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Possibility operators over n-valued Gödel logic

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Abstract

In the area of fuzzy logic, expansions of these logics by Δ operator have been intensively studied; the interest of Δ operator is due to the fact that it presents a fuzzy behavior, the associated systems were studied in propositional and first-order level. On the other hand, the possibility operators that define Łukasiewicz-Moisil algebras have been studied over different classes of algebras; these operators are known as Moisil's operators in the literature. One of these operators coincides with Δ, showing there are other operators with fuzzy behavior. In this paper, we present the study of Moisil's operators over an extension of a fuzzy logic; namely, n-valued Gödel logic, thus opening the possibility to explore more fuzzy operators.

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1 Introduction

Moisil introduced n-valued Łukasiewicz algebras or n-valued Łukasiewicz-Moisil algebras, see, for instance, [3]. Recall that the standard n-valued Łukasiewicz-Moisil algebras is defined by C_n whose universe is

$$\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\},$$

endowed with the operations $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$, $\sim x := 1 - x$ and the operators σ_i are define as follows:

$$\sigma_i(\frac{j}{n}) = \begin{cases} 0 & \text{if } i + j < n \\ 1 & \text{if } i + j \geq n \end{cases}.$$

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The operators σ_i are certain lattice-homomorphisms for $0 \leq i \leq n-1$. Interestingly, certain extensions of n-valued Heyting algebras expanded by Moisil's operators σ_i were studied by Cignoli and Iturrioz. These structures were considered with the intention of presenting termwise equivalent classes to the MV_n-algebras and n-valued Łukasiewicz-Moisil algebras, respectively, [8, 22].

Later on, Canals-Frau and Figallo studied different fragments of the class studied by Cignoli and Iturrios, [4, 5]. Recently, Figallo-Orellano and Slagter studied an implicational fragment with disjunction and presented sound and complete propositional and quantified calculus w.r.t. the class of these algebras, [18]. The adequacy Theorems were given through a new algebraic logic technique developed in this paper; furthermore, this technique was also applied to a family of semisimple varieties studied in literature of algebraic logic, see [18].

On the other hand, the operator σ_0 was studied and called Δ operator by Baaz, [2]; he studied the propositional and the quantified version of Gödel logic expanded by Δ. Later on, Hájek studied the extension of Basic Fuzzy Logic (BL), Łukasiewicz logic, Product logics and other fuzzy logics with Δ operator, [20]. In this setting, Esteva and Godo introduced the logic MTL and its extension MTL_Δ by Δ operator, [10], see also [11, 12]. Furthermore, Hájek and Cintula called all these systems Δ-fuzzy logics and presented their quantified version with the respective soundness and Completeness Theorem in [21]. Their completeness proof for these first-order logics is obtained by adding the axiom of *constant domains* and using a similar Henkin's strategy.

In this paper, we will introduce the class of n-valued σ-Gödel logic, this class of algebras is obtained by taking n-valued Heyting algebras expanded by Moisil's possibility operator. Later on, we will present the propositional and quantified logics that have the class introduced as algebraic counterpart. The soundness and Completeness Theorem will be proved by applying the technique developed in [17] and [18], see also [9].

Recall that n-valued Heyting algebras are also known as n-valued Gödel algebras and they are algebraic counterpart to *intuitionistic logic* with a matrix based on a chain with n elements. To present this logic, let us consider the language of Gödel logic (for short, $\mathcal{G}$) is built as usual from a countable set of propositional V, the constant $\bot$, the binary connectives $\wedge$ and $\to$. Disjunction and negation defined as $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ and $\neg\varphi := \varphi \to \bot$, and the constant $\top$ as $\bot \to \bot$. The axioms of $\mathcal{G}$ are the following:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(A2) $(\varphi \wedge \psi) \to \varphi$,

(A3) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$,

(A4) $(\varphi \wedge (\varphi \to \psi)) \to (\psi \wedge (\psi \to \varphi))$,

(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi))$,

(A5b) $((\varphi \wedge \psi) \to \chi)) \to (\varphi \to (\psi \to \chi))$,

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,

(A7) $\top \to \varphi$,

(A8) $\varphi \to (\varphi \wedge \varphi)$.

The only deduction rule of $\mathcal{G}$ is *modus ponens*. This axiomatic comes from adding (A8) of Hájek's BL logic, [20]. Later on, it was shown that axioms (A2) and (A3) were in fact redundant. It is well-known that the algebraic counterpart of $\mathcal{G}$ is the class of Gödel algebras, which is a variety generated by a Gödel algebra with support of the unit interval $[0,1]$. If we replace the interval by the truth-table set $GV_n = \{0, 1/(n-1), \cdots, (n-2)/(n-1), 1\}$, we obtain the standard algebra of n-valued Gödel logic (for short, $\mathcal{G}_n$), which is the axiomatic extension of $\mathcal{G}$ with the axiom:

(G_n) $(\varphi_1 \to \varphi_2) \vee (\varphi_3 \to \varphi_4) \vee \cdots \vee (\varphi_{n-1} \to \varphi_n)$.

2 The class of n-valued σ-Gödel algebras

We start by recalling that M. Canals-Frau and A. V. Figallo studied the n-valued implicative fragment with Moisil possibility operators, [4, 5]. Figallo-Orellano and Slagter studied the $\{\to, \vee\}$-fragment n-valued Hilbert algebras with Moisil possibility operators in [18]. So, we will introduce a new class of algebras where this can be seen as $\{\to, \vee, \wedge, \bot, \top\}$-fragment n-valued Distributive Hilbert algebras ([14]) with Moisil possibility operators.

Definition 2.1. *We say that the algebra $\langle A, \vee, \wedge, \to, \sigma_0, \cdots, \sigma_{n-1}, 0, 1\rangle$ is an n-valued σ-Gödel algebra (or Hey_n^σ-algebras) if the reduct $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is an n-valued Gödel algebra and the operators $\sigma_0, \cdots, \sigma_{n-1}$ verify the following conditions:*

(σ-He1) $(\sigma_0 x \to y) \to x = x$;

(σ-He2) $\sigma_i(x \to y) \to (\sigma_i x \to \sigma_j y) = 1$, *for any* $0 \leq i \leq j \leq n-1$;

(σ-He3) $(\sigma_i x \to \sigma_i y) \to ((\sigma_{i+1} x \to \sigma_{i+1} y) \to \cdots ((\sigma_{n-1} x \to \sigma_{n-1} y) \to \sigma_i(x \to y)) \cdots) = 1$;

(σ-He4) $\sigma_i(x \to \sigma_j y) = x \to \sigma_j y$;

(σ-He5) $\sigma_{n-1} x = (x \to \sigma_i x) \to \sigma_j x$, *for any* $0 \leq i \leq j \leq n-1$;

(σ-He6) $\sigma_i(x \vee y) = \sigma_i x \vee \sigma_i y$ *for all* $0 \leq i \leq n-1$;

(σ-He7) $\sigma_i(x \wedge y) = \sigma_i x \wedge \sigma_i y$ *for all* $0 \leq i \leq n-1$;

By $\mathbb{H}ey_n^\sigma$, we denote the variety of Hey_n^σ-algebras and as usual, sometimes we shall denote a Hey_n^σ-algebra $\langle A, \vee, \wedge, \to, \sigma_0, \cdots, \sigma_{n-1}, 0, 1\rangle$ by $\mathbf{A}$.

