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Residuated lattices derived from filters(ideals) in double Boolean algebras

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Abstract

Double Boolean algebras (dBas) are algebraic structures $\underline{D} = (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$, introduced by Rudolf Wille to capture the equational theory of the algebra of protoconcepts. Our goal is an algebraic investigation of dBas, based on similar results on Boolean algebras. In this paper, first we characterize filters on dBas as deductive systems and we give many characterization of primary filters(ideals). Second, for a given dBa, we show that the set of its filters $F(\underline{D})$ (resp. ideals $I(\underline{D})$) is endowed with the structure of distributive pseudo-complemented lattices, Heyting algebras and residuated lattices. We finish by introducing the notions of annihilators and co-annihilators on dBas and investigate some relalted properties of them. We show that pseudo-complement of an ideal I (filter F) is the annihilator I^* of I (co-annihilator F^*) and the set of annihilators (co-annihilators) forms a Boolean algebra.

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1 Introduction

The notions of ideals and filters has been introduced in many algebraic structures (such as lattices, rings, MV-algebras, residuated lattices) and the ideal (resp. filter) theory is an effective tool for studying various algebraic and logic systems. Theory of filters plays a very important role in proving completeness with respect to algebraic semantics. For example, in the case of classical propositional logic (CPL), we can show the completeness theorem of the logic by Boolean algebras.

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To do so we use the Lindenbaum-Tarski algebra of CPL, which is a quotient algebra by theories, or equivalently, by filters. The Lindenbaum-Tarski method can be applied to many logic including contextual logic.

Various logical algebras have been proposed and researched as the semantical systems of non-classical logical systems. Among these logical algebras, double Boolean algebra were introduced by Rudolf Wille in order to extend Formal Concept Analysis (FCA) to Contextual Logic. A negation has to be formalized [16], there are many options; one of these wants to preserve the correspondence between negation and set complementation, and leads to the notions of semiconcept, protoconcept and preconcept [16]. To capture their equational theory, Wille proved that each double Boolean algebra "quasi-embeds" into an algebra of protoconcepts. Thus the equational axioms of double Boolean algebras generate the equational theory of the algebras of protoconcepts [16] (Corollary 1). Double Boolean algebras form the variety generated by proto-concept algebras which are one of the fundamental structures of contextual logic. To the best of our knowledge, the investigation of dBas has been so far concentrated on representation problem such as equational theory [16], contextual representation [2] and most recently topological representation [2, 10]. Of course, the prime ideal theorem [12] plays a central role in such representation.

The importance of ideals (filters) and congruences in classification problems, data organization and formal concept analysis allow us to make an intensive study of the properties of the lattice of filters(resp. ideals) of double Boolean algebras. This paper is a continuation of the work begining in [14, 15] and is organized as follows: In Section 2 we recall some basic notions and present proto-concepts algebras as a rich source of examples for dBas. In Section 3, we show that filters on dBas are deductive systems and we give additional characterization of primary filters which extend those of prime filters (resp. ideals) on Boolean algebras. We also introduce the notions of dense and co-dense elements on double Boolean algebras and show that the set of dense (co-dense) elements forms a particular filter (ideal) on dBa and some characterizations of trivial dBas using dense set and co-dense set are obtained. In Section 4 we show that the lattice $F(\underline{D})$(resp $I(\underline{D})$) of filters(resp. ideals) of any dBa $\underline{D}$ is distributive, pseudocomplemented and is endowed with a structure of Heyting algebra, Brouwerian algebra, Gödel algebra and residuated lattice. In Section 5 we introduce the notions of annihilators and co-annihilators on double Boolean algebras and some related properties are studied. We show that co-annihilators(resp. annihilators) are filters (resp. ideals). We show that the co-annihilators (resp. annihilators) of dBAs form a Boolean algebra and pseudo-complement of filters (resp. ideals) are exactly the co-annihilators filters(resp. annihilators ideals).

2 Concepts, protoconcepts and double Boolean algebras

In this section, we provide the reader with some basic notions and notations. For more details we refer to [6, 16]. A **formal context** is a triple $\mathbb{K} := (G, M, I)$ where G is a set of objects, M a set of attributes and $I \subseteq G \times M$, a binary relation to describe if an object of G has an attribute in M. We write gIm for $(g, m) \in I$. To extract clusters, the following derivation operators are defined on subsets $A \subseteq G$ and $B \subseteq M$ by:

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\} \quad \text{and} \quad B' := \{g \in G \mid \ gIm \text{ for all } m \in B\}.$$

The maps $A \mapsto A'$ and $B \mapsto B'$ form a Galois connection between the power set of G and that of M. The composition $''$ is a closure operator.

A **formal concept** is a pair (A, B) with $A' = B$ and $B' = A$. We call A the **extent** and B the **intent** of the formal concept (A, B). They are closed subsets with respect to $''$ (i.e. $X'' = X$).

The set $\mathfrak{B}(\mathbb{K})$ of all formal concepts of the formal context $\mathbb{K}$ can be ordered by

$$(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 \quad \text{(or equivalently, } B_2 \subseteq B_1).$$

The poset $\underline{\mathfrak{B}}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}), \leq)$ is a complete lattice, called the **concept lattice** of the context $\mathbb{K}$. Conversely, each complete lattice is isomorphic to a concept lattice. This basic theorem on concept lattice ([6, Theorem 3]) is a template for contextual representation problems. The lattice operations $\wedge$ (meet) and $\vee$ (join) can be interpreted as a logical conjunction and a logical disjunction for concepts, and are given by:

$$\begin{aligned} \text{meet:} \quad & (A_1, B_1) \wedge (A_2, B_2) = \left(A_1 \cap A_2, (A_1 \cap A_2)'\right), \\ \text{join:} \quad & (A_1, B_1) \vee (A_2, B_2) = \left((B_1 \cap B_2)', B_1 \cap B_2\right). \end{aligned}$$

To extent FCA to contextual logic, we need to define the negation of a concept. Unfortunately, the complement of a closed subset is not always closed. To preserve the correspondence between a set complementation and negation, the notion of concept is extended to that of proto-concept.

Let $\mathbb{K} := (G, M, I)$ be a formal context and $A \subseteq G, B \subseteq M$. The pair (A, B) is called a **semi-concept** if $A' = B$ or $B' = A$, and a **proto-concept** if $A'' = B'$.

The set of all semi-concepts of $\mathbb{K}$ is denoted by $\mathfrak{h}(\mathbb{K})$, and that of all proto-concepts by $\mathfrak{P}(\mathbb{K})$. Note that each semi-concept is a proto-concept; i.e. $\mathfrak{h}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$. Meet and join of proto-concepts are then defined, similar as above for concepts. A negation (resp. opposition) is defined by taking the complement on objects (resp. attributes). More precisely, for proto-concepts (A_1, B_1), (A_2, B_2), (A, B) of $\mathbb{K}$ we define the operations:

$$\begin{aligned} \text{meet:} \quad & (A_1, B_1) \sqcap (A_2, B_2) := (A_1 \cap A_2, (A_1 \cap A_2)'), \\ \text{join:} \quad & (A_1, B_1) \sqcup (A_2, B_2) := ((B_1 \cap B_2)', B_1 \cap B_2), \\ \text{negation:} \quad & \neg(A, B) := (G \setminus A, (G \setminus A)'), \\ \text{opposition:} \quad & \lrcorner(A, B) := ((M \setminus B)', M \setminus B), \\ \text{nothing:} \quad & \bot := (\emptyset, M), \\ \text{all:} \quad & \top := (G, \emptyset). \end{aligned}$$

The algebra $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcap, \sqcup, \urcorner, \lrcorner, \bot, \top)$ is called the **algebra of proto concepts** of $\mathbb{K}$. Note that applying any operation above on proto concepts gives a semi-concept as result. Therefore $\underline{\mathfrak{H}}(\mathbb{K})$ is a sub-algebra of $\underline{\mathfrak{P}}(\mathbb{K})$. For structural analysis of $\underline{\mathfrak{P}}(\mathbb{K})$, we split $\mathfrak{H}(\mathbb{K})$ in $\sqcap$**-semi concepts** and $\sqcup$**-semi concepts**,

$$\mathfrak{P}(\mathbb{K})_\sqcap := \{(A, A') \mid A \subseteq G\}, \quad \text{and} \quad \mathfrak{P}(K)_\sqcup := \{(B', B) \mid B \subseteq M\},$$

and set $x \vee y := \neg(\neg x \sqcap \neg y)$, $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$, $\overline{\top} := \neg\bot$ and $\underline{\bot} := \lrcorner\top$ for $x, y \in \mathfrak{P}(\mathbb{K})$.

$\underline{\mathfrak{P}}(\mathbb{K})_\sqcap := (\mathfrak{P}(\mathbb{K})_\sqcap, \sqcap, \vee, \neg, \bot, \neg\bot)$ (resp. $\underline{\mathfrak{P}}(\mathbb{K})_\sqcup := (\mathfrak{P}(\mathbb{K})_\sqcup, \wedge, \sqcup, \lrcorner, \lrcorner\top, \top)))$ is a Boolean algebra isomorphic (resp. anti-isomorphic) to the powerset algebra of G (resp. M).

