defined as follows:

$$(\forall r, u \in T)(r \leq u \ \Leftrightarrow\ r * u = 0). \tag{1}$$

Every BCI/BCK-algebra T satisfies the conditions below (see [1, 8]):

$$(\forall r \in T)\,(r * 0 = r)\,, \tag{2}$$
$$(\forall r, u, d \in T)\,(r \leq u \ \Rightarrow\ r * d \leq u * d,\ d * u \leq d * r)\,, \tag{3}$$
$$(\forall r, u, d \in T)\,((r * u) * d = (r * d) * u)\,. \tag{4}$$

A subset Z of a BCI/BCK-algebra T is called

- a *subalgebra* of T (see [1, 8]) if it satisfies:

$$(\forall r, u \in Z)(r * u \in Z), \tag{5}$$

- an *ideal* of T (see [1, 8]) if it satisfies:

$$0 \in Z, \tag{6}$$
$$(\forall r, u \in T)(r * u \in Z,\ u \in Z \ \Rightarrow\ r \in Z). \tag{7}$$

A subset Z of a BCK-algebra T is called a *positive implicative ideal* of T (see [8]) if it satisfies (6) and

$$(\forall r, u, d \in T)((r * u) * d \in Z,\ u * d \in Z \ \Rightarrow\ r * d \in Z). \tag{8}$$

Lemma 2.1. [8] *A nonempty subset Z of a BCK-algebra T is a positive implicative ideal of T if and only if Z is an ideal of T that satisfies:*

$$(\forall r, u \in T)((r * u) * u \in Z \ \Rightarrow\ r * u \in Z). \tag{9}$$

2.2 Basic concepts about (Łukasiewicz) fuzzy sets

A fuzzy set g in a set T of the form

$$g(u) := \begin{cases} t \in (0,1] & \text{if } u = r, \\ 0 & \text{if } u \neq r, \end{cases}$$

is said to be a *fuzzy point* with support r and value t and is written as $[r/t]$.

For a fuzzy set g in a set T, we say that a fuzzy point $[r/t]$ is

(i) *contained* in g, written as $[r/t] \in g$, (see [9]) if $g(r) \geq t$.

(ii) *quasi-coincident* with g, written as $[r/t]\, q\, g$, (see [9]) if $g(r) + t > 1$.

If $[r/t]\, \alpha\, g$ is not established for $\alpha \in \{\in, q\}$, it is written as $[r/t]\ \overline{\alpha}\ g$.

A fuzzy set g in a BCI/BCK-algebra T is called

- a *fuzzy subalgebra* of T (see [6]) if it satisfies:

$$(\forall r, u \in T)(g(r * u) \geq \min\{g(r), g(u)\}). \tag{10}$$

- a *fuzzy ideal* of T (see [6, 10]) if it satisfies:

$$(\forall r \in T)(g(0) \geq g(r)), \tag{11}$$
$$(\forall r, u \in T)(g(r) \geq \min\{g(r * u), g(u)\}). \tag{12}$$

A fuzzy set g in a BCK-algebra T is called a *fuzzy positive implicative ideal* of T (see [10]) if it satisfies: (11) and

$$(\forall r, u, d \in T)(g(r * u) \geq \min\{g((r * u) * d), g(u * d)\}). \tag{13}$$

Definition 2.2. [4] *Let g be a fuzzy set in a set T and let $\kappa \in (0,1)$. A function*

$${}^{\kappa}_{g} : T \to [0,1],\ d \mapsto \max\{0, g(d) + \kappa - 1\},$$

is called the Łukasiewicz fuzzy set of g in T.

Definition 2.3. [4] *Let g be a fuzzy set in a BCI/BCK-algebra T and κ an element of $(0,1)$. Then its Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ in T is called a Łukasiewicz fuzzy subalgebra of T if it satisfies:*

$$[d/t_r] \in {}^{\kappa}_{g},\ [r/t_u] \in {}^{\kappa}_{g}\ \Rightarrow [(d * r)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g} \tag{14}$$

for all $d, r \in T$ and $t_r, t_u \in (0,1]$.

Let g be a fuzzy set in T. For the Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ of g in T and $t \in (0,1]$, consider the sets

$$({}^{\kappa}_{g}, t)_{\in} := \{d \in T \mid [d/t] \in {}^{\kappa}_{g}\},$$

$$({}^{\kappa}_{g}, t)_{q} := \{d \in T \mid [d/t]\, q\, {}^{\kappa}_{g}\},$$

which are called the $\in$-*set* and q-*set*, respectively, of ${}^{\kappa}_{g}$ (with value t). Also, consider a set:

$$O({}^{\kappa}_{g}) := \{d \in T \mid {}^{\kappa}_{g}(d) > 0\}, \tag{15}$$

which is called an O-*set* of ${}^{\kappa}_{g}$. It is observed that

$$O({}^{\kappa}_{g}) = \{d \in T \mid g(d) + \kappa - 1 > 0\}.$$

Definition 2.4. [5] *Let g be a fuzzy set in a BCI/BCK-algebra T. Then its Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ in T is called a Łukasiewicz fuzzy ideal of T if it satisfies:*

$$ {}^{\kappa}_{g}(0) \text{ is an upper bound of } \{{}^{\kappa}_{g}(d) \mid d \in T\}, \tag{16}$$

$$ [(d * r)/t_r] \in {}^{\kappa}_{g},\ [r/t_u] \in {}^{\kappa}_{g} \ \Rightarrow [d/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}, \tag{17}$$

for all $d, r \in T$ and $t_r, t_u \in (0, 1]$.