Lemma 2.2. *For each $\mathbf{A} \in \mathbb{H}ey_n^\sigma$, the following properties hold for every $x, y \in A$:*

(σ-He8) $\sigma_0 x \leq x$,

(σ-He9) $\sigma_i(\sigma_j x) = \sigma_j x$,

(σ-He10) $\sigma_j 1 = 1$,

(σ-He11) $\sigma_0 x \leq \sigma_1 x \leq \cdots \leq \sigma_{n-1} x$,

(σ-He12) $x \leq \sigma_{n-1} x$,

(σ-He13) $x \leq y$ *then* $\sigma_i x \leq \sigma_i y$,

(σ-He14) $\sigma_i(\sigma_j x \to y) = \sigma_j x \to \sigma_i y$,

(σ-He15) $x \to \sigma_j(x \to y) = \sigma_j(x \to y)$,

(σ-He16) $x \to \sigma_j y \leq \sigma_j(x \to y)$,

(σ-He17) $\sigma_j(x \to y) \leq \sigma_j x \to \sigma_j y$,

(σ-He18) $(\sigma_0 x \to \sigma_0 y) \to ((\sigma_1 x \to \sigma_1 y) \to ...((\sigma_{n-1} x \to \sigma_{n-1} y) \to (x \to y)) \cdots) = 1$,

(σ-He19) $\sigma_i x = \sigma_i y$ *for all* i, $0 \leq i \leq n-1$, *then* $x = y$,

(σ-He20) $(\sigma_j x \to y) \to \sigma_j x = \sigma_j x$, ($\sigma$-He21) $\sigma_{n-1}x = (x \to \sigma_1 x) \to x$,

(σ-He22) $\sigma_1(\sigma_1 y \to x) = (\sigma_1(\sigma_1 x \to t) \to (\sigma_1 y \to t)) = 1$,

(σ-He23) $\sigma_j(\neg x) = \neg\sigma_j x$ *and* $\sigma_j 0 = 0$ *where* $\neg x := x \to 0$.

Proof. The proof from (σ-He8) to (σ-He22) can be consulted in [4], see also [5]. The proof of (σ-He23) is in [6, Proposition 2.3]. □

Recall that for any Hilbert algebra $\mathbf{A}$, a subset D is said to be a deductive system of A (d.s., for short) if $1 \in D$ and if $x, x \to y \in D$, then $y \in D$. We denote by $\mathcal{D}(A)$ the set of deductive systems of A.

A subset D of $\mathbf{A} \in \mathbb{H}ey_n^\sigma$ is said to be a modal deductive system (m.d.s.) if $D \in \mathcal{D}(A)$, and $x \in D$ implies $\sigma_0 x \in D$. We denote by $\mathcal{D}_m(A)$ the set of all m.d.s. of the Hey_n^σ-algebra $\mathbf{A}$. Suppose M a d.s. of A. We will say M is maximal if for all M_0 d.s., such that $M \subseteq M_0$, then $M = M_0$ or $M_0 = A$. We can define the same concept for m.d.s.. Let us note that it is not hard to prove that for any maximal m.d.s. if $x \in A \backslash M$ and $y \in A$, then $\sigma_0 x \to y \in M$ and $\sigma_k x \to y \in M$, see [18, Lemma 6.4].

For a given algebra $\mathbf{A}$, it is not hard to see that an arbitrary intersection of modal deductive systems is a modal deductive system of A. Then, as usual we will consider the notion of generated modal deductive system by a set X, that we denote $[X]$. Then:

Lemma 2.3. *Let* $\mathbf{A} \in \mathbb{H}ey_n^\sigma$, *and* $M \subseteq A$. *Then:*

$$[M] = \{y \in A : \textit{there are } z_1, \cdots, z_n \in M \textit{ such that}$$
$$\sigma_0 z_1 \to (\sigma_0 z_2 \to (\cdots(\sigma_0 z_{n-1} \to (\sigma_0 z_n \to y)\cdots) = 1\}.$$

Lemma 2.4. *Let* $\mathbf{A} \in \mathbb{H}ey_n^\sigma$, *$B$ be a subalgebra of* $\mathbf{A}$ *and* $D_B \in \mathcal{D}_m(B)$. *Then, there exists* $D \in \mathcal{D}_m(A)$ *such that* $D_B = D \cap B$; *i.e., the variety of* Hey_n^σ*-algebra has the congruence extension property.*

Proof. It follows immediately from Lemma 2.3. □

Theorem 2.5. *For any* $\mathbf{A} \in \mathbb{H}ey_n^\sigma$ *and any* $D \in \mathcal{D}_m(A)$, *we have that* $Con(A) = \{R(D) : D \in \mathcal{D}_m(A)\}$ *where* $R(D) = \{(x, y) \in A^2 : x \to y, y \to x \in D\}$. *Then, there exists a lattice-isomorphism between* $Con(A)$ *and* $D_m(A)$.

Proof. It is an immediately consequence from (HM9), (HM10) and well-known results of Heyting algebras Theory. □

In what follows, we will prove that the variety of n-valued σ-Gödel algebra is in fact a semi-simple variety. To this end, let us start by considering a Hey_n^σ-algebra $\mathbf{A}$, then we can define a new binary operation $\rightarrowtail$ named weak implication such that: $x \rightarrowtail y = \sigma_0 x \to y$ for $x, y \in A$.

Lemma 2.6. [18] *Let* $\mathbf{A} \in \mathbb{H}\mathbf{ey_n^\sigma}$ *and for any* $x, y, z \in A$, *the following properties hold:*

(wi1) $1 \rightarrowtail x = x$, (wi2) $x \rightarrowtail x = 1$,

(wi3) $x \rightarrowtail \sigma_0 x = 1$, (wi4) $x \rightarrowtail (y \rightarrowtail z) = (x \rightarrowtail y) \rightarrowtail (x \rightarrowtail z)$, (wi5) $x \rightarrowtail (y \rightarrowtail x) = 1$, (wi6) $((x \rightarrowtail y) \rightarrowtail x) \rightarrowtail x = 1$.

Definition 2.7. *Let* $\mathbf{A}$ *be a* Hey_n^σ*-algebra and suppose* $D \subseteq A$*, we say that* D *is a weak deductive system (w.d.s.) if* $1 \in D$*, and if* $x, x \rightarrowtail y \in D$*, then* $y \in D$*.*

We denote by $\mathcal{D}_w(A)$ the set of weak deductive systems of a given Hey_n^σ-algebra $\mathbf{A}$. It is not hard to see that the set of modal deductive systems is equal to the set of weak deductive systems.

Now, for every (weak) deductive system D of A, we say that D is maximal if for every (weak) deductive system M such that $D \subseteq M$, then $M = A$ or $M = D$. Besides, let us consider the set of all maximal w.d.s. denoted by $\mathcal{E}_w(A)$.

Definition 2.8. *Let* $\mathbf{A}$ *be a* Hey_n^σ*-algebra,* $D \in \mathcal{D}_w(A)$ *and* $p \in A$*. We say that* D *is a weak deductive system tied to* p *if* $p \notin D$ *and for any* $D' \in \mathcal{D}(A)$ *such that* $D \subsetneq D'$*, then* $p \in D'$*.*

Lemma 2.9. *For a given* Hey_n^σ*-algebra* $\mathbf{A}$*, every modal deductive system is a weak deductive system and vice versa.*

Lemma 2.10. *Let* $\mathbf{A}$ *be a* Hey_n^σ*-algebra and* M *a maximal deductive system of* A*. Then, for every* $x \in A \setminus M$*, we have that* $\sigma_0 x \to y \in A$ *for every* $y \in A$*.*

Now, we are in conditions to prove the principal result of this section, Lemma 2.11.