Theorem 2.1. [16] *The following equations hold in* $\underline{\mathfrak{P}}(\mathbb{K})$*:*

(1a)	$(x \sqcap x) \sqcap y = x \sqcap y$	(1b)	$(x \sqcup x) \sqcup y = x \sqcup y$
(2a)	$x \sqcap y = y \sqcap x$	(2b)	$x \sqcup y = y \sqcup x$
(3a)	$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	(3b)	$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
(4a)	$\neg(x \sqcap x) = \neg x$	(4b)	$\lrcorner(x \sqcup x) = \lrcorner x$
(5a)	$x \sqcap (x \sqcup y) = x \sqcap x$	(5b)	$x \sqcup (x \sqcap y) = x \sqcup x$
(6a)	$x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	(6b)	$x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
(7a)	$x \sqcap (x \vee y) = x \sqcap x$	(7b)	$x \sqcup (x \wedge y) = x \sqcup x.$
(8a)	$\neg\neg(x \sqcap y) = x \sqcap y$	(8b)	$\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
(9a)	$x \sqcap \neg x = \bot$	(9b)	$x \sqcup \lrcorner x = \top$
(10a)	$\neg\bot = \top \sqcap \top$	(10b)	$\lrcorner\top = \bot \sqcup \bot$
(11a)	$\neg\top = \bot$	(11b)	$\lrcorner\bot = \top$

$$(12)\ (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$$

A **double Boolean algebra** (dBa) is an algebra $\underline{D} := (D; \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ that satisfies the equations in Theorem 2.1. Wille showed that these equations generate the equational theory of protoconcept algebras [16].

A double Boolean algebra $\underline{D}$ is called **pure** if it satisfies $x \sqcap x = x$ or $x \sqcup x = x$, for all $x \in D$. In fact, $\underline{\mathfrak{H}}(\mathbb{K})$ is a pure double Boolean algebra. A quasi-order $\sqsubseteq$ is defined on dBas by:

$$x \sqsubseteq y : \text{ if and only if } x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

We set $D_\sqcap := \{x \in D : x \sqcap x = x\}$ and $D_\sqcup := \{x \in D : x \sqcup x = x\}$. The algebra $\underline{D_\sqcap} := (D_\sqcap, \sqcap, \vee, \neg, \bot, \neg\bot)$ (resp. $\underline{D_\sqcup} = (D_\sqcup, \sqcup, \wedge, \lrcorner, \top, \lrcorner\top)$) is a Boolean algebra. In addition $x \sqsubseteq y$ iff $x \sqcap x \sqsubseteq y \sqcap y$ and $x \sqcup x \sqsubseteq y \sqcup y$, for all $x, y \in D$ [16].

Definition 2.2. [12, 16] *A dBa $\underline{D}$ is:*

1. ***contextual*** *if the quasi-order $\sqsubseteq$ is an order relation.*
2. ***complete*** *if and only if its Boolean algebras $\underline{D_\sqcap}$ and $\underline{D_\sqcup}$ are complete lattices.*

Example 2.3. *Consider the six elements dBa $\underline{D_6} = (\{\bot, \gamma, \lambda, \beta, \alpha, \top\}, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ which is a pure dBa in which the operations $\neg$, $\lrcorner, \sqcap$ and $\sqcup$ are defined in the following Cayley tabulars and the Hasse diagram is presented in Figure 1.*

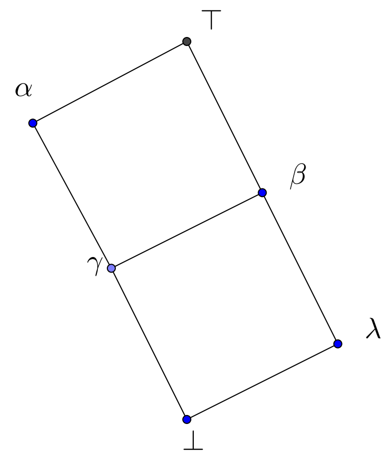

Figure 1. Hasse diagram of $\underline{D}_6$

$\sqcap$	$\perp$	γ	λ	β	α	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
γ	$\perp$	γ	$\perp$	γ	γ	γ
λ	$\perp$	$\perp$	λ	λ	$\perp$	λ
β	$\perp$	γ	λ	β	γ	β
α	$\perp$	γ	$\perp$	γ	γ	γ
$\top$	$\perp$	γ	λ	β	α	β

,

$\sqcup$	$\perp$	γ	λ	β	α	$\top$
$\perp$	γ	γ	β	β	α	$\top$
γ	γ	γ	β	β	$\top$	$\top$
λ	β	β	β	β	$\top$	$\top$
β	β	β	β	β	$\top$	$\top$
α	α	α	$\top$	$\top$	α	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

x	$\perp$	γ	λ	β	α	$\top$
$\neg x$	β	λ	γ	$\perp$	λ	$\perp$
$\lrcorner x$	$\top$	$\top$	α	α	β	γ

.

We have $D_{6,\sqcap} = \{\perp, \lambda, \gamma, \beta\}$ and $D_{6,\sqcup} = \{\beta, \gamma, \alpha, \beta, \top\}$ and $\underline{D}_6$ is a pure dBa.

Additional known algebraic properties of dBas useful for us in these notes are put together in the following two propositions.

Proposition 2.4. [12, 9, 10, 16] *Let $\underline{D}$ be a double Boolean algebra and $x, y, a \in D$, then:*

1. $\perp \sqsubseteq x$ *and* $x \sqsubseteq \top$.
2. $x \sqcap y \sqsubseteq x, y \sqsubseteq x \sqcup y$.
3. $x \sqsubseteq y \implies \begin{cases} x \sqcap a \sqsubseteq y \sqcap a \\ x \sqcup a \sqsubseteq y \sqcup a. \end{cases}$
4. $\neg(x \vee y) = \neg x \sqcap \neg y$.
5. $\neg(x \sqcap y) = \neg x \vee \neg y$.
6. $x \sqsubseteq \lrcorner y$ *if and only if* $y \sqsubseteq \lrcorner x$.
7. $\neg\neg x = x \sqcap x$ *and* $\lrcorner\lrcorner x = x \sqcup x$.
8. $\neg x, x \vee y \in D_\sqcap$ *and* $\lrcorner x, x \wedge y \in D_\sqcup$.
9. $x \sqsubseteq y$ *if and only if* $\begin{cases} \neg y \sqsubseteq \neg x \\ \lrcorner y \sqsubseteq \lrcorner x. \end{cases}$
10. $\lrcorner(x \wedge y) = \lrcorner x \sqcup \lrcorner y$.
11. $\lrcorner(x \sqcup y) = \lrcorner x \wedge \lrcorner y$.
12. $\neg x \sqsubseteq y$ *if and only if* $\neg y \sqsubseteq x$.
13. $\neg x \sqcap \neg x = \neg x$ *and* $\lrcorner x \sqcup \lrcorner x = \lrcorner x$.
14. *(i)* $x \sqcap y \sqsubseteq x \vee y \sqsubseteq x \sqcup y$.
    (ii) $x \sqcap y \sqsubseteq x \wedge y \sqsubseteq x \sqcup y$.

The following proposition gives distributivity-like properties of dBas.

Proposition 2.5. [14, Proposition 3] *Let $\underline{D}$ be a double Boolean algebra. For any $a, b, c, d \in D$, we have:*

(i) $a \vee (b \sqcap c) = (a \vee b) \sqcap (a \vee c)$. (iv) $a \wedge (a \sqcup b) = a \sqcup a$.

(ii) $a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$. (v) $(a \sqcap a) \vee (b \sqcap b) = a \vee b$.

(iii) $a \vee (a \sqcap b) = a \sqcap a$. (vi) $(a \sqcup a) \wedge (b \sqcup b) = a \wedge b$.

(vii) The binary operations $\vee$, $\wedge$, $\sqcap$ and $\sqcup$ are compatible with $\sqsubseteq$, that is, if $a \sqsubseteq b$ and $c \sqsubseteq d$, then $a \vee c \sqsubseteq b \vee d$ (1) , $a \wedge c \sqsubseteq b \wedge d$ (2), $a \sqcap c \sqsubseteq b \sqcap d$ (3) and $a \sqcup c \sqsubseteq b \sqcup d$ (4).

The following proposition gives an additional algebraic properties of dBas.

Proposition 2.6. *Let $\underline{D}$ be a dBa, for any $a, b \in D$, the following equality hold:*

(1) *(i)* $a \sqcap \neg(a \sqcap b) = a \sqcap \neg b$, *(ii)* $a \sqcup \lrcorner(a \sqcup b) = a \sqcup \lrcorner b$.

(2) *(i)* $a \sqcup b = (a \wedge a) \sqcup (b \wedge b)$, *(ii)* $a \sqcap b = (a \vee a) \sqcap (b \vee b)$.

Proof. We give the proof of (1)(i) and (2)(i), that of (1)(ii) and (2)(ii) are obtained dually.