Lemma 2.5. [5] *Let g be a fuzzy set in T. Then its Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ is a Łukasiewicz fuzzy ideal of T if and only if it satisfies:*

$$ (\forall d \in T)(\forall t_r \in (0,1]) \left([d/t_r] \in {}^{\kappa}_{g} \ \Rightarrow \ [0/t_r] \in {}^{\kappa}_{g}\right), \tag{18}$$

$$ (\forall d, r \in T)({}^{\kappa}_{g}(d) \geq \min\{{}^{\kappa}_{g}(d * r), {}^{\kappa}_{g}(r)\}). \tag{19}$$

3 Łukasiewicz fuzzy positive implicative ideals

In what follows, let T be a BCK-algebra, and κ be an element of $(0, 1)$ unless otherwise specified.

Definition 3.1. *Let g be a fuzzy set in T. Then its Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ in T is called a Łukasiewicz positive implicative fuzzy ideal (briefly, ŁPIf-ideal) of T if it satisfies* (16) *(or, equivalently* (18)*) and*

$$ [((d * r) * u)/t_r] \in {}^{\kappa}_{g},\ [(r * u)/t_u] \in {}^{\kappa}_{g} \ \Rightarrow [(d * u)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}, \tag{20}$$

for all $d, r, u \in T$ and $t_r, t_u \in (0, 1]$.

Example 3.2. *Let $T = \{0, r_1, r_2, r_3, r_4\}$ be a set and the binary operation "$*$" in T is given in Table 1.*

Table 1: Cayley table for the binary operation "$*$"

$*$	0	r_1	r_2	r_3	r_4
0	0	0	0	0	0
r_1	r_1	0	r_1	0	0
r_2	r_2	r_2	0	r_2	0
r_3	r_3	r_3	r_3	0	0
r_4	r_4	r_4	r_3	r_2	0

Then T is a BCK-algebra (see [8]). Define a fuzzy set g in T as follows:

$$ g : T \to [0,1],\ d \mapsto \begin{cases} 0.77 & \text{if } d = 0, \\ 0.62 & \text{if } d = r_1, \\ 0.42 & \text{if } d = r_2, \\ 0.59 & \text{if } d = r_3, \\ 0.42 & \text{if } d = r_4. \end{cases}$$

If we take $\kappa := 0.56$, then the Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ of g in T is given as follows:

$$ {}^{\kappa}_{g} : T \to [0,1],\ d \mapsto \begin{cases} 0.33 & \text{if } d = 0, \\ 0.18 & \text{if } d = r_1, \\ 0.15 & \text{if } d = r_3, \\ 0.00 & \text{if } d \in \{r_2, r_4\} \end{cases}$$

and it is simple to check that ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T.

Lemma 3.3. [5] *Every Łukasiewicz fuzzy ideal* ${}^{\kappa}_{g}$ *of* T *satisfies:*

$$(\forall d, r \in T)(\forall t_r \in (0,1])(d \leq r,\ [r/t_r] \in {}^{\kappa}_{g} \ \Rightarrow\ [d/t_r] \in {}^{\kappa}_{g}), \tag{21}$$

$$(\forall d, r, u \in T)(\forall t_u, t_d \in (0,1]) \left(\begin{array}{c} d * r \leq u,\ [r/t_u] \in {}^{\kappa}_{g},\ [u/t_d] \in {}^{\kappa}_{g} \\ \Rightarrow\ [d/\min\{t_u, t_d\}] \in {}^{\kappa}_{g} \end{array} \right). \tag{22}$$

Lemma 3.4. [5] *If* ${}^{\kappa}_{g}$ *is a Łukasiewicz fuzzy ideal of* T*, then the conditions* (21) *and* (22) *are equivalent to*

$$(\forall d, r \in T)(d \leq r \ \Rightarrow\ {}^{\kappa}_{g}(d) \geq {}^{\kappa}_{g}(r)), \tag{23}$$

$$(\forall d, r, u \in T)(d * r \leq u \ \Rightarrow\ {}^{\kappa}_{g}(d) \geq \min\{{}^{\kappa}_{g}(r), {}^{\kappa}_{g}(u)\}). \tag{24}$$

respectively.

Proposition 3.5. *If a Łukasiewicz fuzzy set* ${}^{\kappa}_{g}$ *of a fuzzy set* g *in* T *is a Łukasiewicz fuzzy ideal of* T*, then the following are equivalent to each other.*

$$[((d * r) * r)/t_r] \in {}^{\kappa}_{g} \ \Rightarrow [(d * r)/t_r] \in {}^{\kappa}_{g}, \tag{25}$$

$$[((d * r) * u)/t_r] \in {}^{\kappa}_{g} \ \Rightarrow [((d * u) * (r * z))/t_r] \in {}^{\kappa}_{g}, \tag{26}$$

for all $d, r, u \in T$ *and* $t_r \in (0,1]$.

Proof. Assume that (25) is valid. If ${}^{\kappa}_{g}(r * u) < {}^{\kappa}_{g}((r * u) * u) := t_r$ for some $r, u \in T$, then $[((r * u) * u)/t_r] \in {}^{\kappa}_{g}$ and $[(r * u)/t_r] \,\overline{\in}\, {}^{\kappa}_{g}$. This is a contradiction, and thus

$${}^{\kappa}_{g}(d * r) \geq {}^{\kappa}_{g}((d * r) * r), \tag{27}$$

for all $d, r \in T$. Let $d, r, u \in T$ and $t_r \in (0,1]$ be such that $[((d * r) * u)/t_r] \in {}^{\kappa}_{g}$. Since

$$((d * (r * u)) * u) * u = ((d * u) * (r * u)) * u \leq (d * r) * u,$$

by (I_1), (3) and (4), it follows from (4), (23) and (27) that

$$\begin{aligned} {}^{\kappa}_{g}((d * u) * (r * u)) &= {}^{\kappa}_{g}((d * (r * u)) * u) \\ &\geq {}^{\kappa}_{g}(((d * (r * u)) * u) * u) \\ &\geq {}^{\kappa}_{g}((d * r) * u) \geq t_r. \end{aligned}$$

Hence $[((d * u) * (r * u))/t_r] \in {}^{\kappa}_{g}$.

