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Positive implicative equality algebras and equality algebras with some types

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Abstract

The notion of a positive implicative equality algebras are defined, and related properties are studied. Characterizations of a positive implicative equality algebra is investigated. Conditions for an equality algebra to be positive implicative are provided. Equality algebra with some types is considered, and several properties are investigated. Using equality algebra with some types, we characterize a commutative equality algebra and a positive implicative algebra.

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1 Introduction

A new structure, called equality algebras, is introduced by Jenei in [6]. The basic idea for examining equations of equality algebra is taken from EQ-algebras of Novák et al. [9]. The equality algebra has two connectives, a meet operation and an equivalence, and a constant. Given that equality algebra can be considered as corresponding algebras with fuzzy type theory, it is important to study in this field. In [7], the author studied the relation among equality algebra and BCK-meet

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semilattice and they proved that every $BCK(D)$-meet semilattice and equality algebra equivalent. Equality algebra has been studied by mathematicians in various fields and excellent results have been obtained, including the relation between equality algebra with other algebraic structures are obtained by Zebardast et al. in [14] such as residuated lattice, MTL-algebra, BL-algebra and etc. They showed that under special conditions, they are equivalent. Moreover, commutative equality algebras and characterizations of them are discussed in [4, 5, 13, 14]. Also, in [12], Rezaei and et al. introduced the concept of derivation on equality algebra X by using the notions of inner and outer derivations. Then they investigated some properties of (inner, outer) derivation and introduced some suitable conditions that they help us to define a derivation on X. They introduced kernel and fixed point sets of derivation on X and proved that under which condition they are filters of X. Finally, they proved that the equivalence relations on $(X, \rightarrow, 1)$ coincide with the equivalence relations on X with derivation d. In addition, in [3], Aaly and et al. introduced the concept of an implicative equality algebra and related properties are investigated. Characterizations of an implicative equality algebra and the relation among implicative and positive implicative equality algebras are discussed. Also, they defined the notion of annihilator of equality algebras and investigated some traits of it and proved that annihilator of any nonempty set of equality algebras is a deductive system. Moreover, by using this notion, they define the operation $*$ and proved that for any commutative equality algebra X, the algebraic structure $(DS(X), *, \{1\})$ is a bounded implicative BCK-algebra.

This paper contains the notion of (positive) implicative equality algebras, and we study related properties of them. We discuss characterizations of a positive implicative equality algebra, and provide conditions for an equality algebra to be positive implicative. We consider equality algebra with some types, and investigate several properties. Using equality algebra with some types, we characterize a commutative equality algebra and a positive implicative algebra.

2 Preliminaries

Definition 2.1. [6, 7] *By an equality algebra, we mean an algebra $(X, \wedge, \sim, 1)$ satisfying the next conditions.*

(E1) $(X, \wedge, 1)$ *is a commutative idempotent integral monoid,*

(E2) *The operation "$\sim$" is commutative,*

(E3) $(\forall x \in X)(x \sim x = 1)$,

(E4) $(\forall x \in X)(x \sim 1 = x)$,

(E5) $(\forall x, y, z \in X)(x \leq y \leq z \;\Rightarrow\; x \sim z \leq y \sim z,\; x \sim z \leq x \sim y)$,

(E6) $(\forall x, y, z \in X)(x \sim y \leq (x \wedge z) \sim (y \wedge z))$,

(E7) $(\forall x, y, z \in X)(x \sim y \leq (x \sim z) \sim (y \sim z))$,

where $x \leq y$ iff $x \wedge y = x$.

Note. From now on the symbol X means an equality algebra such as $(X, \wedge, \sim, 1)$.

In X, we define two operations "$\rightarrow$" and "$\leftrightharpoons$" on X as follows:

$$x \rightarrow y := x \sim (x \wedge y), \tag{1}$$

$$x \leftrightharpoons y := (x \rightarrow y) \wedge (y \rightarrow x). \tag{2}$$

Proposition 2.2. [7] *For all $x, y, z \in X$, we have*

$$x \to y = 1 \textit{iff } x \leq y, \tag{3}$$
$$x \to (y \to z) = y \to (x \to z), \tag{4}$$
$$1 \to x = x, \ \ x \to 1 = 1, \ \ x \to x = 1, \tag{5}$$
$$x \leq y \to z \ \textit{ iff } y \leq x \to z, \tag{6}$$
$$x \leq y \to x, \tag{7}$$
$$x \leq (x \to y) \to y, \tag{8}$$
$$x \to y \leq (y \to z) \to (x \to z), \tag{9}$$
$$y \leq x \ \ \Rightarrow \ \ x \leftrightharpoons y = x \to y = x \sim y, \tag{10}$$
$$x \sim y \leq x \leftrightharpoons y \leq x \to y, \tag{11}$$
$$x \leq y \ \ \Rightarrow \begin{cases} y \to z \leq x \to z, \\ z \to x \leq z \to y \end{cases} \tag{12}$$

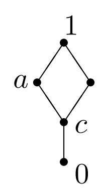
The algebraic structure X is said to be *bounded* if there is a least element such as $0 \in X$ such that $0 \leq x$ for every $x \in X$. If X is bounded, then we introduce the unary operation "$\neg$" on X as $\neg x = x \to 0 = x \sim 0$ for each $x \in X$.