Lemma 2.11. *For a given* Hey_n^σ*-algebra* $\mathbf{A}$*, then* $\{1\} = \bigcap_{M \in \mathcal{E}_w(A)} M$ *where* $\mathcal{E}_w(A)$ *is the set of maximal w.d.s of* $\mathbf{A}$*.*

Proof. To see that for every weak deductive system D, there is a weak deductive system L_p tied to some element $p \in A$ which contains it, let us first consider the set $\mathcal{D}_w(D, p) = \{S \in \mathcal{D}_w : D \subseteq S, p \notin S\}$ where $\mathcal{D}_w$ is the set of all weak deductive systems of A. It is not hard to see that every chain of $\mathcal{D}_w(D, p)$ has an upper bound on it, then by Zorn's Lemma there is a maximal element L_p on it. The set L_p is the desired weak deductive system tied to p such that $D \subseteq L_p$.

Now, it is clear that $D \subseteq \bigcap_{p \in A \setminus D} L_p$ but it is not hard to see that $D = \bigcap_{p \in A \setminus D} L_p$.

It is possible to see that every maximal weak deductive system is a weak deductive system tied to some element of A and vise versa. To see this, we need to take into account Lemma 2.10 and (wi6). Thus, since $\{1\}$ is a weak deductive system, then the proof is complete. □

We will then consider the quotient algebra A/M defined by $a \equiv_M b$ iff $a \to b, b \to a \in M$, and the canonical projection $q_M : A \to A/M$ defined by $q_M = |x|_M$, where $|x|_M$ denotes the equivalence class of x generated by M. From universal algebra results, we have that if M is a maximal deductive system of A, then A/M is a simple Hey_n^σ-algebra. We say that a variety is semisimple if every subdirectly irreducible algebra is simple; or equivalently, every algebra of the variety is a subdirect product of simple algebras. Now, we will show that the variety of Hey_n^σ-algebras is in fact a semisimple one. Indeed:

Lemma 2.12. *Let* $\mathbf{A}$ *be a* Hey_n^σ*-algebra then the map* $\Phi : A \longrightarrow \prod_{M \in \mathcal{E}_w(A)} A/M$*, defined by* $\Phi(x)(M) = q_M(x)$*, is a one-to-one homomorphism.*

Proof. Taking $\prod_{M_\alpha \in \mathcal{E}_w(A)} A/M_\alpha$ where $\mathcal{E}_w(A)$ is the set of maximal w.d.s. defined before. Let us define $\Phi : A \to \prod_{M_\alpha \in \mathcal{E}_w(A)} A/M_\alpha$ such that for every α we have that $\Phi(a) = f_a$ where $f_a(\alpha) =$

$q_\alpha(a) = |a|_\alpha \in A/M_\alpha$ with $a \in A$. It is not hard to see that Φ is a Hey_n^σ-homomorphism in view of the fact that $\equiv_{M_\alpha}$ is a congruence relation. Now, from the fact that $\{1\} = \bigcap_{M \in \mathcal{E}_w(A)} M$, it is possible to see that Φ is a one-to-one function which completes the proof. □

Our next task is to determine the generating algebras. First, for a given Hey_n^σ-algebra, we want to determine the associated partition to a given congruence. Indeed:

Lemma 2.13. *Let $\mathbf{A} \in \mathbb{H}ey_n^\sigma$ which contains more than one element and $M \in \mathcal{E}_w(A)$. Then, the family $\mathcal{F}_M = \{E_j^M\}_{0 \le j \le m}$, $m \le n$ is a partition of A where*

$$E_j^M = \{a \in A : a, \sigma_k a \notin M, 1 \le k \le j, \sigma_{j+1} a \in M\},$$

with $1 \le j \le n-2$,

$$E_{n-1}^M = \{a \in A : a, \sigma_{n-1} a \notin M\},$$

and $E_0^M = M$.

In the next, we will present an important algebra that we call a standard Hey_n^σ-algebra and it is defined as follows:

$$\mathbf{C_n} = \langle \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\}, \vee, \wedge, \to \{\sigma_i\}_{0 \le i \le n-1}, 0, 1 \rangle,$$

where $x \to y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$ and $\sigma_i(\frac{j}{n}) = \begin{cases} 0 & \text{if } i+j < n \\ 1 & \text{if } i+j \ge n \end{cases}$.

It is clear that the operator σ_0 coincides with Baaz's $\triangle$-operator for n-valued Gödel algebras, see [2]. So, we are in a position to prove the following theorem.

Theorem 2.14. *Let $\mathbf{A}$ be a non-trivial Hey_n^σ-algebra, $M \in \mathcal{E}_w(A)$ and $\mathcal{F}_M = \{E_j^M\}_{0 \le j \le m}$, $m \le n-1$ the partition associated with M. Then, the map $h : A \longrightarrow \mathbf{C_n}$ such that $h(x) = \dfrac{n-j}{n}$ if $x \in E_j^M$ is an homomorphism and $h^{-1}(\{1\}) = M$.*

Proof. In paper [18], it was proved that $h(\sigma_k x) = \sigma_k h(x)$, $h(x \to y) = h(x) \to h(y)$ and $h(x \vee y) = h(x) \vee h(y)$. So, we only have to prove that $h(x \wedge y) = h(x) \wedge h(y)$.Indeed, from $(\sigma - He7)$ and the fact that $(x \wedge y) \to x = (x \wedge y) \to y = 1$, it follows that $x, y \in E_j^\Gamma$ implies that $x \wedge y \in E_j^\Gamma$. Furthermore, if $x \in E_j^\Gamma$ and $y \in E_i^\Gamma$, with $0 < i < j < 1$, then it is easy to see that $h(x \wedge y) = h(x) \wedge h(y)$. The latter is obtained by taking into account that $\sigma_k x \notin M$, $\sigma_k y \notin M$ implies $\sigma_k x \wedge \sigma_k y \notin M$. The rest of the proof is left to the reader. Finally, it is not hard to see that $h^{-1}(\{1\}) = M$. □

From the last theorem and well-known results of universal algebra, we have:

Corollary 2.15. *The simple Hey_n^σ-algebras are $\mathbb{C}_n$ and their subalgebras. They are the unique subdirectly irreducible algebras up to isomorphism.*

3 The calculus $\mathcal{H}ey_n^\sigma$

Let Var be a denumerable set of propositional variables. The symbols $\to$, $\vee$, $\wedge$ and $\sigma_0, \cdots, \sigma_{n-1}$ are named implication, supremum, infimum, and Moisil's possibility operators, respectively. We denote by Fm the set of formulas and it is defined as usual. Besides, we denote by $\mathfrak{Fm} = \langle Fm, \vee, \wedge, \to, \sigma_0, \cdots, \sigma_{n-1}, \bot, \top \rangle$ the absolutely free algebra generated by the set Var.