Conversely, (25) is obtained by taking $u = r$ in (26) and using (I_3) and (2). □

Theorem 3.6. *Every ŁPIf-ideal is a Łukasiewicz fuzzy ideal.*

Proof. Let ${}^{\kappa}_{g}$ be a ŁPIf-ideal of T. If we take $u = 0$ in (20) and use (2), then we have

$$[(d * r)/t_r] \in {}^{\kappa}_{g},\ [r/t_u] \in {}^{\kappa}_{g} \ \Rightarrow [d/\min\{t_r, t_u\}] \in {}^{\kappa}_{g},$$

for all $d, r \in T$ and $t_r, t_u \in (0,1]$. Therefore ${}^{\kappa}_{g}$ is a Łukasiewicz fuzzy ideal of T. □

The converse of Theorem 3.6 may not be true as shown in the following example.

Example 3.7. *Let* $T = \{0, r_1, r_2, r_3\}$ *be a set with the binary operation "$*$" which is given in Table 2. Then* T *is a BCK-algebra (see [8]). Define a fuzzy set* g *in* T *as follows:*

$$g : T \to [0,1],\ d \mapsto \begin{cases} 0.79 & \text{if } d = 0, \\ 0.63 & \text{if } d = r_1, \\ 0.63 & \text{if } d = r_2, \\ 0.48 & \text{if } d = r_3. \end{cases}$$

Table 2: Cayley table for the binary operation "$*$"

$*$	0	r_1	r_2	r_3
0	0	0	0	0
r_1	r_1	0	0	r_1
r_2	r_2	r_1	0	r_2
r_3	r_3	r_3	r_3	0

If we take $\kappa := 0.57$*, then the Łukasiewicz fuzzy set* ${}^{\kappa}_{g}$ *of* g *in* T *is given as follows:*

$$ {}^{\kappa}_{g} : T \to [0,1],\ d \mapsto \begin{cases} 0.36 & \text{if } d = 0, \\ 0.20 & \text{if } d = r_1, \\ 0.20 & \text{if } d = r_2, \\ 0.05 & \text{if } d = r_3 \end{cases} $$

and it is simple to check that ${}^{\kappa}_{g}$ *is a Łukasiewicz fuzzy ideal of* T*. But it is not a ŁPIf-ideal of* T *because of* $[((r_2 * r_1) * r_1)/0.32] = [0/0.32] \in {}^{\kappa}_{g}$ *and* $[(r_1 * r_1)/0.24] = [0/0.24] \in {}^{\kappa}_{g}$*, but* $[(r_2 * r_1)/\min\{0.32, 0.24\}] = [r_1/0.24] \,\overline{\in}\, {}^{\kappa}_{g}$*.*

Proposition 3.8. *Every ŁPIf-ideal* ${}^{\kappa}_{g}$ *of* T *satisfies* (25) *and* (26)*.*

Proof. Let ${}^{\kappa}_{g}$ be a ŁPIf-ideal of T. Let $d, r \in T$ and $t_r \in (0,1]$ be such that $[((d * r) * r)/t_r] \in {}^{\kappa}_{g}$. Since $[(r * r)/t_r] = [0/t_r] \in {}^{\kappa}_{g}$, it follows from (20) that $[(d * r)/t_r] \in {}^{\kappa}_{g}$. Hence (25) is valid. Also ${}^{\kappa}_{g}$ satisfies (26) by the combination of Proposition 3.5 and Theorem 3.6. □

We provide conditions for a Łukasiewicz fuzzy ideal to be a ŁPIf-ideal.

Theorem 3.9. *Let* ${}^{\kappa}_{g}$ *be a Łukasiewicz fuzzy ideal of* T*. Then it is a ŁPIf-ideal of* T *if and only if it satisfies:*

$$ (\forall d, r, u \in T)({}^{\kappa}_{g}(d * u) \geq \min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\}). \tag{28} $$

Proof. Assume that ${}^{\kappa}_{g}$ be a ŁPIf-ideal of T. Note that

$$ [((d * r) * u)/{}^{\kappa}_{g}((d * r) * u)] \in {}^{\kappa}_{g} \text{ and } [(r * u)/{}^{\kappa}_{g}(r * u)] \in {}^{\kappa}_{g}, $$

for all $d, r, u \in T$. It follows from (20) that

$$ [(d * u)/\min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * z)\}] \in {}^{\kappa}_{g}, $$

and hence, for all $d, r, u \in T$:

$$ {}^{\kappa}_{g}(d * u) \geq \min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\}. $$

Conversely, let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy ideal of T that satisfies (28). Let $d, r, u \in T$ and $t_r, t_u \in (0,1]$ be such that $[((d * r) * u)/t_r] \in {}^{\kappa}_{g}$ and $[(r * u)/t_u] \in {}^{\kappa}_{g}$. Then ${}^{\kappa}_{g}((d * r) * u) \geq t_r$ and ${}^{\kappa}_{g}(r * u) \geq t_u$, which imply from (28) that

$$ {}^{\kappa}_{g}(d * u) \geq \min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\} \geq \min\{t_r, t_u\}. $$

Thus $[(d * u)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$. Therefore ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T. □

Theorem 3.10. *If a Łukasiewicz fuzzy ideal* ${}^{\kappa}_{g}$ *of* T *satisfies* (25)*, then it is a ŁPIf-ideal of* T*.*

Proof. Let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy ideal of T that satisfies (25). Let $d, r, u \in T$ and $t_r, t_u \in (0,1]$ be such that $[((d * r) * u)/t_r] \in {}^{\kappa}_{g}$ and $[(r * u)/t_u] \in {}^{\kappa}_{g}$. Since

$$ ((d * u) * u) * (r * u) \leq (d * u) * r = (d * r) * u, $$

for all $d, r, u \in T$, it follows from Lemma 3.3 that $[(((d * u) * u) * (r * u))/t_r] \in {}^{\kappa}_{g}$. Hence $[((d * u) * u)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$ by (17), and so $[(d * u)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$ by (25). Therefore ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T. □

We discuss the relationship between a fuzzy positive implicative ideal and an ŁPIf-ideal.