3 Positive implicative equality algebras

Definition 3.1. *The structure X is said to be positive implicative if it satisfies:*

$$(\forall x, y, z \in X)(x \to (y \to z) = (x \to y) \to (x \to z)). \tag{13}$$

Example 3.2. *Suppose $X = \{0, a, b, c, 1\}$ has the next Hasse diagram.*



Then $(X, \wedge, 1)$ is a monoid. Define an operation $\sim$ on X by Table 1. Then $(X, \wedge, \sim, 1)$ is positive

Table 1: Cayley table for the implication "$\sim$"

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	a	b
c	0	b	a	1	c
1	0	a	b	c	1

implicative, and the implication $(\to)$ is shown in Table 2.

Table 2: Cayley table for the implication "$\rightarrow$"

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Theorem 3.3. *The structure X is positive implicative iff*

$$(\forall x, y \in X)(x \rightarrow (x \rightarrow y) = x \rightarrow y). \tag{14}$$

Proof. By using (13) and (5), we have

$$x \rightarrow (x \rightarrow y) = (x \rightarrow x) \rightarrow (x \rightarrow y) = 1 \rightarrow (x \rightarrow y) = x \rightarrow y,$$

for all $x, y \in X$.

Conversely, consider X is satisfying the condition (14). Consider $x, y, z \in X$. Then

$$\begin{aligned}((z \rightarrow y) \rightarrow (z \rightarrow x)) \rightarrow (z \rightarrow (y \rightarrow x)) &= ((z \rightarrow y) \rightarrow (z \rightarrow x)) \rightarrow (y \rightarrow (z \rightarrow x)) \\ &\geq y \rightarrow (z \rightarrow y) = z \rightarrow (y \rightarrow y) = z \rightarrow 1 = 1,\end{aligned}$$

and so $(z \rightarrow y) \rightarrow (z \rightarrow x) \leq z \rightarrow (y \rightarrow x)$. Also,

$$a \rightarrow (y \rightarrow z) \leq a \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = (z \rightarrow x) \rightarrow (a \rightarrow (y \rightarrow x)), \tag{15}$$

for all $a \in X$ by (4), (7) and (12). If we put $x := z \rightarrow x$, $y := z \rightarrow y$, $z := z \rightarrow (z \rightarrow x)$ and $a := z \rightarrow (y \rightarrow x)$ in (15), then

$$\begin{aligned}&((z \rightarrow (z \rightarrow x)) \rightarrow (z \rightarrow x)) \rightarrow ((z \rightarrow (y \rightarrow x)) \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x))) \\ &\geq (z \rightarrow (y \rightarrow x)) \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow (z \rightarrow x))) \\ &\geq (z \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow (z \rightarrow x)) = 1,\end{aligned}$$

by (4), (5), (9) and (12). From (3),(5) and (14) that

$$z \rightarrow (y \rightarrow x) \leq (z \rightarrow y) \rightarrow (z \rightarrow x).$$

Hence $z \rightarrow (y \rightarrow x) \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$, and therefore X is positive implicative. $\square$

Lemma 3.4. *In X, we have*

$$(\forall x, y \in X)((((x \rightarrow y) \rightarrow y) \rightarrow x) \rightarrow x \leq (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)). \tag{16}$$

Proof. Let $x, y \in X$. Since $y \leq (y \rightarrow x) \rightarrow x$ by (8), we have

$$(x \rightarrow y) \rightarrow y \leq (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x),$$

by (12). By (4),

$$\begin{aligned}x \to y &\le ((x \to y) \to y) \to ((y \to x) \to x)\\ &= (y \to x) \to (((x \to y) \to y) \to x)\\ &= (y \to x) \to (((((x \to y) \to y) \to x) \to x) \to x)\\ &= ((((x \to y) \to y) \to x) \to x) \to ((y \to x) \to x),\end{aligned}$$

which implies the $((((x \to y) \to y) \to x) \to x \le (x \to y) \to ((y \to x) \to x))$. □

Proposition 3.5. *If X is positive implicative , then*

$$(\forall x, y \in X)((((x \to y) \to y) \to x) \to x = (x \to y) \to ((y \to x) \to x)). \tag{17}$$

Proof. For any $x, y \in X$, we get

$$\begin{aligned}&(x \to y) \to ((y \to x) \to x) \le (x \to y) \to ((x \to y) \to ((y \to x) \to y))\\ &= (y \to x) \to ((x \to y) \to ((x \to y) \to y))\\ &= (y \to x) \to ((x \to y) \to y)\\ &\le (((x \to y) \to y) \to x) \to ((x \to y) \to y)\\ &\le (((x \to y) \to y) \to x) \to ((((x \to y) \to y) \to x) \to x)\\ &= (((x \to y) \to y) \to x) \to x.\end{aligned}$$

From Lemma 3.4,

$$(((x \to y) \to y) \to x) \to x = (x \to y) \to ((y \to x) \to x),$$

for all $x, y \in X$. □

Proposition 3.6. *Every positive implicative equality algebra X satisfies the next conditions:*

$$(\forall x, y \in X)(((x \to y) \to y) \to (x \to y) = x \to y). \tag{18}$$

$$(\forall x, y \in X)((x \to y) \to ((x \to y) \to y) = (x \to y) \to y). \tag{19}$$

$$(\forall x, y \in X)((x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)). \tag{20}$$

Proof. If we replace x with $x \to y$ in (17) and use (5), then

$$\begin{aligned}x \to y = 1 \to (x \to y) &= ((x \to y) \to (x \to y)) \to (x \to y)\\ &= ((((x \to y) \to y) \to y) \to (x \to y)) \to (x \to y)\\ &= ((x \to y) \to y) \to ((y \to (x \to y)) \to (x \to y))\\ &= ((x \to y) \to y) \to (x \to y),\end{aligned}$$

for all $x, y \in X$. (19) is induced by substituting $x \to y$ for x in (18). Using (4), (12) and (19), we get

$$\begin{aligned}(y \to x) \to ((x \to y) \to y) &= (y \to x) \to ((x \to y) \to ((x \to y) \to y))\\ &\ge (y \to x) \to ((x \to y) \to x)\\ &= (x \to y) \to ((y \to x) \to x).\end{aligned}$$

Similarly, we have the reverse inequality, and so (20) is valid. □

We find conditions for converting an equality algebra to positive implicative one.