Definition 3.1. *We denote by* $\mathcal{H}ey_n^{\sigma}$ *the calculus determined by Gödel logic axioms, and the followings axioms and inference rules where* $\alpha, \beta, \gamma \in Fm$*:*

Axiom schemas

(σ-G1) $((\sigma_0\alpha \to \beta) \to \alpha) \to \alpha$,

(σ-G2) $\sigma_i(\alpha \to \beta) \to (\sigma_i\alpha \to \sigma_j\beta)$, *for every* i, j *such that* $0 \leq i \leq j \leq n-1$,

(σ-G3) $(\sigma_i\alpha \to \sigma_i\beta) \to ((\sigma_{i+1}\alpha \to \sigma_{i+1}\beta) \to \cdots((\sigma_{n-1}\alpha \to \sigma_{n-1}\beta) \to \sigma_i(\alpha \to \beta))\cdots)$ *for every* i *such that* $0 \leq i \leq n-1$,

(σ-G4) $(\sigma_i(\alpha \to \sigma_j\beta)) \leftrightarrow (\alpha \to \sigma_j\beta)$ *for every* i, j *such that* $0 \leq i \leq j \leq n-1$,

(σ-G5) $\sigma_{n-1}\alpha \leftrightarrow ((\alpha \to \sigma_i\alpha) \to \sigma_j\alpha)$, *for every* i, j *such that* $0 \leq i \leq j \leq n-1$,

(σ-G6) $\sigma_i(\alpha \vee \beta) \leftrightarrow (\sigma_i\alpha \vee \sigma_i\beta)$, *for every* i *such that* $0 \leq i \leq n-1$,

(σ-G7) $\sigma_i(\alpha \wedge \beta) \leftrightarrow (\sigma_i\alpha \wedge \sigma_i\beta)$, *for every* i *such that* $0 \leq i \leq n-1$,

(σ-G8) $\sigma_0\alpha \to \sigma_i\alpha$, *for every* i *such that* $1 \leq i \leq n-1$.

By $\alpha \leftrightarrow \beta$*, we denote* $\alpha \to \beta$ *and* $\beta \to \alpha$ *are axioms.*

Inference rules

(MP) $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$ (NEC) $\dfrac{\alpha}{\sigma_0\alpha}$.

We will consider the usual notion of derivation of a formula α in $\mathcal{H}ey_n^{\sigma}$. We say that α is derivable from Γ in $\mathcal{H}ey_n^{\sigma}$, denoted by $\Gamma \vdash \alpha$, if there exists a derivation of α from Γ in $\mathcal{H}ey_n^{\sigma}$. If $\Gamma = \emptyset$, then we denote it by $\vdash \alpha$. In this case, we say that α is a theorem of $\mathcal{H}ey_n^{\sigma}$. The following results can be proven in a standard way.

Proposition 3.2.

(P1) $\vdash \alpha \to \alpha$,

(P2) $\{(\alpha \to \beta) \to (\alpha \to \gamma)\} \vdash \alpha \to (\beta \to \gamma)$,

(P3) $\vdash \sigma_0\alpha \to \alpha$,

(RP1) $\dfrac{\vdash \sigma_0\alpha}{\vdash \sigma_i\alpha}$ *for every* $1 \leq i \leq j \leq n-1$,

(RP2) $\dfrac{\vdash \beta}{\vdash \alpha \to \beta}$,

(RP3) $\dfrac{\vdash \alpha \to (\beta \to \gamma)}{\vdash (\alpha \to \beta) \to (\alpha \to \gamma)}$,

(RP4) $\dfrac{\vdash \alpha \to \beta, \vdash \beta \to \gamma}{\vdash \alpha \to \gamma}$,

(RP5) $\dfrac{\vdash \alpha \to (\beta \to \gamma)}{\vdash \beta \to (\alpha \to \gamma)}$, (RP6) $\dfrac{\vdash \alpha \to \beta}{\vdash (\gamma \to \alpha) \to (\gamma \to \beta)}$, (RP7) $\dfrac{\vdash \alpha \to \beta}{\vdash (\beta \to \gamma) \to (\alpha \to \gamma)}$,

(NM2) $\dfrac{\vdash \sigma_k(\alpha \to \beta)}{\vdash \sigma_k\alpha \to \sigma_k\beta}$, (NM8) $\dfrac{\vdash \sigma_j(\sigma_i\alpha)}{\vdash \sigma_i\alpha}$,

(NM9) $\vdash \sigma_i\alpha \to \sigma_j\sigma_i\alpha$, (NM10) $\dfrac{\vdash \sigma_j(\sigma_i\alpha) \to \beta}{\vdash \sigma_i\alpha \to \beta}$,

Proof. Routine. □

Lemma 3.3. *$\equiv$ is a congruence in $\mathfrak{Fm}$, where $\equiv$ is defined by $\alpha \equiv \beta$ iff $\vdash \alpha \to \beta$ and $\vdash \beta \to \alpha$.*

Proof. We only have to see that if $\alpha \equiv \beta$, then $\sigma_i\alpha \equiv \sigma_i\beta$ for every $0 \leq i \leq i-1$. But this is immediately from (NEC), (σ-G2), (RP1) and (MP). □

From the last Lemma, it is possible to consider the algebra $\mathfrak{Fm}/\equiv$ which is known as Lindenbaum-Tarski algebra; moreover, it is not hard to see that:

Proposition 3.4. *The algebra $\mathfrak{Fm}/\equiv$ is a Hey_n^σ-algebra where $\overline{\alpha \to \alpha}$ is the greatest element, where we denote by $\overline{\alpha}$ the class of α by $\equiv$.*

Now, we will expose necessary notions in order to prove the Completeness Theorem. To this end, let us start by recalling that a logic defined over a language $\mathcal{S}$ is a system $\mathcal{L} = \langle For, \vdash \rangle$, where For is the set of formulas over $\mathcal{S}$ and the relation $\vdash_{\mathcal{L}} \subseteq \mathcal{P}(For) \times For$ and $\mathcal{P}(A)$ is the set of all subsets of A. The logic $\mathcal{L}$ is said to be a Tarskian logic if it satisfies the following properties, for every set $\Gamma \cup \Omega \cup \{\varphi, \beta\}$ of formulas:

(1) if $\alpha \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \alpha$,

(2) if $\Gamma \vdash_{\mathcal{L}} \alpha$ and $\Gamma \subseteq \Omega$, then $\Omega \vdash_{\mathcal{L}} \alpha$,

(3) if $\Omega \vdash_{\mathcal{L}} \alpha$ and $\Gamma \vdash_{\mathcal{L}} \beta$ for every $\beta \in \Omega$, then $\Gamma \vdash_{\mathcal{L}} \alpha$.

A logic $\mathcal{L}$ is said to be finitary if it satisfies the following:

(4) if $\Gamma \vdash_{\mathcal{L}} \alpha$, then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash_{\mathcal{L}} \alpha$.

The following condition is to add the *structurality* to a Tarskian logic:

(5) if $\Gamma \vdash_{\mathcal{L}} \alpha$, then $\sigma[\Gamma] \vdash_{\mathcal{L}} \sigma(\alpha)$ for each $\mathcal{L}$-substitution σ;

in this way, we obtain what is known as *deductive system.*

Definition 3.5. *Let $\mathcal{L}$ be a Tarskian logic and let Γ be a set of formulas. We say that every set of formulas is a theory. Moreover, Γ is said to be a consistent theory if there is a formula φ such that $\Gamma \nvdash_{\mathcal{L}} \varphi$. Besides, we say that Γ is a maximal consistent theory if $\Gamma, \psi \vdash_{\mathcal{L}} \varphi$ for any formula $\psi \notin \Gamma$; and, in this case, we say Γ is maximal respect to φ.*

A set of formulas Γ is closed in $\mathcal{L}$ if the following property holds for every formula φ: $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\varphi \in \Gamma$. It is easy to see that any maximal consistent theory is a closed one.