For (1)(i), let $a, b \in D$, we have

$$\begin{aligned} a \sqcap \neg(a \sqcap b) &= a \sqcap (\neg a \vee \neg b) \text{ (by (4) of Proposition 2.4)} = (a \sqcap \neg a) \vee (a \sqcap \neg b) \text{ (by axiom (6a))} \\ &= \bot \vee (a \sqcap \neg b) \text{ (by axiom 9a)} = a \sqcap \neg b \text{ (by (1) of Proposition 2.4).} \end{aligned}$$

Thus (1)(i) is proved. We recall that $a \vee a = a \sqcap a$ and $a \sqcup a = a \wedge a$.
For (2)(i) we have $a \sqcup b = (a \sqcup a) \sqcup (b \sqcup b)$ (by axiom (1b))$= (a \wedge a) \sqcup (b \wedge b)$. □

Let $\underline{D}$ be a dBa. A nonempty subset F of D is called **filter** if for all $x, y \in D$,

$$x, y \in F \implies x \sqcap y \in F \text{ and } (x \in F,\ x \sqsubseteq y) \implies y \in F.$$

Ideals of dBas are defined dually. We denote by $F(\underline{D})$ (resp. $I(\underline{D})$) the set of filters (resp. ideals) of the dBa $\underline{D}$. These two sets are closed under intersection [12]. Note that $F(\underline{D}) \cap I(\underline{D}) = \{D\}$. For $X \subseteq D$, the smallest filter (resp. ideal) containing X, denoted by Filter$\langle X \rangle$ (resp. Ideal$\langle X \rangle$), is the intersection of all filters (resp. ideals) containing X, and is called the **filter** (resp. **ideal**) **generated by** X. A **principal filter** (resp. **ideal**) is a filter (resp. ideal) generated by a singleton. In this case we omit the curly brackets and set $\mathrm{F}(x) :=$ Filter$\langle \{x\} \rangle$, and $\mathrm{I}(x) :=$ Ideal$\langle \{x\} \rangle$. For any $a \in D$, $F(a \sqcap a) = F(a)$ and $I(a \sqcup a) = I(a)$. The set $F(\underline{D})$ (resp. $I(\underline{D})$) is a complete lattice with least element $F(\top)$ (resp. $I(\bot)$) and greatest element D under set inclusion in which for any $F, G \in F(\underline{D})$ (resp. $I, J \in I(\underline{D})$) $F \cap G$ (resp. $I \cap J$) is the infimum of F and G (resp. I and J) while the supremum is given by $F \vee G$ (resp. $I \vee J$) and is characterized in the following two propositions.

Proposition 2.7. [14] *Let $\underline{D}$ be a dBa, $\emptyset \neq X \subseteq D$, $F_1, F_2 \in \mathcal{F}(\underline{D})$ and $I_1, I_2 \in \mathcal{I}(\underline{D})$. Then*

(a) $\mathrm{I}(a) = \{x \in D \mid x \sqsubseteq a \sqcup a\}$ *and* $\mathrm{F}(a) = \{x \in D \mid a \sqcap a \sqsubseteq x\}$.

(b) Ideal$\langle \emptyset \rangle = \mathrm{I}(\bot) = \{x \in D \mid x \sqsubseteq \bot \sqcup \bot\}$ *and* Filter$\langle \emptyset \rangle = \mathrm{F}(\top) = \{x \in D \mid \top \sqcap \top \sqsubseteq x\}$.

(c) Ideal$\langle X \rangle = \{x \in D \mid x \sqsubseteq b_1 \sqcup \ldots \sqcup b_n$ *for some* $b_1, \ldots, b_n \in X, n \geq 1\}$.

(d) Filter$\langle X \rangle = \{x \in D \mid x \sqsupseteq b_1 \sqcap \ldots \sqcap b_n$ *for some* $b_1, \ldots, b_n \in X, n \geq 1\}$.

(e) $I_1 \vee I_2 =$ Ideal$\langle I_1 \cup I_2 \rangle = \{x \in D \mid x \sqsubseteq i_1 \sqcup i_2$ *for some* $i_1 \in I_1$ *and* $i_2 \in I_2\}$.

(*f*) $F_1 \vee F_2 = \mathrm{Filter}\langle F_1 \cup F_2 \rangle = \{x \in D \mid f_1 \sqcap f_2 \sqsubseteq x \textit{ for some } f_1 \in F_1 \textit{ and } f_2 \in F_2\}$.

Proposition 2.8. [14] *Let $\underline{D}$ be a dBa. For any $a_1, \dots, a_n, a, b, c \in D$, the following hold:*

(1) $\bigvee_{i=1}^{n} \mathrm{F}(a_i) = \mathrm{F}(\bigsqcap_{i=1}^{n} a_i) = \mathrm{Filter}\langle \{a_1, \dots, a_n\} \rangle$.

(2) $\bigvee_{i=1}^{n} \mathrm{I}(a_i) = \mathrm{I}(\bigsqcup_{i=1}^{n} a_i) = \mathrm{Ideal}\langle \{a_1, \dots, a_n\} \rangle$.

(3) $\mathrm{F}(a) \wedge (\mathrm{F}(b) \vee \mathrm{F}(c)) = (\mathrm{F}(a) \wedge \mathrm{F}(b)) \vee (\mathrm{F}(a) \wedge \mathrm{F}(c))$.

(4) $\mathrm{I}(a) \vee (\mathrm{I}(b) \wedge \mathrm{I}(c)) = (\mathrm{I}(a) \vee \mathrm{I}(b)) \wedge (\mathrm{I}(a) \vee \mathrm{I}(c))$.

(5) $\bigcap_{i=1}^{n} \mathrm{F}(a_i) = \mathrm{F}(\bigvee_{i=1}^{n} a_i)$.

(6) $\bigcap_{i=1}^{n} \mathrm{I}(a_i) = \mathrm{I}(\bigwedge_{i=1}^{n} a_i)$.

(7) $\mathrm{F}(a) \wedge \mathrm{F}(\neg a) = \mathrm{F}(\top)$.

(8) $\mathrm{I}(a) \vee \mathrm{I}(\lrcorner a) = \mathrm{I}(\top)$.

(9) $\mathrm{F}(a) \vee \mathrm{F}(\neg a) = \mathrm{F}(\bot)$.

(10) $\mathrm{I}(a) \wedge \mathrm{I}(\lrcorner a) = \mathrm{I}(\bot)$.

(11) $a \sqsubseteq b \implies I(a) \subseteq I(b)$.

(12) $a \sqsubseteq b \implies F(b) \subseteq F(a)$.

3 Filters as deductive systems, some particulars filters on dBas

3.1 Filters as deductive systems

A subset F of a Boolean algebra $\mathcal{B}$ is a deductive filter (system) of $\mathcal{B}$ if $1 \in F$ and $a,\ a \to b \in F$ implies $b \in F$ (see [5], p.188). It is proven in Turunen (2001) that if F is a filter of a BL-algebra, then it satisfies

(a) $1 \in F$ and, $x \in F$ and $x \to y \in F$ imply $y \in F$ (with $x \to y := \neg x \vee y$).

M. Kondo et al in [11] proved a similar result for residuated lattices by showing that deductive systems of Booleans algebras are exactly the nonempty lattice filters of its lattice reduct. Since Boolean algebras form a subclass of dBas we extend this result on arbitrary dBas in this subsection. We denote for any $x, y \in D$, $\neg x \vee y$ by $x \to y$(resp. $\lrcorner x \wedge y$ by $x \rightsquigarrow y$). We have the following proposition.

Proposition 3.1. *Let $\underline{D}$ be a dBa, F and I be a non-empty subsets of D, $\to$ and $\rightsquigarrow$ as above. Then the following statements hold:*

1. *F is a filter of $\underline{D}$ if and only if (†): $\top \sqcap \top \in F$ and $x,\ x \to y \in F$ implies $y \in F$.*
2. *I is an ideal of $\underline{D}$ if and only if (‡): $\bot \sqcup \bot \in I$ and $x,\ x \rightsquigarrow y \in I$ implies $y \in I$.*

Proof. We give the proof of (1) and that of (2) is obtained dually.
($\Rightarrow$) Assume that F is a filter of $\underline{D}$, then $\top \sqcap \top \in F$. Let $x, y \in D$ such that $x,\ x \to y \in F$. Since $x, x \to y \in F$ and F a filter we get $x \sqcap (x \to y) \in F$, and

$$\begin{aligned} x \sqcap (x \to y) &= x \sqcap (\neg x \vee y) \\ &= (x \sqcap \neg x) \vee (x \sqcap y) \text{ (by (i) of Proposition 2.5)} \\ &= \bot \vee (x \sqcap y) = x \sqcap y \sqsubseteq y. \end{aligned}$$

It follows that $x \sqcap (x \to y) = x \sqcap y \in F$ and $x \sqcap y \sqsubseteq y$, as F is a filter we deduce that $y \in F$.
($\Leftarrow$) Conversely, assume that (†) holds. We will show that F is a filter. Let $x, y \in F$. Then

$$\begin{aligned} x \to (y \to (x \sqcap y)) &= \neg x \vee (\neg y \vee (x \sqcap y)) \\ &= (\neg x \vee \neg y) \vee (x \sqcap y) \text{ (by associativity of } \vee) \\ &= \neg(x \sqcap y) \vee (x \sqcap y) \text{ (by (4) of Prop 2.4)} \\ &= \top \sqcap \top \in F. \end{aligned}$$

So $y \to (x \sqcap y) \in F$ (due to $x \in F$ and (†) holds). As $y \in F$ and $y \to (x \sqcap y) \in F$, we obtain $x \sqcap y \in F$. We deduce that $x \sqcap y \in F$. To finish our proof, assume that $x \sqsubseteq y$ and $x \in F$; we will show that $y \in F$. Since $x \sqsubseteq y$, we have $x \sqcap y = x \sqcap x$. Furthermore,

$$\begin{aligned} x \to y &\overset{\text{def}}{=} \neg x \vee y \\ &= \neg(x \sqcap x) \vee y \text{ (due to axiom (4a))} \\ &= \neg(x \sqcap y) \vee y \text{ (due to } \ x \sqcap y = x \sqcap x) \\ &= (\neg x \vee \neg y) \vee y \text{ (by (5) of Prop 2.4)} \\ &= \neg x \vee (\neg y \vee y) \text{ (by associativity of } \ \vee) \\ &= \neg x \vee (\top \sqcap \top) = \top \sqcap \top \in F. \end{aligned}$$

So $x \in F$ and $x \to y \in F$; by assumptions on F we deduce that $y \in F$. Therefore F is a filter. □

From the above Proposition 3.1, each filter (resp. ideal) on dBa is a deductive system.