Theorem 3.7. *If X satisfies the condition* (20)*, then it is positive implicative.*

Proof. Cosider X satisfying the condition (20). If we put $y := y \to x$ in (20), then

$$\begin{aligned} y \to x &= 1 \to (((y \to x) \to x) \to x) \\ &= (x \to (y \to x)) \to (((y \to x) \to x) \to x) \\ &= ((y \to x) \to x) \to ((x \to (y \to x)) \to (y \to x)) \\ &= ((y \to x) \to x) \to (y \to x) \\ &= y \to (((y \to x) \to x) \to x) \\ &= y \to (y \to x). \end{aligned}$$

By Theorem 3.3, X is positive implicative. □

Summarizing the above results induces characterizations of a positive implicative equality algebra which is described in the next theorem.

Theorem 3.8. *A structure X is positive implicative iff any one of conditions* (17), (18), (19) *and* (20) *is true.*

Proposition 3.9. *Every positive implicative equality algebra X satisfies the next assertion.*

$$(\forall x, y \in X)(x \to (x \to y) = 1 \;\Rightarrow\; x \to y = 1). \tag{21}$$

Proof. Assume that $x \to (x \to y) = 1$ for all $x, y \in X$. If we put $y := x$ and $z := y$ in (13) and use (5), then

$$x \to y = 1 \to (x \to y) = (x \to x) \to (x \to y) = x \to (x \to y) = 1,$$

for all $x, y \in X$. □

Theorem 3.10. *If X satisfies the condition* (21)*, then it is positive implicative.*

Proof. Using (4), we get

$$x \to (x \to ((x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1,$$

and so $1 = x \to ((x \to (x \to y)) \to y) = (x \to (x \to y)) \to (x \to y)$. Since $(x \to y) \to (x \to (x \to y)) = 1$, we obtain $x \to (x \to y) = x \to y$. Therefore X is a positive implicative equality algebra by Theorem 3.3. □

Given $a, b \in X$, we define

$$X(a, b) := \{x \in X \mid a \leq b \to x\}. \tag{22}$$

Clearly 1, a and b are contained in $X(a, b)$.

Definition 3.11. *The structure X is called an &-equality algebra if for all $a, b \in X$, the set $X(a, b)$ has the least element which is denoted by $a \diamond b$.*

Easily, we obtain

$$(\forall a, b \in X)(a \diamond b \leq a, a \diamond b \leq b, a \diamond 1 = 1 \diamond a = a), \tag{23}$$

in the &-equality algebra X(See [2]).

Lemma 3.12. [2] *In an &-equality algebra X, we have*

(1) $(x \diamond y) \to z = x \to (y \to z)$,

(2) $(x \to y) \diamond (y \to z) \leq x \to z$,

(3) $x \diamond y \leq x \to y \leq (z \diamond x) \to (z \diamond y)$.

(4) $x \diamond y = y \diamond x$.

(5) $(x \diamond y) \diamond z = x \diamond (y \diamond z)$.

(6) $x \leq y \;\Rightarrow\; x \diamond z \leq y \diamond z$.

Lemma 3.13. [2] *For any elements a and b of X, if $x \in X$ is infimum of $x \diamond a \leq b$, then $x = a \to b$.*

Proposition 3.14. *Suppose X is a positive implicative &-equality algebra. Then*

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; x \diamond y = x). \tag{24}$$
$$(\forall x \in X)(x \diamond x = x). \tag{25}$$
$$(\forall x, y, z \in X)(z \to (y \diamond x) = (z \to y) \diamond (z \to x)). \tag{26}$$
$$(\forall x, y \in X)(x \diamond y = (x \to y) \diamond x). \tag{27}$$

Proof. Consider $x, y \in X$ is such that $x \leq y$. Then $x \to y = 1$. Since $x \leq y \to (x \diamond y)$, by (4), (5), (9) and (13), we consequence

$$x \to (x \diamond y) = 1 \to (x \to (x \diamond y)) = (x \to y) \to (x \to (x \diamond y)) = x \to (y \to (x \diamond y)) = 1.$$

Hence $x \diamond y = x$, and (24) is valid. (25) is obtained by taking $y := x$ in (24). Using Lemma 3.12(3), we get

$$z \to x \leq (y \diamond z) \to (y \diamond x) = (z \diamond y) \to (y \diamond x). \tag{28}$$