Lemma 3.6 (Lindenbaum-Łoś)**.** *Let $\mathcal{L}$ be a Tarskian and finitary logic. Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that $\Gamma \nvdash_{\mathcal{L}} \varphi$. Then, there exists a set of formulas Ω such that $\Gamma \subseteq \Omega$ with Ω maximal consistent theory with respect to the formula φ in $\mathcal{L}$.*

Proof. The proof can be found [23, Theorem 2.22]. □

Going back to our logic, we can affirm that $\mathcal{H}ey_n^\sigma$ is a Tarskian and finitary logic. Now, we are in condition to see the following:

Proposition 3.7. [18] *Let $\Gamma \cup \{\alpha\}$ be a set of formulas where Γ is a maximal theory with respect to α, then:*

(NM11) *If $\sigma_i\alpha \in \Gamma$, then $\sigma_{i+1}\alpha, \cdots, \sigma_{n-1}\alpha \in \Gamma$ with $1 \leq i < n-1$,*

(NM12) *If $\sigma_{n-1}\alpha \notin \Gamma$, then $\sigma_i\alpha \notin \Gamma$ with $1 \leq i \leq n-1$.*

(NM13) *$\alpha \wedge \beta \in \Gamma$ if and only if $\alpha \in \Gamma$ and $\beta \in \Gamma$.*

Proof. The proof of (NM11) and (NM12) are in [18, Proposition 4.5]. The proof of (NM13) follows immediately from that the formula $\alpha \to (\beta \to (\alpha \wedge \beta))$ is a theorem in Gödel logic taking it as an extension of the logic BL. □

We will consider a *consequence relation* $\vDash$ as follows: for a given function $v : \mathfrak{Fm} \to \mathbf{A}$, we say that v is a valuation for $\mathcal{H}ey_n^\sigma$ if it satisfies $v(\alpha\#\beta) = v(\alpha)\#v(\beta)$ with $\# \in \{\to, \vee\}$, $v(\sigma_i\alpha) = \sigma_i v(\alpha)$ for every $0 \leq i \leq n-1$. Besides, we say that α is a semantically valid formula if, for all valuation v and for all Hey_n^σ-algebras $\mathbf{A}$, $v(\alpha) = 1$ and we denote it by $\vDash \alpha$. Moreover, we say $\Gamma \vDash \alpha$ if for every valuation v and every Hey_n^σ-algebra $\mathbf{A}$, if $v(\beta) = 1$ for every $\beta \in \Gamma$, then $v(\alpha) = 1$.

Now, for a given maximal theory Γ with respect to φ, we denote by $\Gamma/\equiv$ the set $\{\overline{\alpha} : \alpha \in \Gamma\}$. It is clear that $\Gamma/\equiv$ is a subset of the n-valued Gödel modal algebra $\mathfrak{Fm}/\equiv$. Then:

Theorem 3.8. [18] *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}$, with Γ non-trivial maximal respect to φ in $\mathcal{H}ey_n^\sigma$. Then:*

(i) *if $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$, then $\beta \in \Gamma$;*

(ii) *$\Gamma/\equiv$ is a modal deductive system tied to $\overline{\varphi}$ of $\mathfrak{Fm}/\equiv$.*

It is important to note that from the last theorem, we have that $\Gamma/\equiv$ is a maximal deductive system in the sense of Definition 2.8. Now, the following lemma can be proven using Theorem 3.8 and Lemma 2.10.

Lemma 3.9. *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}$, with Γ non-trivial maximal with respect to φ in $\mathcal{H}ey_n^\sigma$. If $\alpha \notin \Gamma$ then, $\sigma_0\alpha \to \beta \in \Gamma$ for any $\beta \in \mathfrak{Fm}$.*

The following theorem is an adaptation of Theorem 2.14 to the syntactic context where we use the algebra $\mathbf{C_n}$ mentioned in this theorem.

Theorem 3.10. *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}$, with Γ non-trivial maximal respect to φ in $\mathcal{H}ey_n^\sigma$. Consider the map $v : \mathfrak{Fm} \to \mathbf{C_n}$, defined by $v(\alpha) = \dfrac{n-j}{n}$ if $\alpha \in E_j^\Gamma$ where $\mathbf{C_n} = \langle \mathbf{C_n}, \to, \vee, \sigma_0, \cdots, \sigma_{n-1}, 1\rangle$ and*

$$E_j^\Gamma = \{\alpha \notin \Gamma : \sigma_k\alpha \notin \Gamma, 0 \leq k \leq j, \sigma_{j+1}\alpha \in \Gamma\},$$

with $0 \leq j < n-1$ and $E_0^\Gamma = \Gamma$ and

$$E_{n-1}^\Gamma = \{\alpha \notin \Gamma : \sigma_{n-1}\alpha \notin \Gamma\}.$$

Then, v is homomorphism in $\mathcal{H}ey_n^\sigma$.

Proof. We have to prove that v is an homomorphism. In paper [18], it was proved that $v(\alpha \to \beta) = v(\alpha) \to v(\beta)$, $v(\sigma_j \alpha) = \sigma_j v(\alpha)$ and $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$. So, we only show that $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$. Let us suppose that $v(\alpha \wedge \beta) = \dfrac{n-j}{n}$. Then, $\alpha \wedge \beta \in E_j^\Gamma$ and $\sigma_s(\alpha \wedge \beta) \notin \Gamma$ (with $0 \leq s \leq j$), $\sigma_{j+1}(\alpha \wedge \beta) \in \Gamma$. Thus, we have to prove that $\sigma_s(\alpha), \sigma_s(\beta) \notin \Gamma$ (with $0 \leq s \leq j$) and $\sigma_{j+1}(\alpha), \sigma_{j+1}(\beta) \in \Gamma$. Indeed, if $\sigma_s(\alpha), \sigma_s(\beta) \in \Gamma$, then from $(\sigma - G7)$ we obtain that $\sigma_s(\alpha \wedge \beta) \in \Gamma$ (with $0 \leq s \leq j$), which is a contradiction. Therefore, $\sigma_s(\alpha \wedge \beta) \notin \Gamma$ with $0 \leq s \leq j$. Taking into account (NM13), it is easy to see that $\sigma_{j+1}(\alpha), \sigma_{j+1}(\beta) \in \Gamma$ implies $\sigma_{j+1}(\alpha) \wedge \sigma_{j+1}(\beta) \in \Gamma$ and then $\sigma_{j+1}(\alpha \wedge \beta) \in \Gamma$ as desired. □

Theorem 3.11. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}$, $\Gamma \vdash \varphi$ *if and only if* $\Gamma \vDash \varphi$.

Proof. It is not hard to see that every axiom of $\mathcal{H}ey_n^\sigma$ is valid; furthermore, satisfaction is preserved by the inference rules.