3.2 Some characterization of primary filters (ideals)

To prove that each double Boolean algebra can be quasi-embedded into the algebra of protoconcepts of a suitable context, Rudolf Wille (see. [16]) consider the set $\mathcal{F}_P(\underline{D})$ of filter F of the dBa $\underline{D}$ for which $F \cap D_\sqcap$ is a prime filter of the Boolean algebra $\underline{D}_\sqcap$, and the set $\mathcal{I}_P(\underline{D})$ of ideal I of $\underline{D}$ for which $I \cap D_\sqcup$ is a prime ideal of the Boolean algebra $\underline{D}_\sqcup$. To establish the prime ideal theorem for dBas, Kwuida in [12] introduced the set $\mathcal{F}_{pr}(\underline{D})$ of filters F such that F is proper and for any $x \in D$, $x \in F$ or $\neg x \in F$ called a **primary filter** of $\underline{D}$ and the set $\mathcal{I}_{pr}(\underline{D})$ of ideals I such that I is proper and for any $x \in D$, $\lrcorner x \in I$ or $x \in I$ called a **primary ideal** of $\underline{D}$. Prosenjit et al in [9] shown that the primary filters(ideals) of a dBa $\underline{D}$ introduced by Kwuida in [12] are exactly the extension of prime filters(ideals) of the Boolean algebras $\underline{D_\sqcap}(\underline{D_\sqcup})$ as presented in the following proposition.

Proposition 3.2. [9] *Let $\underline{D}$ be a double Boolean algebra. Then the following statements hold:*
(1) $\mathcal{F}_{pr}(\underline{D}) = F_P(\underline{D})$.
(2) $\mathcal{I}_{pr}(\underline{D}) = \mathcal{I}_P(\underline{D})$.

Following this proposition, we retain that the role of prime filter(resp. ideal) of Boolean algebras is assumed by the primary filters(resp. ideals) introduced by Kwuida for dBas. The filter F is called **maximal** or **ultrafilter** if F is not contained in another proper filter of $\underline{D}$. The ideal I is called a **maximal ideal** if I is proper and not contained in another proper ideal of $\underline{D}$. We say that an ideal I of a lattice $\underline{L}$ is **prime** if $I \neq L$ and $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Dually, we say that a filter F of a lattice $\underline{L}$ is **prime** if $F \neq L$ and

$$a \vee b \in F \implies a \in F \text{ or } b \in F.$$

Knowing that primary ideals (resp. filters) are very important for the study of algebraic structures and topology on algebraic structures, we give many characterizations of primary filters (resp. primary ideals) on dBas in the following theorem that extend those existing in Boolean algebras.

Theorem 3.3. *Let $\underline{D}$ be a dBa and let F be a filter of $\underline{D}$. Then the following statements are equivalent:*

(1) *F is a primary filter.*

(2) *F is an ultrafilter.*

(3) *For any $x, y \in D$, $x \vee y \in F$ implies $x \in F$ or $y \in F$.*

(4) *For any $x, y \in D$, $x \vee y \in F$ implies $x \to y \in F$ or $y \to x \in F$.*

(5) *If F_1 and F_2 are filters and $F_1 \cap F_2 \subseteq F$, then $F_1 = F$ or $F_2 = F$.*

Proof. We recall that for any $x \in D$, $x \to y = \neg x \vee y$, $\neg x \vee x = \top \sqcap \top$, and $x \sqcap x = \neg\neg x$.

(1)$\Rightarrow$ (2) We assume that F is a primary filter of $\underline{D}$, then $F \sqcap D_\sqcap$ is a prime filter of $\underline{D}_\sqcap$ and so a maximal filter of $\underline{D}_\sqcap$. Let G be a filter of $\underline{D}$ such that $F \subseteq G$, then $F \cap D_\sqcap \subseteq G \cap D_\sqcap$, as $F \cap D_\sqcap$ is maximal in $\underline{D}_\sqcap$ we get $F \cap D_\sqcap = G \cap D_\sqcap$ or $G \cap D_\sqcap = D_\sqcap$. Assume that $G \cap D_\sqcap = D_\sqcap$, then $D_\sqcap \subseteq G$. Let $x \in D$, then $x \sqcap x \in D_\sqcap \subseteq G$, since G is a filter, we have $x \in G$, so $G = D$. Now we suppose that $F \cap D_\sqcap = G \cap D_\sqcap$, then for $x \in F$, $x \sqcap x \in F \cap D_\sqcap = G \cap D_\sqcap$; so $x \sqcap x \in G$, and $x \in G$(due to G a filter); therefore $F \subseteq G$, similarly one can show that $G \subseteq F$; hence $F = G$. Thus F is an ultrafilter.

(2) $\Rightarrow$ (3) Assume that F is an ultrafilter. Let $x, y \in D$ such that $x \vee y \in F$, in addition assume that $x \notin F$. We will show that $y \in F$. We define $G = \{z \in D : x \vee z \in F\}$. Clearly $F \subseteq G$ (due to $z \sqcap z \sqsubseteq x \vee z$ for all $z \in F$). Assume that $a, b \in G$, then $x \vee a, x \vee b \in F$, as F is a filter, we get $(a \sqcap b) \vee x = (a \vee x) \sqcap (b \vee x) \in F$(by (iii) Proposition 2.5), hence $a \sqcap b \in G$. If $a \sqsubseteq b$ and $a \in G$, then $a \vee x \sqsubseteq b \vee x$ (using (vii) of Proposition 2.5 and $b \vee x \in F$); so $b \in G$. Thus G is a filter containing F. Furthermore $x \vee x = x \sqcap x \notin F$, so $x \notin G$, therefore $G \neq D$, and as F is maximal we get $F = G$. Since $x \vee y \in F$, we deduce that $y \in G = F$.

(3) $\Rightarrow$(4) Assume that (3) holds. Let $x, y \in D$ such that $x \vee y \in F$. We will show that $x \to y \in F$ or $y \to x \in F$. From the assumption we get $x \in F$ or $y \in F$. If $x \in F$, then $x \sqcap x \in F$ and $y \to x = \neg y \vee x = (\neg y) \vee (x \sqcap x) \in F$ (due to $x \sqcap x \sqsubseteq (\neg y) \vee (x \sqcap x)$ and F a filter). If $y \in F$, then a similar argument shows that $x \to y \in F$ and (4) holds.

(4)$\Rightarrow$(1) Assume that (4) holds. Let $x \in D$, we will show that $x \in F$ or $\neg x \in F$. We have $x \vee \neg x = \top \sqcap \top \in F$, so by assumption $\neg x \to x \in F$ or $x \to \neg x \in F$, that is $\neg\neg x \vee x = x \sqcap x \in F$ or $\neg x \vee \neg x = \neg x \in F$; hence $x \in F$ or $\neg x \in F$ and we are done.

(3)$\Rightarrow$(5) Assume that (3) holds. Let F_1, F_2 be two filters such that $F_1 \cap F_2 \subseteq F$. We will show that $F_1 \subseteq F$ or $F_2 \subseteq F$. Assume that F_i is not contained in F $(i = 1, 2)$, then there exist $x \in F_1 \setminus F$ and $y \in F_2 \setminus F$. Since $x \vee y \in F_1 \cap F_2 \subseteq F$ and (3) holds for F, we get $x \in F$ or $y \in F$, contradicting the assumption on x and y. It follows that $F_1 \subseteq F$ or $F_2 \subseteq F$.

(5)$\Rightarrow$(3). Assume that (5) holds. Let $x, y \in D$ such that $x \vee y \in D$, then using (7) of Proposition 2.8, $F(x) \cap F(y) = F(x \vee y) \subseteq F$. Applying (5) we get that $F(x) \subseteq F$ or $F(y) \subseteq F$, and we deduce that $x \in F$ or $y \in F$. $\square$

The dual result of the above Theorem 3.3 holds for ideals on dBAs.

3.3 Filter dense and ideal co-dense in dBas

A dense set and a dense element were studied on a double p-algebras by R. Beaser (see [1]), we introduce the dense set and co-dense set on double Boolean algebras and study some related properties. Let $\underline{D}$ be a dBa, we consider the following sets:

$$F_D := \{x \in D : \neg x = \bot\}, \quad I_D = \{x \in D : \lrcorner x = \top\}.$$

The elements of F_D are called **dense** and those of I_D **co-dense**. In the following proposition we show that F_D is a filter and I_D is an ideal.