Lemma 3.11. [5] *If g is a fuzzy ideal of T, then its Łukasiewicz fuzzy set ${}_{g}^{\kappa}$ in T is a Łukasiewicz fuzzy ideal of T.*

Theorem 3.12. *If g is a fuzzy positive implicative ideal of T, then its Łukasiewicz fuzzy set ${}_{g}^{\kappa}$ in T is a ŁPIf-ideal of T.*

Proof. If g is a fuzzy positive implicative ideal of T, then it is a fuzzy ideal of T, and so its Łukasiewicz fuzzy set ${}_{g}^{\kappa}$ in T is a Łukasiewicz fuzzy ideal of T by Lemma 3.11. Let $d, r \in T$ and $t_r \in (0, 1]$ be such that $[((d * r) * r)/t_r] \in {}_{g}^{\kappa}$. Then

$$\begin{aligned} {}_{g}^{\kappa}(d * r) &= \max\{0, g(d * r) + \kappa - 1\} \\ &\geq \max\{0, g((d * r) * r) + \kappa - 1\} \\ &= {}_{g}^{\kappa}((d * r) * r) \geq t_r, \end{aligned}$$

and so $[(d * r)/t_r] \in {}_{g}^{\kappa}$. Therefore ${}_{g}^{\kappa}$ is a ŁPIf-ideal of T by Theorem 3.10. □

The converse of Theorem 3.12 may not be true as seen in the following example.

Example 3.13. *Let $T = \{0, r_1, r_2, r_3, r_4\}$ be a set with the binary operation "$*$" which is given in Table 3.*

Table 3: Cayley table for the binary operation "$*$"

$*$	0	r_1	r_2	r_3	r_4
0	0	0	0	0	0
r_1	r_1	0	r_1	r_1	0
r_2	r_2	r_2	0	r_2	0
r_3	r_3	r_3	r_3	0	0
r_4	r_4	r_4	r_4	r_4	0

Then T is a BCK-algebra (see [8]). Define a fuzzy set g in T as follows:

$$g : T \to [0, 1],\ d \mapsto \begin{cases} 0.92 & \text{if } d = 0, \\ 0.47 & \text{if } d = r_1, \\ 0.83 & \text{if } d = r_2, \\ 0.79 & \text{if } d = r_3, \\ 0.51 & \text{if } d = r_4. \end{cases}$$

If we take $\kappa := 0.48$, then the Łukasiewicz fuzzy set ${}_{g}^{\kappa}$ of g in T is given as follows:

$${}_{g}^{\kappa} : T \to [0, 1],\ d \mapsto \begin{cases} 0.40 & \text{if } d = 0, \\ 0.00 & \text{if } d = r_1, \\ 0.31 & \text{if } d = r_2, \\ 0.27 & \text{if } d = r_3, \\ 0.00 & \text{if } d = r_4, \end{cases}$$

and it is simple to check that ${}_{g}^{\kappa}$ is a ŁPIf-ideal of T. But g is not a fuzzy positive implicative ideal of T since

$$g(r_1 * r_2) = 0.47 < 0.51 = \min\{g((r_1 * r_4) * r_2), g(r_4 * r_2)\}.$$

Theorem 3.14. *If a Łukasiewicz fuzzy ideal ${}_{g}^{\kappa}$ of T satisfies (26), then it is a ŁPIf-ideal of T.*

Proof. Let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy ideal of T that satisfies (26). Since

$$[((d*r)*u)/{}^{\kappa}_{g}((d*r)*u)] \in {}^{\kappa}_{g},$$

for all $d, r, u \in T$, we have $[((d*u)*(r*u))/{}^{\kappa}_{g}((d*r)*u)] \in {}^{\kappa}_{g}$ by (26). It follows from (19) that

$$\begin{aligned} {}^{\kappa}_{g}(d*u) &\geq \min\{{}^{\kappa}_{g}((d*u)*(r*u)), {}^{\kappa}_{g}(r*u)\} \\ &\geq \min\{{}^{\kappa}_{g}((d*r)*u), {}^{\kappa}_{g}(r*u)\}, \end{aligned}$$

for all $d, r, u \in T$. Hence ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T by Theorem 3.9. □

Theorem 3.15. *Let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy ideal of T. Then it is a ŁPIf-ideal of T if and only if it satisfies:*

$$[(((d*r)*r)*u)/t_r] \in {}^{\kappa}_{g},\ [u/t_u] \in {}^{\kappa}_{g} \ \Rightarrow\ [(d*r)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}, \tag{29}$$

for all $d, r, u \in T$ and $t_r, t_u \in (0, 1]$.

Proof. Assume that ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T and let $d, r, u \in T$ and $t_r, t_u \in (0, 1]$ be such that

$$[(((d*r)*r)*u)/t_r] \in {}^{\kappa}_{g}\ \ ,\ \ [u/t_u] \in {}^{\kappa}_{g}.$$

Then

$$\begin{aligned} {}^{\kappa}_{g}(d*r) &\geq \min\{{}^{\kappa}_{g}((d*r)*u), {}^{\kappa}_{g}(u)\} \\ &= \min\{{}^{\kappa}_{g}(((d*u)*r)*(r*r)), {}^{\kappa}_{g}(u)\} \\ &\geq \min\{{}^{\kappa}_{g}(((d*u)*r)*r), {}^{\kappa}_{g}(u)\} \\ &= \min\{{}^{\kappa}_{g}(((d*r)*r)*u), {}^{\kappa}_{g}(u)\} \\ &\geq \min\{t_r, t_u\}, \end{aligned}$$

and so $[(d*r)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$.