Conversely, if $\Gamma \nvdash \varphi$, then by Theorem 3.6, there exists Ω a non-trivial maximal respect to φ such that $\Gamma \subseteq \Omega$. By Theorem 3.10, there exists a valuation $v : \mathfrak{Fm} \to \mathbf{C_n}$ such that $v(\psi) = 1$ iff $\psi \in \Omega$. By hypothesis, we know that $\varphi \notin \Omega$. So, $v(\varphi) \neq 1$, then $\Omega \nvDash \varphi$. Since $\Gamma \subseteq \Omega$, then $\Gamma \nvDash \varphi$, which is a contradiction. □

4 The first-order logics of $\mathcal{H}ey_n^\sigma$: logic $\mathcal{QH}ey_n^\sigma$

In this section, the first-order logic of $\mathcal{H}ey_n^\sigma$ will be introduced. To this end, let us start by assuming Θ the propositional signature of $\mathcal{H}ey_n^\sigma$, as well as two quantified symbols $\forall$ and $\exists$, together with the punctuation marks, commas and parentheses. Furthermore, let us consider Var be a denumerable set of individual variables. We denote by $\mathfrak{Fm}_\Sigma$ the set of the formulas and denote by Ter the absolutely free algebra of the terms. Next, we will consider a complete Hey_n^σ-algebra $\mathbf{A}$ as a lattice in which all subsets have both a supremum and an infimum.

As usual, a first-order signature Σ is a triple $\langle \mathcal{P}, \mathcal{F}, \mathcal{C} \rangle$, where $\mathcal{P}$ denotes a non-empty set of predicate symbols, $\mathcal{F}$ is a set of function symbols and $\mathcal{C}$ denotes a set of of individual constants. The notions of bound and free variables, closed terms, sentences, and substitutability are also defined in the standard way.

A Σ-structure $\mathfrak{A}$ is a pair $\langle \mathbf{A}, \mathbf{S} \rangle$ where $\mathbf{A}$ is a complete Hey_n^σ-algebra, $\mathbf{S} = \langle S, \{P_\mathbf{S}\}_{P \in \mathcal{P}}, \{f_\mathbf{S}\}_{f \in \mathcal{F}}, \mathcal{C}, \cdot^\mathfrak{A} \rangle$, S is a non-empty domain and $\cdot^\mathfrak{A}$ is an interpretation map which assigns:

- to each individual constant $c \in \mathcal{C}$, an element $c^\mathfrak{A}$ of S;
- to each functional symbol f, a function $f^\mathfrak{A} : S^n \to S$;
- to each predicate symbol P of arity n, a function $P^\mathfrak{A} : S^n \to A$.

By $\varphi(x/t)$, we denote the formula that results from φ by replacing simultaneously all the free occurrences of the variable x by t.

Let Σ be a first-order signature. The logic $\mathcal{QH}ey_n^\sigma$ over Σ is obtained by extending $\mathcal{H}ey_n^\sigma$ to the new language and adding the following axioms and rules:

Axioms Schemas

(Q1) $\varphi(x/t) \to \exists x \varphi$, if t is a term free for x in φ,

(Q2) $\forall x \varphi \to \varphi(x/t)$, if t is a term free for x in φ,

(Q3) $\sigma_i \exists x \varphi \leftrightarrow \exists x \sigma_i \varphi$, with $1 \leq i \leq n-1$, (Q4) $\sigma_i \forall x\varphi \leftrightarrow \forall x\sigma_i\varphi$, with $1 \leq i \leq n-1$.

Inference rules

(QR1) $\dfrac{\alpha \to \beta}{\exists x\alpha \to \beta}$, and x does not occur free in β,

(QR2) $\dfrac{\alpha \to \beta}{\alpha \to \forall x\beta}$, and x does not occur free in α,

We denote by $\vdash \alpha$ the derivation of a formula α in $\mathcal{QH}ey_n^\sigma$, and with $\Gamma \vdash \alpha$ the derivation of α from the set of premises Γ. These notions are defined as usual. We denote $\vdash \varphi \leftrightarrow \psi$ as an abbreviation of $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$.

A $\mathfrak{A}$-valuation is a mapping $v : Var \to S$. By $v[x \to a]$ we denote the the following $\mathfrak{A}$-valuation, $v[x \to a](x) = a$ and $v[x \to a](y) = v(y)$ for any $y \in V$ such that $y \neq x$.

It is important to note that the axiom (Q3) and (Q4) comes from the definition of monadic MV_n-algebras, see [15, Section 7]. Furthermore, this request is present on monadic version of algebraic structures with possibility operators or simply unary operators where these operators commute with the quantifiers as we can see in the paper [1, 16, 19].

Returning to our logic, let $\mathfrak{S} = \langle \mathbf{A}, \mathbf{S}\rangle$ be a Σ-structure and v a $\mathfrak{S}$-valuation. We define the values of the terms and the truth values of the formulas in $\mathfrak{S}$ for a valuation v as follows:

$$||c||_v^{\mathfrak{S}} = c^{\mathfrak{S}} \text{ if } c \in S,$$
$$||x||_v^{\mathfrak{S}} = v(x) \text{ if } x \in Var,$$
$$||f(t_1, \cdots, t_n)||_v^{\mathfrak{S}} = f^{\mathfrak{S}}(||t_1||_v^{\mathfrak{S}}, \cdots, ||t_n||_v^{\mathfrak{S}}), \text{ for any } f \in \mathcal{F},$$
$$||P(t_1, \cdots, t_n)||_v^{\mathfrak{S}} = P^{\mathfrak{S}}(||t_1||_v^{\mathfrak{S}}, \cdots, ||t_n||_v^{\mathfrak{S}}), \text{ for any } P \in \mathcal{P},$$
$$||\alpha \to \beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \to ||\beta||_v^{\mathfrak{S}},$$
$$||\alpha \wedge \beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \wedge ||\beta||_v^{\mathfrak{S}},$$
$$||\alpha \vee \beta||_v^{\mathfrak{S}} = ||\alpha||_v^{\mathfrak{S}} \vee ||\beta||_v^{\mathfrak{S}},$$
$$||\neg\alpha||_v^{\mathfrak{S}} = \neg||\alpha||_v^{\mathfrak{S}},$$
$$||\sigma_i\alpha||_v^{\mathfrak{S}} = \sigma_i||\alpha||_v^{\mathfrak{S}},$$
$$||\forall x\alpha||_v^{\mathfrak{S}} = \bigwedge_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{S}},$$
$$||\exists x\alpha||_v^{\mathfrak{S}} = \bigvee_{a\in S} ||\alpha||_{v[x\to a]}^{\mathfrak{S}}.$$

Now, it is easy to see that the following property $||\varphi(x/t)||_v^{\mathfrak{A}} = ||\varphi||_{v[x\to ||t||_v^{\mathfrak{A}}]}^{\mathfrak{A}}$ holds.

Now, we say that $\mathfrak{A}$ and v **satisfy** a formula φ, denoted by $\mathfrak{A} \vDash \varphi[v]$, if $||\varphi||_v^{\mathfrak{A}} = 1$. Besides, we say that φ is **true** $\mathfrak{S}$ if $||\varphi||_v^{\mathfrak{A}} = 1$ for each $\mathfrak{A}$-valuation v and we denote it by $\mathfrak{A} \vDash \varphi$. We say that φ is a **semantical consequence** of Γ in $\mathcal{QH}_n^\sigma$, if, for any structure $\mathfrak{A}$: if $\mathfrak{A} \vDash \gamma$ for each $\gamma \in \Gamma$, then $\mathfrak{A} \vDash \varphi$. In this case, we denote it by $\Gamma \vDash \varphi$.

The following technical result is essential to prove the Soundness Theorem.