Proposition 3.4. *Let $\underline{D}$ be a dBas and F_D, I_D as above, then the following hold:*
(1) *F_D is a filter of $\underline{D}$.*
(2) *I_D is an ideal of $\underline{D}$.*

Proof. (1) is dual of (2), we show (1) and the proof of (2) is obtained dually. We have $F_D \neq \emptyset$ (due to $\neg\top = \bot$ (axiom (11a))); so $\top \in F_D$. Let $x, y \in F_D$, then $\neg x = \bot$ and $\neg y = \bot$; furthermore $\neg(x \sqcap y) = \neg x \vee \neg y$(by (5) of Proposition 2.4) $= \bot \vee \bot$ (due to $x, y \in F_D$) $= \bot \sqcap \bot = \bot$; so $x \sqcap y \in F_D$. Let $x \in F_D$ and $y \in D$ such that $x \sqsubseteq y$, then $\neg y \sqsubseteq \neg x = \bot$ (by (9) of Proposition 2.4 and $\neg x = \bot$), therefore using (1) of Proposition 2.4 we get $\bot \sqsubseteq \neg y \sqsubseteq \bot$ and $\neg y = \bot$, so $y \in F_D$. Thus F_D is a filter of $\underline{D}$. $\square$

The set F_D is called a **dense filter** and I_D is called a **co-dense** ideal of $\underline{D}$. In [12], Kwuida defined a **trivial double Boolean algebra** as a double Boolean algebra in which $\top \sqcap \top = \bot \sqcup \bot$. As example of trivial dBa we get the three elements chain $\underline{D_3} = (\{\bot, a, \top\}; \sqcup, \sqcap, \neg, \lrcorner, \bot, \top)$ with $a \sqcap a = a \sqcup a = a$ and $\bot \sqsubseteq a \sqsubseteq \top$.

Theorem 3.5. *Let $\underline{D}$ be a dBa, F_D and I_D as above. Then $I_D \cap F_D \neq \emptyset$ if and only if $\underline{D}$ is trivial.*

Proof. Assume that $\underline{D}$ is trivial; then $D_\sqcap \cap D_\sqcup = \{e\}$ with $e = \bot \sqcup \bot = \top \sqcap \top$ and $\neg e = \bot$, $\lrcorner e = \top$. Hence $F_D \cap I_D \neq \emptyset$. Conversely, assume that $F_D \cap I_D \neq \emptyset$. We will show that $\underline{D}$ is trivial. Let $a \in F_D \cap I_D$, then $\neg a = \bot$ and $\lrcorner a = \top$. Furthermore $\neg\neg a = \neg\bot = \top \sqcap \top$ (axiom (10a)) and $\lrcorner\lrcorner a = \lrcorner\top = \bot \sqcup \bot$ (axiom (10b)), that is $a \sqcap a = \top \sqcap \top$ and $a \sqcup a = \bot \sqcup \bot$. Knowing that $\bot \sqsubseteq \top \sqcap \top$ we have by (viii) of Proposition 2.5 that $\bot \sqcup \bot \sqsubseteq \top \sqcap \top$. Since $a \sqcup a = \bot \sqcup \bot$ and $a \sqcap a \sqsubseteq a \sqcup a$, we deduce that $\top \sqcap \top \sqsubseteq \bot \sqcup \bot \sqsubseteq \top \sqcap \top$, using the fact that $\top \sqcap \top, \bot \sqcup \bot \in D_\sqcap \cap D_\sqcup$, we deduce that $\bot \sqcup \bot = \top \sqcap \top$ and $\underline{D}$ is trivial. $\square$

Following the above Theorem 3.5 and the prime ideal theorem [12], we get that if $\underline{D}$ is not trivial, then $F_D \cap I_D = \emptyset$ and there exists a maximal ideal I and a maximal filter F such that $F_D \subseteq F$, $I_D \subseteq I$ and $F \cap I = \emptyset$.

4 The structure of the set of filters(ideals) on dBas

In [14], we have shown that the lattice of filters(resp. ideals) of a dBa form an algebraic lattice in which compact elements are principal filters(resp. ideals) and principal filters (resp. ideals) form a Boolean algebras. In this section, we give additional properties of these lattices by showing that they are distributive, Brouwerian, pseudocomplemented lattices and form a Heyting algebra. We finish this section by showing that the lattice of filters(resp. ideals) of a double Boolean algebra possess a structure of residuated lattice. We need the following definitions and properties in lattice theory for the sequel.

Definition 4.1. *A lattice $\underline{L}$ is called **distributive** if the following distributive law holds:*

$$(D)\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Infinite analogue of distributive law in a complete lattice is given by:

$$(JID)\quad x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i) \quad \text{join-infinite distributive law.}$$

The distributivity of a complete lattice does not always imply the law (JID)(see [5], p.20). An algebra $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0)$ of type $(2, 2, 2, 0)$ is called a **Heyting algebra** if $\mathcal{A} = (A, \wedge, \vee, 0)$ is a lattice with a smallest element 0 and for all $x, y, z \in A$ the following holds:

$$x \wedge y \leq z \text{ iff } y \leq (x \rightarrow z) \quad \text{(the law of } \wedge \text{ residuation)},$$

that is $x \rightarrow z = \max\{y : x \wedge y \leq z\}$(see [5], p.21). A complete lattice $(L, \wedge, \vee)$ is **Brouwerian** iff it satisfies the above "join-infinite distributive law" whenever the arbitrary joins exist.

Let $\underline{L}$ be a lattice. An element $a \in L$ is **compact** if whenever $\bigvee A$ exists and $a \leq \bigvee A$ for $A \subseteq L$, we have $a \leq \bigvee B$ for some finite $B \subseteq A$. The lattice $\underline{L}$ is **compactly generated** if every element in L is a supremum of compact elements; $\underline{L}$ is **algebraic** if $\underline{L}$ is complete and compactly generated(see) [3], p.17).

In a bounded distributive lattice L, if b is a complement of a, then b is the largest element x of L with $a \wedge x = 0$. More generally, let L be a lattice with zero; an element a^* is a **pseudocomplement** of a $(\in L)$ if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. An element can have at most one pseudocomplement (see [7], p.99). A **pseudocomplement lattice** is one in which every element has a pseudocomplement ([8]). A homomorphism ϕ of a pseudocomplemented lattice into another pseudocomplemented lattice is a lattice homomorphism additionnally preserving $0, 1, ^*$, that is

$$\phi(0) = 0,\ \phi(1) = 1 \text{ and } \phi(x)^* = \phi(x^*).$$

We will use the following remark in the sequel.

Remark 4.2. *Let $\underline{D}$ be a dBa. Let J be an arbitrary nonempty set, $\{F_j : j \in J\} \subseteq F(\underline{D})$ and $\{I_j : j \in J\} \subseteq I(\underline{D})$. Set $\tilde{J} := \{(i_1, \ldots, i_n) \in J^n : n \geq 1, n \in \mathbb{N}\}$. The following hold:*

$$(*)\ \underset{j \in J}{\vee} F_j = \bigcup_{(i_1,\ldots,i_n) \in \tilde{J}} (F_{i_1} \vee \ldots \vee F_{i_n}), \qquad (**)\ \underset{j \in J}{\vee} I_j = \bigcup_{(i_1,\ldots,i_n) \in \tilde{J}} (I_{i_1} \vee \ldots \vee I_{i_n}).$$

Lemma 4.3. *Let $\underline{D}$ be a dBa, $\{F, F_i, i \in I\} \subseteq F(\underline{D})$, $\{I, I_j, j \in J\} \subseteq I(\underline{D})$, then*
(1) $I \wedge (\underset{j \in J}{\vee} I_j) = \underset{j \in J}{\vee} (I \wedge I_i)$.
(2) $F \wedge (\underset{j \in J}{\vee} F_j) = \underset{j \in J}{\vee} (F \wedge F_i)$.