From (6), clearly $z \diamond y \leq (z \to x) \to (y \diamond x)$. If we take $x := y$ and $y := z$ in (28) and use (25), (12) and (4), then

$$\begin{aligned} z \to y &\leq (z \diamond z) \to (z \diamond y) = z \to (z \diamond y) \\ &\leq z \to ((z \to x) \to (y \diamond x)) \\ &= (z \to x) \to (z \to (y \diamond x)), \end{aligned}$$

that is, $z \to (y \diamond x) \in X(z \to y, z \to x)$. Therefore

$$(z \to y) \diamond (z \to x) \leq z \to (y \diamond x).$$

Since $y \diamond x \leq x$ and $y \diamond x \leq y$, we have $z \to (y \diamond x) \leq z \to x$ and $z \to (y \diamond x) \leq z \to y$ by (12). Hence

$$\begin{aligned} z \to (y \diamond x) &= (z \to (y \diamond x)) \diamond (z \to (y \diamond x)) \\ &\leq (z \to y) \diamond (z \to (y \diamond x)) \\ &\leq (z \to y) \diamond (z \to x), \end{aligned}$$

by (25) and (12). Therefore $z \to (y \diamond x) = (z \to y) \diamond (z \to x)$ for all $x, y, z \in X$. If $x, y \in X$, then obviously $y \diamond x$ is a lower bound of $\{x, y\}$. Suppose $z \in X$ is a lower bound of $\{x, y\}$. Then $z \to x = 1$ and $z \to y = 1$. By (25) and (26),

$$z \to (y \diamond x) = (z \to y) \diamond (z \to x) = 1 \diamond 1 = 1,$$

i.e., $z \leq y \diamond x$. Therefore $y \diamond x$ is the greatest lower bound of $\{x, y\}$. Lemma 3.13 implies that $(x \to y) \diamond x \leq y$. Thus

$$(x \to y) \diamond x = (x \to y) \diamond (x \diamond x) = ((x \to y) \diamond x) \diamond x \leq y \diamond x,$$

by (25) and Lemma 3.12. Obviously, $y \diamond x \leq (x \to y) \diamond x$. Therefore (27) is valid. □

Theorem 3.15. *Every &-equality algebra X satisfying the condition* (27) *is positive implicative.*

Proof. Assume that an &-equality algebra X satisfies the condition (27). Using Lemma 3.12(1) and (27), we get

$$y \to x = (1 \diamond y) \to x = ((y \to y) \diamond y) \to x = (y \diamond y) \to x = y \to (y \to x),$$

for all $x, y \in X$. By Theorem 3.3, we have X is positive implicative. □

Corollary 3.16. *If an &-equality algebra X satisfies one of the conditions* (24)*,* (25) *and* (26)*, then it is positive implicative.*

Theorem 3.17. *Every bounded commutative equality algebra is an &-equality algebra.*

Proof. Consider X is a bounded commutative equality algebra. Suppose $\diamond$ is an operation on X defined by

$$\diamond : X \times X \to X, \ (a, b) \mapsto \neg(b \to \neg a).$$

Then $b \leq (b \to \neg a) \to \neg a = a \to \neg(b \to \neg a) = a \to (a \diamond b)$, and so $a \diamond b \in X(a, b)$. If $x \in X(a, b)$, then $b \leq a \to x$. Thus

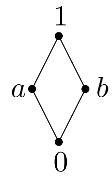
$$1 = b \to (a \to x) = b \to (a \to \neg\neg x) = \neg x \to (b \to \neg a),$$

that is, $\neg x \leq b \to \neg a$. Hence $a \diamond b = \neg(b \to \neg a) \leq \neg\neg x = x$. This shows that $a \diamond b$ is the least element of $X(a, b)$ for all $a, b \in X$. Therefore X is an &-equality algebra. □

Corollary 3.18. *Every bounded commutative equality algebra X satisfying the condition* (27) *is positive implicative.*

The next example illustrates Theorem 3.17.

Example 3.19. *Assume $X = \{0, a, b, 1\}$ has the next Hasse diagram.*



Now, we define a binary operation $\sim$ on X by Table 3. Then the implication $(\to)$ is given by Table

Table 3: Cayley table for the implication "$\sim$"

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

4. Also we can verity that it is an &-equality algebra in which the &-operation is given by Table 5.

Table 4: Cayley table for the implication "$\to$"

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Table 5: Cayley table for the binary operation "&"

&	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Now, by an example we show that a bounded &-equality algebra which is not commutative exists.

Example 3.20. *Consider X which is given in Example 3.2. Then the &-operation in X is given by Table 6. Hence X is a bounded &-equality algebra. But it is not commutative since $(c \to 0) \to 0 \neq (0 \to c) \to c$.*

Table 6: Cayley table for the operation “&”

&	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

The next theorem is an immediate consequence of Proposition 3.14 and Lemma 3.12.

Theorem 3.21. *Every positive implicative &-equality algebra is a Brouwerian semi-lattice.*

Corollary 3.22. *If every bounded commutative equality algebra is positive implicative, then it is a semi-Brouwerian algebra.*

4 Equality algebras of type $(m, n; i, j)$

For any $x, y \in X$, we define

$$X_{(1,1)}(x,y) := (y \to x) \to x, \tag{29}$$

$$X_{(i+1,j)}(x,y) := (y \to x) \to X_{(i,j)}(x,y), \tag{30}$$

$$X_{(i,j+1)}(x,y) := (x \to y) \to X_{(i,j)}(x,y), \tag{31}$$

where i and j are natural numbers. Easily $x \leq X_{(1,1)}(y,x)$, $y \leq X_{(1,1)}(x,y)$, and $X_{(1,1)}(x,y) \to x = y \to x$.