Conversely, let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy ideal of T that satisfies (29). If we take $u = 0$ in (29), then

$$[(((d*r)*r)*0)/t_r] \in {}^{\kappa}_{g},\ [0/t_u] \in {}^{\kappa}_{g} \ \Rightarrow\ [(d*r)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}.$$

It follows from (2) and (18) that

$$[((d*r)*r)/t_r] \in {}^{\kappa}_{g} \ \Rightarrow [(d*r)/t_r] \in {}^{\kappa}_{g}.$$

Therefore ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T by Theorem 3.10. □

Lemma 3.16. *If a Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ satisfies the condition* (24)*, then it is a Łukasiewicz fuzzy ideal of T.*

Proof. Since $0*d \leq d$ for all $d \in T$, we have ${}^{\kappa}_{g}(0) \geq \min\{{}^{\kappa}_{g}(d), {}^{\kappa}_{g}(d)\} = {}^{\kappa}_{g}(d)$ for all $d \in T$ by (24). Hence ${}^{\kappa}_{g}(0)$ is an upper bound of $\{{}^{\kappa}_{g}(d) \mid d \in T\}$. Let $d, r \in T$ and $t_r, t_u \in (0, 1]$ be such that $[(d*r)/t_r] \in {}^{\kappa}_{g}$ and $[r/t_u] \in {}^{\kappa}_{g}$. Then ${}^{\kappa}_{g}(d*r) \geq t_r$ and ${}^{\kappa}_{g}(r) \geq t_u$. Since $d*(d*r) \leq r$ for all $d, r \in T$, it follows from (24) that

$${}^{\kappa}_{g}(d) \geq \min\{{}^{\kappa}_{g}(d*r), {}^{\kappa}_{g}(r)\} \geq \min\{t_r, t_u\}.$$

Hence $[d/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$, and therefore ${}^{\kappa}_{g}$ is a Łukasiewicz fuzzy ideal of T. □

Theorem 3.17. *Let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy set of a fuzzy set g in T. Then it is a ŁPIf-ideal of T if and only if it satisfies:*

$$[r/t_r] \in {}^{\kappa}_{g},\ [u/t_u] \in {}^{\kappa}_{g} \ \Rightarrow\ [(d*r)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}, \tag{30}$$

for all $t_r, t_u \in (0, 1]$ and $d, r, r, u \in T$ with $((d*r)*r)*r \leq u$.

Proof. Assume that ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T. Let $t_r, t_u \in (0,1]$ and $d, r, r, u \in T$ be such that $((d*r)*r)*r \leq u$, $[r/t_r] \in {}^{\kappa}_{g}$ and $[u/t_u] \in {}^{\kappa}_{g}$. Then ${}^{\kappa}_{g}$ is a Łukasiewicz fuzzy ideal of T (see Theorem 3.6), and so

$${}^{\kappa}_{g}(d*r) \geq {}^{\kappa}_{g}((d*r)*r) \geq \min\{{}^{\kappa}_{g}(r), {}^{\kappa}_{g}(u)\} \geq \min\{t_r, t_u\},$$

by Lemma 3.4 and (25). Hence $[(d*r)/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$.

Conversely, suppose that ${}^{\kappa}_{g}$ satisfies (30) for all $t_r, t_u \in (0,1]$ and $d, r, r, u \in T$ with $((d*r)*r)*r \leq u$. Let $d, r, u \in T$ be such that $d*r \leq u$. Then $((d*0)*0)*r \leq u$ by (2). Since $[r/{}^{\kappa}_{g}(r)] \in {}^{\kappa}_{g}$ and $[u/{}^{\kappa}_{g}(u)] \in {}^{\kappa}_{g}$, it follows from (2) and (30) that

$$[d/\min\{{}^{\kappa}_{g}(r), {}^{\kappa}_{g}(u)\}] = [(d*0)/\min\{{}^{\kappa}_{g}(r), {}^{\kappa}_{g}(u)\}] \in {}^{\kappa}_{g}.$$

Thus ${}^{\kappa}_{g}(d) \geq \min\{{}^{\kappa}_{g}(r), {}^{\kappa}_{g}(u)\}$, and hence ${}^{\kappa}_{g}$ is a Łukasiewicz fuzzy ideal of T by Lemma 3.16. Let $x, r \in T$ and $t_r \in (0,1]$ be such that $[((d*r)*r)/t_r] \in {}^{\kappa}_{g}$. Note that $((d*r)*r)*((d*r)*r) \leq 0$, $[((d*r)*r)/{}^{\kappa}_{g}((d*r)*r)] \in {}^{\kappa}_{g}$ and $[0/{}^{\kappa}_{g}(0)] \in {}^{\kappa}_{g}$. Hence

$$[(d*r)/{}^{\kappa}_{g}((d*r)*r)] = [(d*r)/\min\{{}^{\kappa}_{g}((d*r)*r), {}^{\kappa}_{g}(0)\}] \in {}^{\kappa}_{g},$$

by (16) and (30), and therefore ${}_{g}(d*r) \geq {}^{\kappa}_{g}((d*r)*r) \geq t_r$, i.e., $[(d*r)/t_r] \in {}^{\kappa}_{g}$. Consequently, ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T by Theoem 3.10. □

Theorem 3.18. *Let ${}^{\kappa}_{g}$ be a Łukasiewicz fuzzy set of a fuzzy set g in T. Then it is a ŁPIf-ideal of T if and only if it satisfies:*

$$[r/t_r] \in {}^{\kappa}_{g},\ [u/t_u] \in {}^{\kappa}_{g} \ \Rightarrow\ [((d*u)*(r*u))/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}, \tag{31}$$

for all $t_r, t_u \in (0,1]$ and $d, r, u, r, u \in T$ with $((d*r)*u)*r \leq u$.