Lemma 4.1. [18] *Let* $\mathbf{A}$ *be a complete* Hey_n^σ*-algebra and the set* $\{a_i\}_{i\in I}$ *of elements of* A *for any non-empty set* I*. Then, if there exists* $\bigvee_{i\in I} a_i$ $(\bigwedge_{i\in I} a_i)$*, then there exists* $\bigvee_{i\in I} \sigma_j a_i$ $(\bigwedge_{i\in I} \sigma_j a_i)$*, and also* $\bigvee_{i\in I} \sigma_j a_i = \sigma_j \bigvee_{i\in I} a_i$ *and* $\bigwedge_{i\in I} \sigma_j a_i = \sigma_j \bigwedge_{i\in I} a_i$ *hold, for every* $0 \leq j \leq n-1$*.*

Theorem 4.2. *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, if $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$*

Proof. Let us consider the fixed structure $\mathfrak{M} = \langle \mathbf{A}, \mathbf{S} \rangle$. Let φ be a formula such that $\Gamma \vdash \varphi$. Then, there exists $\alpha_1, \cdots, \alpha_n$ a derivation of φ from Γ. If $n = 1$ then φ is an axiom or $\varphi \in \Gamma$. If $\varphi \in \Gamma$, then it is easy to see that $\Gamma \vDash \varphi$. If φ is an axiom we have the trueness of (σ-G1) to (σ-G8). Now, let us Suppose that φ is $\alpha(x/t) \to \exists x\alpha$. Then, $||\varphi||^{\mathfrak{M}}_v = ||\alpha||^{\mathfrak{M}}_{v[x \to ||t||^{\mathfrak{M}}_v]} \to ||\exists x\alpha||^{\mathfrak{M}}_v$. It is clear that $||\alpha||^{\mathfrak{M}}_{v[x \to ||t||^{\mathfrak{M}}_v]} \leq \bigvee_{a \in S} ||\alpha||^{\mathfrak{M}}_{v[x \to a]}$, then $||\alpha||^{\mathfrak{M}}_{v[x \to ||t||^{\mathfrak{M}}_v]} \leq ||\exists x\alpha||^{\mathfrak{M}}_v$. Therefore $||\alpha(x/t) \to \exists x\alpha||^{\mathfrak{M}}_v = 1$. So, we have the axiom (Q1) is valid on $\mathfrak{M} = \langle \mathbf{A}, \mathbf{S} \rangle$. In an analogous way, we have the axiom (Q2) is also valid. To prove the validity of (Q3) and (Q4), we need to use Lemma 4.1. Besides, it is not difficult to see that satisfaction is preserved by the inference rules. □

In what follows, we will prove a strong version of Completeness Theorem for $\mathcal{QH}^\sigma_n$ using the Lindenbaum-Tarski algebra in a similar way to the propositional case. First, let us consider the notion of (maximal) consistent and closed theories with respect to some formula in the same way as the propositional case. Therefore, we have that Lindenbaum- Łoś Theorem holds for $\mathcal{QH}^\sigma_n$, see Section 3. The relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash \beta \to \alpha$ and $\vdash \alpha \to \beta$. Thus, we have the algebra $\mathfrak{Fm}_\Sigma/\equiv$ is a Hey^σ_n-algebra and the proof is exactly the same as in the propositional case. On the other hand, it is clear that $\mathcal{QH}^\sigma_n$ is a Tarskian and finitary logic, see Section 3. Then, we have the following:

Lemma 4.3. [18] *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Sigma$, with Γ non-trivial maximal with respect to φ in $\mathcal{QH}^\sigma_n$. Let $\Gamma/\equiv = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of $\mathfrak{Fm}_\Sigma/\equiv$, then:*

(i) *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$, then $\beta \in \Gamma$. Besides, it is verified that $\Gamma/\equiv = \{\overline{\alpha} : \Gamma \vdash \alpha\}$ in this case we say that it is closed.*

(ii) *$\Gamma/\equiv$ is a modal deductive system of $\mathfrak{Fm}_\Sigma/\equiv$. Also, if $\overline{\varphi} \notin \Gamma/\equiv$ and for any modal deductive system $\overline{D}$ being closed in the sense of 1 and containing properly to $\Gamma/\equiv$, then $\overline{\varphi} \in \overline{D}$.*

The previous lemma is essential in the proof of Completeness Theorem because it allow us to prove the following technical result:

Proposition 4.4. *Let $\mathfrak{Fm}_\Sigma/\Gamma$ be the Hey^σ_n-algebra defined as: $\alpha \equiv_\Gamma \beta$ iff $\alpha \to \beta, \beta \to \alpha \in \Gamma$. Then, $\mathfrak{Fm}_\Sigma/\Gamma$ is a finite chain which is a simple Hey^σ_n-algebra.*

Proof. For a given maximal consistent theory Γ of $\mathfrak{Fm}_\Sigma$, we have $\Gamma/\equiv$ is a maximal modal deductive system of $\mathfrak{Fm}_\Sigma/\equiv$, this is thanks to Lemma 4.3. Let us denote $A := \mathfrak{Fm}_\Sigma/\equiv$ and $\theta := \Gamma/\equiv$ by well-known results of Universal algebra, we have the quotient algebra A/θ is a simple algebra, see Theorem 2.14.

From the latter and by adapting the first isomorphism theorem, we have that A/θ is isomorphic to $\mathfrak{Fm}_\Sigma/\Gamma$ where it is defined by the congruence $\alpha \equiv_\Gamma \beta$ iff $\alpha \to \beta, \beta \to \alpha \in \Gamma$ as desired. □

Now, we are in conditions to prove the following central theorem:

Theorem 4.5. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas sentences, if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. Let us suppose $\Gamma \vDash \varphi$ and $\Gamma \nvdash \varphi$. Then, by Lindenbaum- Łoś Lemma, there exists Δ maximal consistent theory with respect to φ such that $\Gamma \subseteq \Delta$. Now, consider the algebra $\mathfrak{Fm}_\Sigma/\Delta$ defined by the congruence $\alpha \equiv_\Delta \beta$ iff $\alpha \to \beta, \beta \to \alpha \in \Delta$. We know that $\mathfrak{Fm}_\Sigma/\Delta$ is isomorphic to a subalgebra of $\mathbf{C_n}$ (by Proposition 4.4) and so complete as a lattice, in view of the above observations.

Let us consider the function $\pi_\Delta : \mathfrak{Fm} \to \mathfrak{Fm}_\Sigma/\Delta$ (the canonical projection) and the structure $\mathfrak{M} = \langle \mathfrak{Fm}_\Sigma/\Delta, Ter, \cdot^{Ter}\rangle$ where Ter is a set of terms defined at the beginning of the section. So, it is clear that for every $t \in Ter$ we have a constant $\hat{t}$ of Σ. Now, we can consider a function $\mu : Var \to Ter$ defined by $\mu(x) = x$ and the interpretation $||\cdot||^{\mathfrak{M}}_\mu : \mathfrak{Fm} \to \mathfrak{Fm}_\Sigma/\Delta$ defined by:

- if $\hat{t}$ is a constant, then $||\hat{t}||^{\mathfrak{M}}_\mu := t$;
- if $f \in \mathcal{F}$, then $||f(t_1, \cdots, t_n)||^{\mathfrak{M}}_\mu = f(t_1, \cdots, t_n)$;
- if $P \in \mathcal{P}$, then $||P(t_1, \cdots, t_n)||^{\mathfrak{M}}_\mu = \pi_\Delta(P(t_1, \cdots, t_n))$.