Definition 4.1. *Suppose m, n, i and j are natural numbers. The structure X is said to be of type $(m,n;i,j)$ if $X_{(m,n)}(x,y) = X_{(i,j)}(y,x)$ for all $x, y \in X$.*

Example 4.2. *(1) Consider X which is given in Example 3.19. It is routine to check that it is an equality algebra of type $(2,1;2,1)$.*

(2) Assume $X = \{0, a, b, 1\}$ is a chain such that $0 \leq a \leq b \leq 1$. Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$ on X by Table 7. Then $(X, \wedge, \sim, 1)$

Table 7: Cayley table for the implication “$\sim$”

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

is a positive implicative equality algebra, and the implication “$\to$” is given by Table 8. By routine calculation, we can see that X is an equality algebra of type $(2,2;1,2)$.

Table 8: Cayley table for the implication "→"

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Example 4.3. *Every commutative equality algebra is an equality algebra of type* $(1,1;1,1)$.

By Example 4.3, we know that equality algebra of type $(m,n;i,j)$ is a generalization of commutative equality algebra.

Proposition 4.4. *In X, we have*

(1) *If m is fixed, then $X_{(m,n)}(x,y) \leq X_{(m,1)}(x,y)$ and $X_{(m,n)}(y,x) \leq X_{(m,1)}(y,x)$,*

(2) *If n is fixed, then $X_{(m,n)}(x,y) \leq X_{(1,m)}(x,y)$ and $X_{(m,n)}(y,x) \leq X_{(1,m)}(y,x)$,*

(3) *If $m \leq i$ and $n \leq j$, then $X_{(m,n)}(x,y) \leq X_{(i,j)}(x,y)$,*

(4) $X_{(i,j-1)}(y,x) \to X_{(m-1,n)}(x,y) \leq X_{(i,j)}(y,x) \to X_{(m,n)}(x,y)$,

(5) $X_{(i-1,j)}(y,x) \to X_{(m,n-1)}(x,y) \leq X_{(i,j)}(y,x) \to X_{(m,n)}(x,y)$,

(6) *If $n \geq 2$, then $X_{(1,1)}(y,x) \leq X_{(m,n)}(x,y)$, $X_{(1,1)}(x,y) \leq X_{(m,n)}(y,x)$,*

for all $x,y \in X$.

Proof. (1) and (2) are clear. Easily, we consequence $X_{(m,n)}(x,y) \leq X_{(m+1,n)}(x,y)$ and $X_{(m,n)}(x,y) \leq X_{(m,n+1)}(x,y)$. Hence the mathematical inducetion induces (3).

(4) and (5). Using (4) and (9), we get

$$\begin{aligned} X_{(i,j-1)}(y,x) \to X_{(m-1,n)}(x,y) &\leq ((y \to x) \to X_{(i,j-1)}(y,x)) \to ((y \to x) \to X_{(m-1,n)}(x,y)) \\ &= X_{(i,j)}(y,x) \to X_{(m,n)}(x,y), \end{aligned}$$

and

$$\begin{aligned} X_{(i-1,j)}(y,x) \to X_{(m,n-1)}(x,y) &\leq ((x \to y) \to X_{(i-1,j)}(y,x)) \to ((x \to y) \to X_{(m,n-1)}(x,y)) \\ &= X_{(i,j)}(y,x) \to X_{(m,n)}(x,y), \end{aligned}$$

for all $x,y \in X$.

(6) We first prove that $X_{(1,1)}(y,x) \leq X_{(m,2)}(x,y)$ for all $x,y \in X$. Using (8) and (12), we have

$$\begin{aligned} X_{(1,1)}(y,x) = (x \to y) \to y &\leq (x \to y) \to ((y \to x) \to x) \\ &= (x \to y) \to X_{(1,1)}(x,y) = X_{(1,2)}(x,y). \end{aligned}$$

Suppose that $X_{(1,1)}(y,x) \leq X_{(k,2)}(x,y)$ for any $k(\neq 1) \in \mathbb{N}$. Then

$$X_{(1,1)}(y,x) \leq X_{(k,2)}(x,y) \leq (y \to x) \to X_{(k,2)}(x,y) = X_{(k+1,2)}(x,y),$$

by (7). Hense, $X_{(1,1)}(y,x) \leq X_{(m,2)}(x,y)$ for all $m \in \mathbb{N}$ by mathematical induction on m. Now, suppose that $X_{(1,1)}(y,x) \leq X_{(m,n)}(x,y)$ for all $m,n \in \mathbb{N}$ with $n \neq 1$. Then

$$\begin{aligned} X_{(1,1)}(y,x) &\leq X_{(m,n)}(x,y) \\ &\leq (y \to x) \to X_{(m,n)}(x,y) \\ &\leq (x \to y) \to ((y \to x) \to X_{(m,n)}(x,y)) \\ &\leq (y \to x) \to ((x \to y) \to X_{(m,n)}(x,y)) \\ &\leq (y \to x) \to X_{(m,n+1)}(x,y) \\ &= X_{(m+1,n+1)}(x,y). \end{aligned}$$

Therefore $X_{(1,1)}(y,x) \leq X_{(m,n)}(x,y)$ by mathematical induction. Similarly, we have $X_{(1,1)}(x,y) \leq X_{(m,n)}(y,x)$. □

We consider conditions for equality algebra of type $(m,n;i,j)$ to be commutative and/or positive implicative.