Proof. Suppose that ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T. Then it is a Łukasiewicz fuzzy ideal of T (see Theorem 3.6). Let $d, r, u, r, u \in T$ be such that $((d*r)*u)*r \leq u$. Assume that $[r/t_r] \in {}^{\kappa}_{g}$ and $[u/t_u] \in {}^{\kappa}_{g}$ for $t_r, t_u \in (0,1]$. Using Lemma 3.4, (26) and Proposition 3.8, we have

$${}^{\kappa}_{g}((d*u)*(r*u)) \geq {}^{\kappa}_{g}((d*r)*u) \geq \min\{{}^{\kappa}_{g}(r), {}^{\kappa}_{g}(u)\} \geq \min\{t_r, t_u\},$$

and thus $[((d*u)*(r*u))/\min\{t_r, t_u\}] \in {}^{\kappa}_{g}$.

Conversely, assume that ${}^{\kappa}_{g}$ satisfies (31). Let $[r/t_r] \in {}^{\kappa}_{g}$ and $[u/t_u] \in {}^{\kappa}_{g}$ for all $d, r, r, u \in T$ with $((d*r)*r)*r \leq u$ and $t_r, t_u \in (0,1]$. Then we get

$$[(d*r)/\min\{t_r, t_u\}] = [((d*r)*(r*r))/\min\{t_r, t_u\}] \in {}^{\kappa}_{g},$$

by putting $u = r$ in (31), and using (I_3) and (2). Therefore ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T by Theorem 3.17. □

Theorem 3.19. *Let ${}^{\kappa}_{g}$ be the Łukasiewicz fuzzy set of a fuzzy set g in T. Then the $\in$-set $({}^{\kappa}_{g}, t)_{\in}$ of ${}^{\kappa}_{g}$ with value $t \in (0.5, 1]$ is a positive implicative ideal of T if and only if the following assertions are valid.*

$$(\forall d \in T)\left(\max\{{}^{\kappa}_{g}(0), 0.5\} \geq {}^{\kappa}_{g}(d)\right), \tag{32}$$

$$(\forall d, r, u \in T)\left(\max\{{}^{\kappa}_{g}(d*u), 0.5\} \geq \min\{{}^{\kappa}_{g}((d*r)*u), {}^{\kappa}_{g}(r*u)\}\right). \tag{33}$$

Proof. Assume that the $\in$-set $({}^{\kappa}_{g}, t)_{\in}$ of ${}^{\kappa}_{g}$ with value $t \in (0.5, 1]$ is a positive implicative ideal of T. If there exists $r \in T$ such that $\max\{{}^{\kappa}_{g}(0), 0.5\} < {}^{\kappa}_{g}(r)$, then ${}^{\kappa}_{g}(r) \in (0.5, 1]$ and ${}^{\kappa}_{g}(r) > {}^{\kappa}_{g}(0)$. If we take $t = {}^{\kappa}_{g}(r)$, then $[r/t] \in {}^{\kappa}_{g}$, that is, $r \in ({}^{\kappa}_{g}, t)_{\in}$, and $0 \notin ({}^{\kappa}_{g}, t)_{\in}$. This is a contradiction, and so ${}^{\kappa}_{g}(d) \leq \max\{{}^{\kappa}_{g}(0), 0.5\}$ for all $d \in T$. Now, if the condition (33) is not valid, then there exist $r, u, d \in T$ such that

$$\min\{{}^{\kappa}_{g}((r*u)*d), {}^{\kappa}_{g}(u*d)\} > \max\{{}^{\kappa}_{g}(r*d), 0.5\}.$$

If we take $s := \min\{{}^{\kappa}_{g}((r*u)*d), {}^{\kappa}_{g}(u*d)\}$, then $s \in (0.5, 1]$, $[((r*u)*d)/s] \in ({}^{\kappa}_{g}, s)_{\in}$ and $[(u*d)/s] \in ({}^{\kappa}_{g}, s)_{\in}$, i.e., $(r*u)*d,\ u*d \in ({}^{\kappa}_{g}, s)_{\in}$. Since $({}^{\kappa}_{g}, s)_{\in}$ is a positive implicative ideal of T, we have $r*d \in ({}^{\kappa}_{g}, s)_{\in}$. But ${}^{\kappa}_{g}(r*d) < s$ implies $r*d \notin ({}^{\kappa}_{g}, s)_{\in}$, a contradiction. Hence the condition (33) is valid.

Conversely, suppose that ${}^{\kappa}_{g}$ satisfies (32) and (33). For every $t \in (0.5, 1]$, we have $0.5 < t \leq {}^{\kappa}_{g}(d) \leq \max\{{}^{\kappa}_{g}(0), 0.5\}$ for all $d \in ({}^{\kappa}_{g}, t)_{\in}$ by (32). Thus $0 \in ({}^{\kappa}_{g}, t)_{\in}$. Let $d, r, u \in T$ be such that $(d * r) * u \in ({}^{\kappa}_{g}, t)_{\in}$ and $r * u \in ({}^{\kappa}_{g}, t)_{\in}$. Then ${}^{\kappa}_{g}((d * r) * u) \geq t$ and ${}^{\kappa}_{g}(r * u) \geq t$, which imply from (33) that

$$0.5 < t \leq \min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\} \leq \max\{{}^{\kappa}_{g}(d * u), 0.5\}.$$

Hense $[(d*u)/t] \in {}^{\kappa}_{g}$, i.e., $d*u \in ({}^{\kappa}_{g}, t)_{\in}$. Therefore $({}^{\kappa}_{g}, t)_{\in}$ is a positive implicative ideal of T for $t \in (0.5, 1]$. □

Theorem 3.20. *If the Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ of a fuzzy set g in T is a ŁPIf-ideal of T, then the q-set $({}^{\kappa}_{g}, t)_q$ of ${}^{\kappa}_{g}$ with value $t \in (0, 1]$ is a positive implicative ideal of T.*