Proof. (2) is dual to (1), we give the proof of (1). Let $I \in I(\underline{D})$ and $\{I_j : j \in J\} \subseteq I(\underline{D})$, We show that $I \wedge (\underset{j \in J}{\vee} I_j) = \underset{j \in J}{\vee} (I \wedge I_j)$. Since $\mathcal{I}(\underline{D})$ is a complete lattice, clearly $\underset{j \in J}{\vee} (I \cap I_j) \subseteq I \cap (\underset{j \in J}{\vee} I_j)$. Let now $x \in I \cap (\underset{j \in J}{\vee} I_j)$, then $x \in I$ and by the Remark 4.2 there exist $j_1, \ldots, j_m \in J, x_{j_k} \in I_{j_k}, (1 \leq k \leq m)$ such that $x \sqsubseteq x_{i_1} \sqcup \ldots \sqcup x_{i_m}$, so by (vii) of Proposition 2.5, $x \wedge x \sqsubseteq x \wedge (x_{i_1} \sqcup \ldots \sqcup x_{i_m})$, in addition, by (ii) of Proposition 2.5 $x \wedge (x_{i_1} \sqcup \ldots \sqcup x_{i_m}) = (x \wedge x_{i_1}) \sqcup \ldots \sqcup (x \wedge x_{j_m})$, and (by (6) of Proposition 2.8) $I(x) \cap I(x_{i_k}) = I(x \wedge x_{i_k}) \subseteq I \cap I_{i_k}, (1 \leq k \leq m)$; furthermore $x \sqsubseteq x \sqcup x = x \wedge x$; we deduce that $x \in \underset{j \in J}{\wedge} (I \cap I_j)$. Hence, (1) is proved. □

Theorem 4.4. *Let $\underline{D}$ be a dBa. The following statements hold:*

1. *$\mathcal{F}(\underline{D}) = (F(\underline{D}), \cap, \vee)$ is a Brouwerian lattice, algebraic lattice in which compact elements are principal filters of $\underline{D}$,*

2. *$\mathcal{I}(\underline{D}) = (I(\underline{D}), \cap, \vee)$ is a Brouwerian lattice, algebraic lattice in which compact elements are principal ideals of $\underline{D}$.*

Proof. We give the proof of (1) and that of (2) is obtained dually. For (1), $\mathcal{F}(\underline{D})$ is an algebraic lattice with compact elements, principal filters (see [14, 15]). (1) of Lemma 4.3 achieves the proof. □

Corollary 4.5. *The lattices $\mathcal{F}(\underline{D})$ and $\mathcal{I}(\underline{D})$ are distributive.*

Proof. It follows from Lemma 4.3 and Theorem 4.4 □

Proposition 4.6. *Let $\underline{D}$ be a dBa and P be a proper filter of $\underline{D}$. If $\{F \in F(\underline{D}) : P \subseteq F\}$ is a chain, then P is primary.*

Proof. Set $\Sigma = \{F \in F(\underline{D}) : P \subseteq F\}$ and suppose that Σ is a chain and P is not a primary filter, then there exist $x, y \in D$ such that $x \vee y \in P, x \notin P$ and $y \notin P$. Since $F(P \cup \{x\}), F(P \cup \{y\}) \in \Sigma$, without loss of generality let $F(P \cup \{x\}) \subseteq F(P \cup \{y\})$. By distributivity of $F(\underline{D})$ and (5) of Proposition 2.8 we have the following sequence formulas:

$$\begin{aligned} P = P \vee F(x \vee y) &= P \vee (F(x) \cap F(y)) \text{ (by (5) of Proposition 2.8)} \\ &= [P \vee F(x)] \cap [P \vee F(y)] \text{ (by distributivity of } \vee \text{ over } \cap) \\ &= F(P \cup \{x\}) \cap F(P \cup \{y\}) \text{ (using (f) of Proposition 2.7)} \\ &= F(P \cup \{x\}). \end{aligned}$$

This shows that $x \in P$, contradiction. Hence P must be a primary filter. □

In ordrer to show that for a given double Boolean algebra, the set of all its filters (resp. ideals) is endowed with a structure of pseudocomplemented distributive lattice and the structure of Heyting algebra, we define the following operations. Let $\underline{D}$ be a dBa and $F_1, F_2 \in F(\underline{D}), I_1, I_2 \in I(\underline{D})$, we set

$$I_1 \to I_2 = \{x \in D : I(x) \cap I_1 \subseteq I_2\}, \quad F_1 \to F_2 = \{x \in D : F(x) \cap F_1 \subseteq F_2\}.$$

Lemma 4.7. *Let $\underline{D}$ be a dBa, $I_j, I \in I(\underline{D}), F_j, F \in F(\underline{D}), j = 1, 2$ and $F_1 \to F_2, I_1 \to I_2$ as above, then the following statements hold:*

1. *(i) $I_1 \to I_2 \in I(\underline{D})$. (ii) $F_1 \to F_2 \in F(\underline{D})$.*

2. *$I_1 \cap I \subseteq I_2$ iff $I \subseteq I_1 \to I_2$, that is $I_1 \to I_2 = \sup\{I \in I(\underline{D}) : I_1 \cap I \subseteq I_2\}$.*

3. *$F_1 \cap F \subseteq F_2$ iff $F \subseteq F_1 \to F_2$, that is $F_1 \to F_2 = \sup\{F \in F(\underline{D}) : F_1 \cap F \subseteq F_2\}$.*

Proof. (1)(i) is dual to (1)(ii) and (2) is dual to (3). We show (1)(i) and (2).

For (1)(i) we show that $I_1 \to I_2$ is an ideal. Since $I(\bot) \cap I_1 = I(\bot) \subseteq I_2$, we get $\bot \in I_1 \to I_2$ and $I_1 \to I_2 \neq \emptyset$. Let $x \in D$ and $y \in I_1 \to I_2$, then $I(y) \cap I_1 \subseteq I_2$; furthermore $x \sqsubseteq y$ implies that $I(x) \subseteq I(y)$ (by (11) of Proposition 2.7); so $I(x) \cap I_1 \subseteq I(y) \cap I_1 \subseteq I_2$ and by transitivity of $\subseteq$ we get that $I(x) \cap I_1 \subseteq I_2$, therefore $x \in I_1 \to I_2$. Now assume that $x, y \in I_1 \to I_2$, then $I(x) \cap I_1 \subseteq I_2$ and

$I(y) \cap I_1 \subseteq I_2$, which imply that $(I(x) \cap I_1) \vee (I(y) \cap I_1) \subseteq I_2$; furthermore, by (1) of Lemma 4.3 taking $I(x) = I_1,\ I(y) = I_2$ and $J = \{1, 2\}$ we get $(I(x) \cap I_1) \vee (I(y) \cap I_1) = I_1 \cap (I(x) \vee I(y)) = I_1 \cap I(x \sqcup y)$ (by (2) of Proposition 2.7) $\subseteq I_2$; therefore $x \sqcup y \in I_1 \to I_2$. Thus $I_1 \to I_2 \in I(\underline{D})$ and (1)(i) is proved.

(2) ($\Rightarrow$) Assume that $I_1 \cap I \subseteq I_2$, we will show that $I \subseteq I_1 \to I_2$. Let $x \in I$, we show that $x \in I_1 \to I_2$, to do this we show that $I(x) \cap I_1 \subseteq I_2$. Let $u \in I(x) \cap I_1 \subseteq I \cap I_1$, then $u \in I(x) \subseteq I$ and $u \in I_1$. Since $u \in I(x) \subseteq I$, we have $u \sqsubseteq x \sqcup x$ (by (1) of Proposition 2.7), and $x \sqcup x \in I$, then $u \in I \cap I_1 \subseteq I_2$ (by assumption), therefore $x \in I_1 \to I_2$ and $I \subseteq I_1 \to I_2$.

($\Leftarrow$) Assume that $I \subseteq I_1 \to I_2$, we will show that $I_1 \cap I \subseteq I_2$. Let $x \in I \cap I_1$, we will show that $x \in I_2$. Since $x \in I \cap I_1$, we get $x \in I$ and $x \in I_1$, by $I \subseteq I_1 \to I_2$, $x \in I$ implies that $I(x) \cap I_1 \subseteq I_2$, but $I(x) \subseteq I_1$ (due to $x \in I_1$), therefore $x \in I_2$, hence $I \cap I_1 \subseteq I_2$. Thus (2) is proved. □

Let $I_\sqcup \in \mathcal{I}(\underline{D_\sqcup}), I \in \mathcal{I}(\underline{D})$, $F_\sqcap \in \mathcal{F}(\underline{D_\sqcap})$, and $F \in \mathcal{F}(\underline{D})$, we set

$$I_\sqcup^\star := \{x \in D_\sqcup : \forall i \in I_\sqcup : i \wedge x = \bot \sqcup \bot\},\ I^\star := \{x \in D : \forall i \in I : i \wedge x = \bot \sqcup \bot\},$$

$$F_\sqcap^* := \{x \in D_\sqcap : \forall j \in F_\sqcap, x \vee j = \top \sqcap \top\},\ F^* := \{x \in D : \forall j \in F, x \vee j = \top \sqcap \top\}.$$

It is shown in [14, 15], that $\mathcal{F}(\underline{D})$, $\mathcal{I}(\underline{D})$, $\mathcal{I}(\underline{D_\sqcup})$, and $\mathcal{F}(\underline{D_\sqcap})$ are algebraic lattices. We consider the following maps: $\Phi : F(\underline{D_\sqcap}) \to F(\underline{D}),\ E \mapsto \Phi(E) = \{x \in D : \exists u \in E, u \sqsubseteq x\}$ and $\Psi : I(\underline{D_\sqcup}) \to I(\underline{D}),\ I \mapsto \Psi(I) = \{x \in D : \exists x_0 \in I,\ x \sqsubseteq x_0\}$ defined in [14].

For any $a \in D, I(a)$ (resp. $F(a)$) is the principal ideal (resp. filter) of $\underline{D}$ generate by a and for any $a \in D_\sqcup$ (resp. $a \in D_\sqcap$), $I_\sqcup(a)$ (resp. $F_\sqcap(a)$) denoted the principal ideal (resp. filter) of $\underline{D_\sqcup}$ (resp. $\underline{D_\sqcap}$). We have the following known result.