Lemma 4.5. *If for $x,y \in X$, $x \leq y$, then $X_{(1,1)}(y,x) = y$ and $X_{(m,n)}(x,y) = X_{(m,1)}(x,y)$ for all natural numbers m and n.*

Proof. Straightforward. □

Lemma 4.6 ([14]). *The structure X is commutative iff*

$$(\forall x,y \in X)(y \leq x \ \Rightarrow \ x = (x \to y) \to y).$$

Theorem 4.7. *If X is of types $(1,1;i,j)$, then it is commutative.*

Proof. Assume that X is of type $(1,1;i,j)$ and suppose $x,y \in X$ such that $y \leq x$. Then

$$x = X_{(1,1)}(x,y) = X_{(i,j)}(y,x) = X_{(i,1)}(y,x) \geq X_{(1,1)}(y,x) \geq x,$$

and so $x = X_{(1,1)}(y,x) = (x \to y) \to y$. Therefore X is commutative by Lemma 4.6. □

Corollary 4.8. *If X is of type $(m,1;i,j)$, then it is commutative.*

Proof. Assume that X is of type $(m,1;i,j)$ and suppose $x,y \in X$ such that $y \leq x$. Then $X_{(m,1)}(x,y) = X_{(i,j)}(y,x)$ and

$$X_{(m,1)}(x,y) = (y \to x)^{m-1} \to X_{(1,1)}(x,y) = 1 \to X_{(1,1)}(x,y) = X_{(1,1)}(x,y).$$

Moreover, by Theorem 4.7, X is commutative. □

Corollary 4.9. *If X is of type $(m,n;i,1)$, then it is commutative.*

Proof. Assume that X is of type $(m,n;i,1)$ and let $x,y \in X$ such that $y \leq x$. Then $X_{(m,n)}(x,y) = X_{(i,1)}(y,x)$ and

$$X_{(i,1)}(y,x) = X_{(m,n)}(x,y) = (y \to x)^{m-1} \to X_{(1,n)}(x,y) = X_{(1,n)}(x,y).$$

Then X is of type $(i,1;1,n)$. By Theorem 4.7, X is commutative.

□

For any x and y of X, we define

$$y^1 \to x := y \to x \text{ and } y^n \to x := y \to (y^{n-1} \to x). \tag{32}$$

Proposition 4.10. *For any natural numbers m and n with $m < n$, we have*

$$(\forall x, y \in X)(y^m \to x = y^n \to x \;\Rightarrow\; (\forall k \in \mathbb{N})(y^{m+k} \to x = y^m \to x)). \tag{33}$$

Proof. Consider $x, y \in X$ such that $y^m \to x = y^n \to x$ for all $m, n \in \mathbb{N}$ with $m < n$. Using (7), we have $y^m \to x \leq y^{m+1} \to x \leq y^n \to x = y^m \to x$ for all $x, y \in X$. Hence

$$y^m \to x = y^{m+1} \to x = y^{m+2} \to x = \cdots = y^{m+k} \to x,$$

for any $k \in \mathbb{N}$. □

Proposition 4.11. *Every X satisfies:*

$$(\forall n \in \mathbb{N})(\forall x, y \in X)(y^n \to x = (X_{(1,1)}(x, y))^n \to x). \tag{34}$$

Proof. Assume $x, y \in X$ and $n \in \mathbb{N}$. If $n = 1$, then

$$(X_{(1,1)}(x, y)) \to x = ((y \to x) \to x) \to x = y \to x.$$

Suppose $y^k \to x = (X_{(1,1)}(x, y))^k \to x$ for $k \in \mathbb{N}$. Then

$$\begin{aligned}
(X_{(1,1)}(x, y))^{k+1} \to x &= ((y \to x) \to x)^{k+1} \to x \\
&= ((y \to x) \to x)^k \to (((y \to x) \to x) \to x) \\
&= ((y \to x) \to x)^k \to (y \to x) \\
&= y \to (((y \to x) \to x)^k \to x) \\
&= y \to ((X_{(1,1)}(x, y))^k \to x) \\
&= y \to (y^k \to x) \\
&= y^{k+1} \to x.
\end{aligned}$$

By mathematical induction, we get $y^n \to x = (X_{(1,1)}(x, y))^n \to x$ for all $n \in \mathbb{N}$ and $x, y \in X$. □

Proposition 4.12. *Every X of type $(m, n; i, j)$ satisfies:*

$$(\forall x, y \in X)(y^n \to x = y^i \to x). \tag{35}$$

Proof. Using (34), we have

$$\begin{aligned}
&X_{(m,n)}(y \to x, x) \\
&= (x \to (y \to x))^{m-1} \to X_{(1,n)}(y \to x, x) \\
&= (x \to (y \to x))^{m-1} \to (((y \to x) \to x)^{n-1} \to X_{(1,1)}(y \to x, x)) \\
&= (x \to (y \to x))^{m-1} \to (((y \to x) \to x)^{n-1} \to ((x \to (y \to x)) \to (y \to x))) \\
&= (x \to (y \to x))^m \to (((y \to x) \to x)^{n-1} \to (y \to x)) \\
&= 1 \to (y \to (X_{(1,1)}(x, y))^{n-1} \to x) \\
&= y \to (y^{n-1} \to x) = y^n \to x,
\end{aligned}$$

and

$$\begin{aligned}
&X_{(i,j)}(x, y \to x) \\
&= (x \to (y \to x))^{j-1} \to X_{(i,1)}(x, y \to x) \\
&= (x \to (y \to x))^{j-1} \to (((y \to x) \to x)^{i-1} \to X_{(1,1)}(x, y \to x)) \\
&= ((y \to x) \to x)^{i-1} \to (((y \to x) \to x) \to x) \\
&= ((y \to x) \to x)^{i-1} \to (y \to x) \\
&= y \to (((y \to x) \to x)^{i-1} \to x) \\
&= y \to ((X_{(1,1)}(x, y))^{i-1} \to x) \\
&= y \to (y^{i-1} \to x) \\
&= y^i \to x,
\end{aligned}$$

for all $x, y \in X$. Thus $y^n \to x = y^i \to x$ for all $x, y \in X$. $\square$