Our interpretation is defined for atomic formulas, but it is easy to see that $||\alpha||^{\mathfrak{M}}_\mu = \pi_\Delta(\alpha)$ for every quantifier-free formula α. Moreover, it is easy to see that for every formula $\phi(x)$ and every term t, we have $||\phi(x/\hat{t})||^{\mathfrak{M}}_\mu = ||\phi(x/t)||^{\mathfrak{M}}_\mu$. Therefore, from the latter property and by (Q1) and (RQ1), we have $||\forall x\alpha||^{\mathfrak{M}}_\mu = \bigwedge_{a \in T_\Theta} ||\alpha||^{\mathfrak{M}}_{\mu[x\to a]}$ and now using (Q2) and (RQ2), we obtain $||\exists x\alpha||^{\mathfrak{M}}_\mu = \bigvee_{a \in T_\Theta} ||\alpha||^{\mathfrak{M}}_{\mu[x\to a]}$. So, $||\cdot||^{\mathfrak{M}}_\mu$ is an interpretation map such that $||\alpha||^{\mathfrak{M}}_\mu = 1$ iff $\alpha \in \Delta$. On the other hand, it is not hard to see for every formula $\beta \in \Gamma \cup \{\alpha\}$, we have $||\beta||^{\mathfrak{M}}_\mu = ||\beta||^{\mathfrak{M}}_v$ for every $\mathfrak{M}$-valuation v. Therefore, $\mathfrak{M} \vDash \gamma$ for every $\gamma \in \Gamma$ but $\mathfrak{M} \nvDash \varphi$. □

Given a formula φ and suppose $\{x_1, \cdots, x_n\}$ is the set of variables of φ, the *universal closure* of φ is defined by $\forall x_1 \cdots \forall x_n \varphi$. Thus, it is clear that if φ is a sentence, then the universal closure of φ is itself. Now, we are in position to prove the following Completeness Theorem for formulas:

Theorem 4.6. *Let $\Gamma \cup \{\varphi\}$ be a set formulas. If $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.*

Proof. Let us suppose $\Gamma \vDash \varphi$ and consider the set $\forall\Gamma$ the universal closure of Γ. From the latter and definition of $\vDash$, we have $\forall\Gamma \vDash \forall x_1 \cdots \forall x_n \varphi$. Then, according to Theorem 4.5, $\forall\Gamma \vdash \forall x_1 \cdots \forall x_n \varphi$. Now, from latter and (Q1) and (RQ1), we have $\Gamma \vdash \varphi$ as desired. □

5 Concluding remarks

In this paper, we have studied logics associated to the class of n-valued σ-Gödel logic. These logics have been presented in the propositional and first-order versions. The axiomatic for the Gödel logic is displayed by extending to Basic Fuzzy logic (BL) with a special axiom. Furthermore, Adequacy theorem for the quantified versions is not based in adding axiom of *constant domain*, in general, the proof is different to the one given for Δ-fuzzy logics. As future work, we are interested in studying BL logic expanded by σ_i operators presented here. Recall that the BL logic has as axiomatic extension Łukasiewicz logic and, in n-valued case, this logic has expressive power enough to define the σ_i operator in terms of the connective of the language, see, for instance, [15, Section 7]. In our paper, the axiomatization for σ_i operators is different to the one given by Baaz for Δ operator. So, we will explore if our axiomatization for Gödel logic is good enough to the BL finite-valued logic.

References

[1] F. Almiñana, G. Pelaitay, *Monadic $k \times j$-rough Heyting algebras*, Archive for Mathematical Logic, (2021), DOI:10.1007/s00153-021-00802-6.

[2] M. Baaz, *Infinite-valued Gödel logics with* 0-1-*projections and relativizations*. In GÖDEL 96, LNL **6**, Hájek P. (Ed.), Springer-Velag, (1996), 23-33.

[3] V. Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu, *Łukasiewicz - Moisil algebras*, Annals of Discrete Mathematics, 49, North - Holland, 1991.

[4] M. Canals Frau, A.V. Figallo, $(n+1)$*-valued Hilbert modal algebras*, Notas de la Sociedad Matematica de Chile, X(1) (1991), 143–149.

[5] M. Canals Frau, A.V. Figallo, $(n+1)$*-valued modal implicative semilattices*, 1992 Proceedings The 22nd International Symposium on Multiple-Valued Logic, 190–196. IEEE Computer Society, 1992.

[6] M.C. Canals Frau, A.V. Figallo, G. Pelaitay, *Congruences on bounded Hilbert algebras with Moisil possibility operators*, South American Journal of Logic, (2021), URL: http://www.sa-logic.org/aaccess/Canals-Frau-Figallo-Pelaitay.pdf.

[7] R. Cignoli, *Estudio algebraico de lógicas polivalentes. Algebras de Moisil de orden* n, Ph. D. thesis, Universidad Nacional del Sur, Bahia Blanca, 1969.

[8] R. Cignoli, *Proper* n*-valued Łukasiewicz algebras as* S*-algebras of Łukasiewicz* n*-valued propositional calculi*, Studia Logica, 41(1) (1982), 3–16.

[9] M. Coniglio, A. Figallo-Orellano, A. Hernández-Tello, M. Pérez-Gaspar, *G'3 as the logic of modal 3-valued Heyting algebras*, Journal of Applied Logic, 9(1) (2022), 175–197.

[10] F. Esteva, L. Godo. *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, 124 (2001), 271–288.

[11] F. Esteva, L. Godo, P. Hájek, M. Navara, *Residuated fuzzy logics with an involutive negation*, Archive for Mathematical Logic, 39 (2000), 103–124.

[12] F. Esteva, L. Godo, F. Montagna, *The* $Ł\Pi$ *and the* $Ł\Pi\frac{1}{2}$ *logics: Two complete fuzzy systems joining Łukasiewicz and product logics*, Archive for Mathematical Logic, 40(1) (2001), 39–67.

[13] A. V. Figallo, G. Pelaitay, J. Sarmiento, C_n *algebras with Moisil possibility operators*, Logic Journal of the IGPL, 28(6) (2020), 1141–1154.

[14] A. Figallo Orellano, *Notes on Hilbert lattices*, International Mathematical Forum, 6(68) (2011), 3371–3379.

[15] A. Figallo Orellano, *A preliminary study of MV-algebras with two quantifiers which commute*, Studio Logica, 104 (2016), 931–956.

[16] A. Figallo Orellano, I. Pascual, *On monadic operators on modal pseudo-complemented De Morgan algebras and tetravalent modal algebras*, Studio Logica, 107 (2019), 591–611.

[17] A. Figallo-Orellano, J. Slagter, *An algebraic study of the first order version of some implicational fragments of the three-valued Łukasiewicz logic*, Computación y Sistemas, (2022), URL: https://arxiv.org/abs/2011.09868.

[18] A. Figallo-Orellano, J. Slagter, *Monteiro's algebraic notion of maximal consistent theory for Tarskian logics*, Fuzzy Sets and Systems, (2022), DOI:10.1016/j.fss.2022.04.007.

[19] C. Gallardo, A. Ziliani, *A generalization of monadic n-valued Łukasiewicz algebras*, Studio Logica, (2021), DOI: 10.1007/s11225-021-09968-9.

[20] P. Hájek, *Metamathematics of fuzzy logic*, volume 4 of Trends in Logic. Kluwer, Dordrecht, 1998.

[21] P. Hájek, P. Cintula. *On theories and models in fuzzy predicate logics*, Journal of Symbolic Logic, 71(3) (2006), 863–880.

[22] L. Iturrioz, *Łukasiewicz and symmetrical Heyting algebras*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 23 (1997), 131–136.

[23] R. Wójcicki, *Lectures on propositional calculi*, Ossolineum, Warsaw, 1984.