Proof. Assume that the ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T and let $t \in (0, 1]$. If $0 \notin ({}^{\kappa}_{g}, t)_q$, then $[0/t]\, \overline{q}\, {}^{\kappa}_{g}$, that is, ${}^{\kappa}_{g}(0) + t \leq 1$. Since ${}^{\kappa}_{g}(0) \geq {}^{\kappa}_{g}(d)$ for $d \in ({}^{\kappa}_{g}, t)_q$, it follows that ${}^{\kappa}_{g}(d) \leq {}^{\kappa}_{g}(0) \leq 1 - t$. Hence $[d/t]\, \overline{q}\, {}^{\kappa}_{g}$, and so $d \notin ({}^{\kappa}_{g}, t)_q$. This is a contadiction, and thus $0 \in ({}^{\kappa}_{g}, t)_q$. Let $d, r, u \in T$ be such that $(d * r) * u \in ({}^{\kappa}_{g}, t)_q$ and $r * u \in ({}^{\kappa}_{g}, t)_q$. Then $[((d * r) * u)/t]\, q\, {}^{\kappa}_{g}$ and $[(r * u)/t]\, q\, {}^{\kappa}_{g}$, that is, ${}^{\kappa}_{g}((d * r) * u) > 1 - t$ and ${}^{\kappa}_{g}(r * u) > 1 - t$. It follows from (28) that ${}^{\kappa}_{g}(d * u) \geq \min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\} > 1 - t$. Thus $[(d * u)/t]\, q\, {}^{\kappa}_{g}$ and so $d * u \in ({}^{\kappa}_{g}, t)_q$. Therefore $({}^{\kappa}_{g}, t)_q$ is a positive implicative ideal of T. □

The next corollary is obtained by the combination of Theorems 3.12 and 3.20.

Corollary 3.21. *Let ${}^{\kappa}_{g}$ be the Łukasiewicz fuzzy set of a fuzzy set g in T. If g is a fuzzy positive implicative ideal of T, then the q-set $({}^{\kappa}_{g}, t)_q$ of ${}^{\kappa}_{g}$ with value $t \in (0, 1]$ is a positive implicative ideal of T.*

Theorem 3.22. *Let g be a fuzzy set in T. For the Łukasiewicz fuzzy set ${}^{\kappa}_{g}$ of g in T, if the q-set $({}^{\kappa}_{g}, t)_q$ of ${}^{\kappa}_{g}$ is a positive implicative ideal of T, then the following arguments are valid.*

$$0 \in ({}^{\kappa}_{g}, t_r)_{\in}, \tag{34}$$

$$[((d * r) * u)/t_r]\, q\, {}^{\kappa}_{g},\ [(r * u)/t_u]\, q\, {}^{\kappa}_{g} \ \Rightarrow\ d * u \in ({}^{\kappa}_{g}, \max\{t_r, t_u\})_{\in}, \tag{35}$$

for all $d, r, u \in T$ and $t_r, t_u \in (0, 0.5]$.

Proof. Let $d, r, u \in T$ and $t_r, t_u \in (0, 0.5]$. If $0 \notin ({}^{\kappa}_{g}, t_r)_{\in}$, then $[0/t_r]\ \overline{\in}\ {}^{\kappa}_{g}$ and so ${}^{\kappa}_{g}(0) < t_r \leq 1 - t_r$ since $t_r \leq 0.5$. Hence $[0/t_r]\ \overline{q}\ {}^{\kappa}_{g}$ and thus $0 \notin ({}^{\kappa}_{g}, t_r)_q$. This is a contradiction, and therefore $0 \in ({}^{\kappa}_{g}, t_r)_{\in}$. Let $[((d * r) * u)/t_r]\, q\, {}^{\kappa}_{g}$ and $[(r * u)/t_u]\, q\, {}^{\kappa}_{g}$. Then

$$(d * r) * u \in ({}^{\kappa}_{g}, t_r)_q \subseteq ({}^{\kappa}_{g}, \max\{t_r, t_u\})_q \quad , \quad r * u \in ({}^{\kappa}_{g}, t_u)_q \subseteq ({}^{\kappa}_{g}, \max\{t_r, t_u\})_q.$$

Hence $d * u \in ({}^{\kappa}_{g}, \max\{t_r, t_u\})_q$, and so

$${}^{\kappa}_{g}(d * u) > 1 - \max\{t_r, t_u\} \geq \max\{t_r, t_u\},$$

that is, $[(d * u)/\max\{t_r, t_u\}] \in {}^{\kappa}_{g}$. Therefore $d * u \in ({}^{\kappa}_{g}, \max\{t_r, t_u\})_{\in}$. □

Theorem 3.23. *Given a fuzzy set g in T, let ${}^{\kappa}_{g}$ be the Łukasiewicz fuzzy set of g in T. If g is a fuzzy positive implicative ideal of T, then the O-set $O({}^{\kappa}_{g})$ of ${}^{\kappa}_{g}$ is a positive implicative ideal of T.*

Proof. Assume that g is a fuzzy positive implicative ideal of T. Then ${}^{\kappa}_{g}$ is a ŁPIf-ideal of T by Theorem 3.12. It is clear that $0 \in O({}^{\kappa}_{g})$. Let $d, r, u \in T$ be such that $(d * r) * u \in O({}^{\kappa}_{g})$ and $r * u \in O({}^{\kappa}_{g})$. Then $g((d * r) * u) + \kappa - 1 > 0$ and $g(r * u) + \kappa - 1 > 0$. It follows from (28) that

$$\begin{aligned} {}^{\kappa}_{g}(d * u) &\geq \min\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\} \\ &= \min\{g((d * r) * u) + \kappa - 1, g(r * u) + \kappa - 1\} > 0. \end{aligned}$$

Hence $d * u \in O({}^{\kappa}_{g})$, and therefore $O({}^{\kappa}_{g})$ is a positive implicative ideal of T. □

Theorem 3.24. *Let ${}^{\kappa}_{g}$ be the Łukasiewicz fuzzy set of a fuzzy set g in T. If the image of T under ${}^{\kappa}_{g}$ is nonzero and ${}^{\kappa}_{g}$ satisfies:*

$$[((d * r) * u)/t_r] \in {}^{\kappa}_{g},\ [(r * u)/t_u] \in {}^{\kappa}_{g} \ \Rightarrow\ [(d * u)/\max\{t_r, t_u\}]\, q\, {}^{\kappa}_{g}, \tag{36}$$

for all $d, r, u \in T$ and $t_r, t_u \in (0, 1]$, then the O-set $O({}^{\kappa}_{g})$ of ${}^{\kappa}_{g}$ is a positive implicative ideal of T.