Proposition 4.13. *Every X of type $(i, i; i, i)$ is an equality algebra of types $(i, m; m, i)$ and $(m, i; i, m)$, where $i, m \in \mathbb{N}$ and $m \geq i$.*

Proof. Suppose $i, m \in \mathbb{N}$ with $m \geq i$ and $m = i + k$. Since X is of type $(i, i; i, i)$, we have $X_{(i,i)}(x, y) = X_{(i,i)}(y, x)$ for all $x, y \in X$. Then

$$\begin{aligned}
X_{(i,m)}(x, y) &= (x \to y) \to X_{(i,m-1)}(x, y) \\
&= (x \to y) \to ((x \to y) \to X_{(i,m-2)}(x, y)) \\
&= (x \to y)^2 \to X_{(i,m-2)}(x, y) \\
&\vdots \\
&= (x \to y)^k \to X_{(i,m-k)}(x, y) \\
&= (x \to y)^k \to X_{(i,i)}(x, y) \\
&= (x \to y)^k \to X_{(i,i)}(y, x) \\
&= (x \to y)^{k-1} \to ((x \to y) \to X_{(i,i)}(y, x)) \\
&= (x \to y)^{k-1} \to X_{(i+1,i)}(y, x) \\
&\vdots \\
&= (x \to y) \to X_{(i+k-1,i)}(y, x) \\
&= X_{(i+k,i)}(y, x) \\
&= X_{(m,i)}(y, x).
\end{aligned}$$

Hence X is of type $(i, m; m, i)$. By the similar way, we can prove that X is of type $(m, i; i, m)$. $\square$

Corollary 4.14. *Every commutative equality algebra is of types $(1, m; m, 1)$ and $(m, 1; 1, m)$ for all $m \in \mathbb{N}$.*

Theorem 4.15. *The structure X is positive implicative iff it is of type $(1, 2; 1, 2)$.*

Proof. If X is positive implicative, then

$$X_{(1,2)}(x, y) = (x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y) = X_{(1,2)}(y, x),$$

for all $x, y \in X$ by (20). Hence X is of type $(1,2;1,2)$.

Consider X is of types $(1,2;1,2)$. Then

$$(x \to y) \to ((y \to x) \to x) = X_{(1,2)}(x,y) = X_{(1,2)}(y,x) = (y \to x) \to ((x \to y) \to y),$$

for all $x, y \in X$. Therefore X is positive implicative by Theorem 3.7. □

Theorem 4.16. *Suppose X is a positive implicative equality algebra. If it is of type $(m,n;i,j)$ for all $m,n,i,j \in \mathbb{N}$, iff it can be reduced of type $(2,2;2,2)$.*

Proof. Let X be positive implicative of type $(m,n;i,j)$. Then by Theorem 3.3, we get that,

$$\begin{aligned} X_{(m,n)}(x,y) &= (x \to y)^{n-1} \to X_{(m,1)}(x,y) \\ &= (x \to y) \to X_{(m,1)}(x,y) \\ &= (x \to y) \to ((y \to x)^{m-1} \to X_{(1,1)}(x,y)) \\ &= (x \to y) \to ((y \to x) \to X_{(1,1)}(x,y)) \\ &= X_{(2,2)}(x,y), \end{aligned}$$

and

$$\begin{aligned} X_{(i,j)}(y,x) &= (y \to x)^{j-1} \to X_{(i,1)}(y,x) \\ &= (y \to x) \to X_{(i,1)}(y,x) \\ &= (y \to x) \to ((x \to y)^{i-1} \to X_{(1,1)}(y,x)) \\ &= (y \to x) \to ((x \to y) \to X_{(1,1)}(y,x)) \\ &= X_{(2,2)}(y,x). \end{aligned}$$

Hence $X_{(2,2)}(x,y) = X_{(m,n)}(x,y) = X_{(i,j)}(y,x) = X_{(2,2)}(y,x)$. Then X is of type $(2,2;2,2)$.