Theorem 4.8. [14] *Let $\underline{D}$ be a double Boolean algebra, then:*

(1) *$\mathcal{F}(\underline{D_\sqcap})$ and $\mathcal{F}(\underline{D})$ are isomorphic lattices via Φ,*

(2) *$\mathcal{I}(\underline{D_\sqcup})$ and $\mathcal{I}(\underline{D})$ are isomorphic lattices via Ψ.*

We need the following lemma for the proof of Theorem 4.10.

Lemma 4.9. *Let $\underline{D}$ be a dBa, $I, J \in I(\underline{D})$, $F, G \in F(\underline{D})$, $I_\sqcup \in I(\underline{D_\sqcup})$, $F_\sqcap \in F(\underline{D_\sqcap})$ and $I^\star, F^*$, $I_\sqcup^\star$, $F_\sqcap^*$ as above, then the following statements hold.*

(1) *(i) $I^\star \in I(\underline{D})$, (ii) $I \cap I^\star = I(\bot \sqcup \bot)$, (iii) If $J \cap I = I(\bot \sqcup \bot)$, then $J \subseteq I^\star$.*

(2) *(i) $F^* \in F(\underline{D})$, (ii) $F^* \cap F = F(\top \sqcap \top)$, (iii) If $G \cap F = F(\top \sqcap \top)$, then $G \subseteq F^*$.*

(3) *(i) $I_\sqcup^\star \in I(\underline{D})$, (ii) $I_\sqcup \cap I_\sqcup^\star = I_\sqcup(\bot \sqcup \bot)$, (iii) If $J_\sqcup \cap I_\sqcup = I_\sqcup(\bot \sqcup \bot)$, then $J_\sqcup \subseteq I_\sqcup^\star$.*

(4) *(i) $F_\sqcap^* \in F(\underline{D_\sqcap})$. (ii) $F_\sqcap^* \cap F_\sqcap = F_\sqcap(\top \sqcap \top)$, (iii) If $G_\sqcap \cap F_\sqcap = F_\sqcap(\top \sqcap \top)$, then $G_\sqcap \subseteq F_\sqcap^*$.*

(5) *(i) $\Psi(I_\sqcup^\star) = (\Psi(I_\sqcup))^\star$. (ii) $\Phi(F_\sqcap^*) = (\Phi(F_\sqcap))^*$.*

Proof. For (1)(i), we have $I^\star \neq \emptyset$ (due to $\bot \wedge i = \bot \sqcup \bot$ for any $i \in I$). Assume that $x \sqsubseteq y$ with $y \in I^\star$. Since $y \in I^\star$, for any $i \in I$ we get $y \wedge i = \bot \sqcup \bot$. Furthermore, $\sqsubseteq$ is compatible with $\sqcup$ and $\wedge$, so from $x \sqsubseteq y$ we get $\bot \sqcup \bot \sqsubseteq x \sqcup x \sqsubseteq y \sqcup y$ and $\bot \sqcup \bot \sqsubseteq (x \sqcup x) \wedge (i \sqcup i) \sqsubseteq (y \sqcup y) \wedge (i \sqcup i) = \bot \sqcup \bot$. It follows that $\bot \sqcup \bot = (x \sqcup x) \wedge (i \sqcup i) = x \wedge i$ and $x \in I^\star$. Let $x,\ y \in I^\star$, then for $i \in I$ we have $(x \sqcup x) \wedge (i \sqcup i) = \bot \sqcup \bot$ and $(y \sqcup y) \wedge (i \sqcup i) = \bot \sqcup \bot$. In addition,

$$\begin{aligned}(i \sqcup i) \wedge (x \sqcup y) &= [(i \sqcup i) \wedge (x \sqcup x)] \sqcup [(i \sqcup i) \wedge (y \sqcup y)] \\ &= (\bot \sqcup \bot) \sqcup (\bot \sqcup \bot) = \bot \sqcup \bot.\end{aligned}$$

So $x \sqcup y \in I$. Thus $I^\star$ is an ideal of $\underline{D}$. For (1)(ii), since $I^\star$ and I are ideals we have $I(\bot \sqcup \bot) \subseteq I \cap I^\star$. Let $x \in I \cap I^\star$, then $(x \sqcup x) \wedge (x \sqcup x) = \bot \sqcup \bot$, that is $x \sqcup x = \bot \sqcup \bot$ and $x \sqsubseteq \bot \sqcup \bot$, so $x \in I(\bot \sqcup \bot)$. Thus $I \cap I^\star = I(\bot \sqcup \bot)$ and (1)(ii) holds.

For (1)(iii) let J be an ideal of $\underline{D}$ such that $J \cap I = I(\bot \sqcup \bot)$, we will show that $J \subseteq I^\star$. Let $x \in J$ and let $i \in I$, then $i \wedge x = (i \sqcup i) \wedge (x \sqcup x) \sqsubseteq x \sqcup x, i \sqcup i$ (due to $x \sqcup x,\ i \sqcup i \in D_\sqcup$) and $x \sqcup x \in J$, $i \sqcup i \in I$, so $x \wedge i \in I \cap J = I(\bot \sqcup \bot)$, therefore $\bot \sqcup \bot \sqsubseteq x \wedge i \sqsubseteq \bot \sqcup \bot$ and $x \wedge i = \bot \sqcup \bot$, thus $x \in I^\star$ and (iii) holds. The proof of (2) is dual to that of (1).

(3) A similar argument use in the proof of (1) (resp.(2)) show that (3) (resp. (4)) holds.

(5) We show that $\Psi(I_\sqcup^*) = (\Psi(I_\sqcup))^\star$. First we show that $\Psi(I_\sqcup^\star) \subseteq (\Psi(I_\sqcup))^\star$. Let $y \in \Psi(I_\sqcup^\star)$, then there exists $u \in I_\sqcup^\star$ such that $y \sqsubseteq u$ (1.1). As $u \in I_\sqcup^\star$, then for any $v \in I_\sqcup$, $u \wedge v = \bot \sqcup \bot$. Let $z \in \Psi(I_\sqcup)$, then there exists $t \in I_\sqcup$ such that $z \sqsubseteq t$ (1.2). From (1.1) and (1.2), using (vii) of Proposition 2.5 we get $\bot \sqcup \bot \sqsubseteq y \wedge z \sqsubseteq u \wedge t = \bot \sqcup \bot$ (by (2)), so $y \wedge z = \bot \sqcup \bot$ for any $z \in \Psi(I_\sqcup)$, therefore $y \in \Psi(I_\sqcup)^\star$ and $\Psi(I_\sqcup^\star) \subseteq \Psi(I_\sqcup)^\star$. It remains to show that $\Psi(I_\sqcup)^\star \subseteq \Psi(I_\sqcup^\star)$. Let $y \in [\Psi(I_\sqcup)]^\star$, we show that $y \in \Psi(I_\sqcup^\star)$. As $y \in [\Psi(I_\sqcup)]^\star$, then for every $v \in \Psi(I_\sqcup), y \wedge v = \bot \sqcup \bot$. Furthermore $I_\sqcup \subseteq \Psi(I_\sqcup)$ (by definition of Ψ), then for any $v \in I_\sqcup$, $y \wedge v = \bot \sqcup \bot$, it follows that $(y \sqcup y) \wedge (v \sqcup v) = \bot \sqcup \bot$ for any $v \in I_\sqcup$. Thus $y \sqcup y \in I_\sqcup^\star$, furthermore $y \sqsubseteq y \sqcup y$, so $y \sqcup y \in \Psi(I_\sqcup^\star)$. Therefore $y \in \Psi(I_\sqcup^\star)$ and $\Psi(I_\sqcup)^\star \subseteq \Psi(I_\sqcup^\star)$. Thus (5) holds.

(6) A similar argument use in the proof of (5) show that (6) holds. □

From the above two Lemmas 4.9 and 4.7, the binary operation $\to$ is well defined on $F(\underline{D})$ and on $I(\underline{D})$, and the unary operation * (resp. $\star$) is also well defined on $F(\underline{D})$ (resp. $I(\underline{D})$). We consider the following algebras:

$$\mathcal{F}(\underline{D}) = (F(\underline{D}), \wedge, \vee, \to, ^*, F(\top), D), \quad \mathcal{F}(\underline{D_\sqcap}) = (F(D_\sqcap), \wedge, \vee, \to, ^*, F_\sqcap(\top \sqcap \top), D_\sqcap),$$

$$\mathcal{I}(\underline{D}) = (I(\underline{D}), \wedge, \vee, \to, ^\star, I(\bot), D),\ \mathcal{I}(\underline{D_\sqcup}) = (I(D_\sqcup), \wedge, \vee, \to, ^\star, I_\sqcup(\bot \sqcup \bot), D_\sqcup).$$

Theorem 4.10. *Let $\underline{D}$ be a dBa, Ψ and Φ as above, then the following statement hold:*

1. *$\mathcal{I}(\underline{D_\sqcup})$ and $\mathcal{I}(\underline{D})$ are distributive lattices, pseudocomplemented lattices, algebraic lattices, Brouwerian and Heyting algebras isomorphic via Ψ.*
2. *$\mathcal{F}(\underline{D_\sqcap})$ and $\mathcal{F}(\underline{D})$ are distributive lattices, pseudocomplemented lattices, Brouwerian lattices, algebraic lattices and Heyting algebras isomorphic via Φ.*