Proof. Assume that ${}^{\kappa}_{g}(d) \neq 0$ for all $d \in T$ and the condition (36) is valid for all $d, r, u \in T$ and $t_r, t_u \in (0, 1]$. It is clear that $0 \in O({}^{\kappa}_{g})$. Let $d, r, u \in T$ be such that $(d * r) * u \in O({}^{\kappa}_{g})$ and $r * u \in O({}^{\kappa}_{g})$. Then $g((d * r) * u) + \kappa - 1 > 0$ and $g(r * u) + \kappa - 1 > 0$. Since

$$[((d * r) * u)/{}^{\kappa}_{g}((d * r) * u)] \in {}^{\kappa}_{g} \quad , \quad [(r * u)/{}^{\kappa}_{g}(r * u)] \in {}^{\kappa}_{g},$$

it follows from (36) that

$$[(d * u)/ \max\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\}] \, q \, {}^{\kappa}_{g}. \tag{37}$$

If $d * u \notin O({}^{\kappa}_{g})$, then ${}^{\kappa}_{g}(d * u) = 0$ and so

$$\begin{aligned}
&{}^{\kappa}_{g}(d * u) + \max\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\} = \max\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\} \\
&= \max\{\max\{0, g((d * r) * u) + \kappa - 1\}, \max\{0, g(r * u) + \kappa - 1\}\} \\
&= \max\{g((d * r) * u) + \kappa - 1, g(r * u) + \kappa - 1\} \\
&= \max\{g((d * r) * u), g(r * u)\} + \kappa - 1 \\
&\leq 1 + \kappa - 1 = \kappa \leq 1,
\end{aligned}$$

that is, $[(d * u)/ \max\{{}^{\kappa}_{g}((d * r) * u), {}^{\kappa}_{g}(r * u)\}] \, \overline{q} \, {}^{\kappa}_{g}$. This is impossible, and thus $d * u \in O({}^{\kappa}_{g})$. Therefore $O({}^{\kappa}_{g})$ is a positive implicative ideal of T. □

Theorem 3.25. *Let ${}^{\kappa}_{g}$ be the Łukasiewicz fuzzy set of a fuzzy set g in T. If it satisfies $[0/\kappa] \, q \, g$ and the condition* (35) *for all $d, r \in T$ and $t_r, t_u \in (0, 1]$, then the O-set $O({}^{\kappa}_{g})$ of ${}^{\kappa}_{g}$ is a positive implicative ideal of T.*

Proof. It is clear that $0 \in O({}^{\kappa}_{g})$ by the condition $[0/\kappa] \, q \, g$. Let $d, r, u \in T$ be such that $(d * r) * u \in O({}^{\kappa}_{g})$ and $r * u \in O({}^{\kappa}_{g})$. Then $g((d * r) * u) + \kappa - 1 > 0$ and $g(r * u) + \kappa - 1 > 0$. Hence

$$\begin{aligned}
{}^{\kappa}_{g}((d * r) * u) + 1 &= \max\{0, g((d * r) * u) + \kappa - 1\} + 1 \\
&= g((d * r) * u) + \kappa - 1 + 1 \\
&= g((d * r) * u) + \kappa > 1,
\end{aligned}$$

and

$${}^{\kappa}_{g}(r * u) + 1 = \max\{0, g(r * u) + \kappa - 1\} + 1 = g(r * u) + \kappa - 1 + 1 = g(r * u) + \kappa > 1,$$

that is, $[((d * r) * u)/1] \, q \, {}^{\kappa}_{g}$ and $[(r * u)/1] \, q \, {}^{\kappa}_{g}$. It follows from (35) that

$$d * u \in ({}^{\kappa}_{g}, \max\{1, 1\})_{\in} = ({}^{\kappa}_{g}, 1)_{\in}.$$

Hence $d * u \in O({}^{\kappa}_{g})$ because if not, then $g(d * u) + \kappa - 1 \leq 0$ and so $g(d * u) \leq 1 - \kappa < 1$, which is a contradiction. Therefore $O({}^{\kappa}_{g})$ is a positive implicative ideal of T. □

4 Conclusion

Based on on Łukasiewicz t-norm, Jun addressed so called a Łukasiewicz fuzzy set and applied it to BCK-algebras and BCI-algebras. In this paper, we dealt with the concept of Łukasiewicz fuzzy positive implicative ideals in BCK-algebras and investigated several properties. We considered characterization of a Łukasiewicz fuzzy positive implicative ideal, and discussed the relationship between Łukasiewicz fuzzy ideals and Łukasiewicz fuzzy positive implicative ideals. We provided a condition for a Łukasiewicz fuzzy ideal to be a Łukasiewicz fuzzy positive implicative ideal. We also provided conditions for the $\in$-set, q-set and O-set to be positive implicative ideals. Using the ideas and results of this paper, we will study various sub-structures in several algebraic systems, for example, BCC-algebras, BCH-algebras, equality algebras, EQ-algebras, hoop algebras, BE-algebras, GE-algebras, etc., in the future. We will also explore Łukasiewicz intuitionistic fuzzy sets, Łukasiewicz bipolar fuzzy sets, Łukasiewicz Pythagorean fuzzy sets, Łukasiewicz picture fuzzy sets, etc. as the generalization of Łukasiewicz fuzzy sets.

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