The proof of other side is similar. □

Example 4.17. *Let X be an equality algebra as in Example 3.19. This example approve Theorem 4.16.*

Theorem 4.18. *The next statements are equivalent.*

(1) *X is positive implicative of type $(1,n;1,j)$ for all $n, j \in \mathbb{N}$.*

(2) *X satisfies $y^n \to x = y \to x$ for all $x, y \in X$, and is of type $(m,n;i,j)$ for all $m,n,i,j \in \mathbb{N}$.*

Proof. (2) ⇒ (1). If $n = 2$, then $y \to (y \to x) = y \to x$ for all $x, y \in X$. From Theorem 3.3, X is positive implicative . Since X is of type $(m,n;i,j)$, we have $X_{(m,n)}(x,y) = X_{(i,j)}(y,x)$ for all $x, y \in X$. Hence

$$(x \to y)^{n-1} \to [(y \to x)^{m-1} \to X_{(1,1)}(x,y)] = (x \to y)^{i-1} \to [(y \to x)^{j-1} \to X_{(1,1)}(y,x)].$$

In particular,

$$(x \to y) \to ((y \to x) \to ((y \to x) \to x)) = (x \to y) \to ((y \to x) \to ((x \to y) \to y)),$$

and so $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$. Thus

$$\begin{aligned}
X_{(1,n)}(x,y) &= (x \to y) \to X_{(1,n-1)}(x,y) \\
&= (x \to y)^{n-1} \to X_{(1,1)}(x,y) \\
&= (x \to y)^{n-1} \to ((y \to x) \to x) \\
&= (x \to y) \to ((y \to x) \to x) \\
&= (y \to x) \to ((x \to y) \to y) \\
&= (y \to x) \to X_{(1,1)}(y,x) \\
&= (y \to x)^{j-1} \to X_{(1,1)}(y,x) \\
&= X_{(1,j)}(y,x).
\end{aligned}$$

(1) $\Rightarrow$ (2). Suppose that X is positive implicative of type $(1,n;1,j)$ for all $n,j \in \mathbb{N}$. Then $y^n \to x = y \to x$ for all $x,y \in X$ and $n \in \mathbb{N}$ by Theorem 3.3, and $X_{(1,n)}(x,y) = X_{(1,j)}(y,x)$, which implies that

$$\begin{aligned}
X_{(m,n)}(x,y) &= (y \to x)^{m-1} \to X_{(1,n)}(x,y) \\
&= (y \to x)^{m-1} \to X_{(1,j)}(y,x) \\
&= (y \to x) \to X_{(1,j)}(y,x) \\
&= (y \to x) \to ((y \to x)^{j-1} \to ((x \to y) \to y)) \\
&= (y \to x)^{j} \to ((x \to y) \to y) \\
&= (y \to x) \to ((x \to y) \to y) \\
&= (x \to y) \to ((y \to x) \to y) \\
&= (x \to y)^{i} \to ((y \to x) \to y) \\
&= (x \to y)^{i-1} \to ((x \to y) \to ((y \to x) \to y)) \\
&= (x \to y)^{i-1} \to ((y \to x) \to ((x \to y) \to y)) \\
&= (x \to y)^{i-1} \to ((y \to x)^{j-1} \to ((x \to y) \to y)) \\
&= X_{(i,j)}(y,x).
\end{aligned}$$

Therefore X is of type $(m,n;i,j)$. $\square$

Lemma 4.19. *If X is positive implicative , then $X_{(m,n)}(x,y) = X_{(2,2)}(x,y)$ for all $x,y \in X$ and $m,n \in \mathbb{N}$.*

Proof. Assume that X is positive implicative . Using Theorem 3.3, we have

$$\begin{aligned}
X_{(m,n)}(x,y) &= (x \to y) \to X_{(m,n-1)}(x,y) \\
&= (x \to y) \to ((x \to y) \to X_{(m,n-2)}(x,y)) \\
&= (x \to y) \to X_{(m,n-2)}(x,y) \\
&\;\vdots \\
&= (x \to y) \to X_{(m,1)}(x,y) \\
&= (x \to y) \to ((y \to x) \to X_{(m-1,1)}(x,y)) \\
&= (x \to y) \to ((y \to x) \to ((y \to x) \to X_{(m-2,1)}(x,y)))
\end{aligned}$$

$$
\begin{aligned}
&= (x \to y) \to ((y \to x) \to X_{(m-2,1)}(x,y)) \\
&\vdots \\
&= (x \to y) \to ((y \to x) \to X_{(1,1)}(x,y)) \\
&= (x \to y) \to X_{(2,1)}(x,y) \\
&= X_{(2,2)}(x,y),
\end{aligned}
$$

for all $x, y \in X$ and $m, n \in \mathbb{N}$. □

Theorem 4.20. *If X is both positive implicative and commutative, then it is of type $(m, n; i, j)$ for all $m, n, i, j \in \mathbb{N}$.*

Proof. Consider X is both positive implicative and commutative. Then $X_{(1,1)}(x, y) = X_{(1,1)}(y, x)$ for all $x, y \in X$ and using Theorem 3.3 and Lemma 4.19 induces

$$
\begin{aligned}
X_{(m,n)}(x,y) &= X_{(2,2)}(x,y) = (y \to x) \to ((x \to y) \to X_{(1,1)}(x,y)) \\
&= (y \to x) \to ((x \to y)^{i-1} \to X_{(1,1)}(y,x)) \\
&= (y \to x) \to X_{(i,1)}(y,x) \\
&= (y \to x) \to ((y \to x) \to X_{(i,1)}(y,x)) \\
&= (y \to x)^{j-1} \to X_{(i,1)}(y,x) \\
&= X_{(i,j)}(y,x)),
\end{aligned}
$$

for all $m, n, i, j \in \mathbb{N}$ and $x, y \in X$. This completes the proof. □

5 Conclusion

The notion of a positive implicative equality algebras are defined, and related properties are studied. Characterizations of a positive implicative equality algebra is investigated. Conditions for an equality algebra to be positive implicative are provided. Equality algebra with some types is considered, and several properties are investigated. Using equality algebra with some types, we characterize a commutative equality algebra and a positive implicative algebra.

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