Proof. (1) By (1) of Lemma 4.9 we get that $I^\star$ is a pseudocomplement of $I \in \mathcal{I}(\underline{D})$ and $I_\sqcup^\star$ is a pseudocomplement of $I_\sqcup \in \mathcal{I}(\underline{D_\sqcup})$. In addition, from Theorem 4.8 we get that Ψ is an isomorphism of Bounded lattices. It remains to show that $\Psi(I_\sqcup^*) = \Psi(I_\sqcup)^*$ for any $I_\sqcup \in \mathcal{I}(\underline{D_\sqcup})$ what is true by (5)(i) of Lemma 4.9. Thus Ψ is an isomorphism of pseudocomplemented lattices.
The proof of (2) is similar to that of (1). □

Definition 4.11. *An algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a* **residuated lattice** *if it satisfies:*

$(LR1)$ *$(A, \wedge, \vee, 0, 1)$ is a Bounded lattice.*

$(LR2)$ *$(A, \odot, 1)$ is a commutatives monoid.*

$(LR3)$ *$\odot$ and $\to$ form an adjoint pair .i.e $c \le a \to b$ if and only if $a \odot c \le b$, for all $a, b, c \in A$.*

A residuated lattice A is called a **Gödel algebra** if $x^2 = x \odot x = x$, for all $x \in A$. Taking $\odot = \cap$ on $\mathcal{F}(\underline{D})$ (resp. $\mathcal{I}(\underline{D})$) and $\to$ as above, we consider the algebras

$$\mathcal{F}(\underline{D}) = (F(\underline{D}), \cap, \vee, \odot, \to, F(\top), D) \text{ and } \mathcal{I}(\underline{D}) = (I(\underline{D}), \cap, \vee, \odot, \to, I(\perp), D).$$

Lemma 4.12. *Let $\underline{D}$ be a dBa, $I \in I(\underline{D})$ and $F \in F(\underline{D})$, then the following statements hold:*
(i) $F^* = F \to F(\top \sqcap \top)$.
(ii) $I^\star = I \to I(\perp \sqcup \perp)$.

Proof. (i) First, we show that $F^* \subseteq F \to F(\top)$. Let $x \in F^*$, then for any $j \in F$, $x \vee j = \top \sqcap \top$ (1). We will show that $x \in F \to F(\top)$, that is $F(x) \cap F \subseteq F(\top)$. Let $y \in F(x) \cap F$, then $y \in F$ and $x \sqcap x \sqsubseteq y$. Using (1) we get $x \vee y = \top \sqcap \top$ and using (VII) of Proposition 2.5 we get $(x \sqcap x) \vee y = \top \sqcap \top \sqsubseteq y \sqcap y \sqsubseteq y$, so $y \in F(\top)$ and $F(x) \cap F \subseteq F(\top)$. Hence $x \in F \to F(\top)$ and $F^* \subseteq F \to F(\top)$. Now let $x \in F \to F(\top)$, then $F(x) \cap F \subseteq F(\top)$. We will show that $x \in F^*$. Let $j \in F$, then $x \vee j \in F(x \vee j) = F(x) \cap F(j) \subseteq F(x) \cap F \subseteq F(\top)$, so $x \vee j \in F(\top)$, that is $\top \sqcap \top \sqsubseteq x \vee j \sqsubseteq \top \sqcap \top$, hence $x \vee j = \top \sqcap \top$ and $x \in F^*$. Therefore $F \to F^* \sqsubseteq F^*$. Thus $F^* = F \to F(\top)$.
The proof of (ii) is similar to that of (i). □

Proposition 4.13. *Let $\underline{D}$ be a dBa, $\odot$ and $\to$ the binary operations defined on $F(\underline{D})$, $F(\underline{D}_\sqcap)$, $I(\underline{D})$ and $I(\underline{D}_\sqcup)$ as above, then:*

1. *the algebras $\mathcal{F}(\underline{D}) = (F(\underline{D}), \cap, \vee, \odot, \to, F(\top), D)$ and $\mathcal{F}(\underline{D}_\sqcap) = (F(\underline{D}_\sqcap), \cap, \vee, \odot, \to, F(\top \sqcap \top), D_\sqcap)$ are residuated lattices;*

2. *the algebras $\mathcal{I}(\underline{D}) = (I(\underline{D}), \cap, \vee, \odot, \to, I(\perp), D)$ and $\mathcal{I}(\underline{D}_\sqcup) = (F(\underline{D}_\sqcup), \cap, \vee, \odot, \to, I(\perp), D_\sqcup)$ are residuated lattices.*

Proof. (1) It is clear that $(F(\underline{D}), \cap, D)$ is a commutative monoid and $(F(\underline{D}), \cap, \vee, F(\top), D)$ is a bounded lattice. In addition, (RL3) holds in $\mathcal{F}(\underline{D})$ by (3) of Lemma 4.7; $\cap$ is commutative and $F \cap D = F$ for all $F \in F(\underline{D})$. Hence $\mathcal{F}(\underline{D}) = (F(\underline{D}), \cap, \vee, \odot, \to, F(\top), D)$ is a residuated lattice. A similar argument show that $\mathcal{F}(\underline{D}_\sqcap) = (F(\underline{D}_\sqcap), \cap, \vee, \odot, \to, F_\sqcap(\top \sqcap \top), D_\sqcap)$ is a residuated lattice.
(2) A similar argument use in the proof of (1) shows that (2) holds. □

Lemma 4.14. *Let $\underline{D}$ be a double Boolean algebra and Φ, Ψ as above, then the following statements hold:*

1. *Φ is an isomorphism of residuated lattices (Heyting) algebras.*

2. *Ψ is an isomorphism of residuated lattice, (Heyting) algebras.*

Proof. We show (1) and the proof of (2) can be obtained using a similar argument.
For (1), it is known that Φ is an isomorphism of bounded lattices. It remains to show that for any $F, G \in F(\underline{D}_\sqcap)$, $\Phi(F \odot G) = \Phi(F) \odot \Phi(G)$ $(*_1)$ and $\Phi(F \to G) = \Phi(F) \to \Phi(G)$ $(*_2)$. It is easy to see that $(*_1)$ holds (due to $\odot = \cap$ and Φ is a morphism of lattices).
To prove $(*_2)$, let $F, G \in F(\underline{D}_\sqcap)$, first we show that $\Phi(F \to G) \subseteq \Phi(F) \to \Phi(G)$. Let $x \in \Phi(F \to G)$, then there exists $u \in F \to G$ such that $u \sqsubseteq x$. As $u \in F \to G$, $F_\sqcap(u) \cap F \subseteq G$. Now we show that $x \in \Phi(F) \to \Phi(G)$, that is $F(x) \cap \Phi(F) \subseteq \Phi(G)$. Let $t \in F(x) \cap \Phi(F)$, then $t \in F(x)$ and $t \in \Phi(F)$ and by definition of $F(x)$ we have $x \sqcap x \sqsubseteq t$ and there exists $v \in F$ such that $v \sqsubseteq t$. In addition, $u \vee v \in F(u \vee v) \subseteq F(u) \cap F \subseteq G$; so $u \vee v \in G$ and by the fact that $u, v \sqsubseteq t$ we get $u \vee v \sqsubseteq t$ and $t \in \Phi(G)$, therefore $F(x) \cap \Phi(F) \subseteq \Phi(G)$. Now we show that

$\Phi(F) \to \Phi(G) \subseteq \Phi(F \to G)$. Let $x \in \Phi(F) \to \Phi(G)$. We show that $x \in \Phi(F \to G)$, that is we find $u \in F \to G$ such that $u \sqsubseteq x$. We have

$$\begin{aligned}
x \in \Phi(F_\sqcap) \to \Phi(G_\sqcap) &\implies F_\sqcap(x) \cap \Phi(F_\sqcap) \subseteq \Phi(G_\sqcap)\\
&\implies \Phi(F(x) \cap D_\sqcap) \cap F_\sqcap) \subseteq \Phi(G_\sqcap) \text{ (due to } \Phi \text{ compatible with } \cap)\\
&\implies \Phi(F_\sqcap(x \sqcap x) \cap F_\sqcap) \subseteq \Phi(G_\sqcap)\\
&\implies F_\sqcap(x \sqcap x) \cap F \subseteq G \text{ (due to } \Phi \text{ is an isomorphism of lattices)}\\
&\implies x \sqcap x \in F_\sqcap \to G_\sqcap \text{ and } x \sqcap x \sqsubseteq x.
\end{aligned}$$

We deduce that $x \in \Phi(F_\sqcap \to G_\sqcap)$, so $\Phi(F_\sqcap) \to \Phi(G_\sqcap) \subseteq \Phi(F_\sqcap \to G_\sqcap)$. Thus $\Phi(F_\sqcap \to G_\sqcap) = \Phi(F_\sqcap) \to \Phi(G_\sqcap)$ and Φ is an isomorphism of (Heyting algebras) residuated lattices. □

We have the following theorem which present residuated lattices derived from Boolean algebras and double Boolean algebras.

Theorem 4.15. *Let $\underline{D}$ be a dBa. The following conditions hold.*

1. *The algebras $\mathcal{F}(\underline{D}) = (F(\underline{D}), \cap, \vee, \odot, \to, F(\top), D)$ and $\mathcal{F}(\underline{D}_\sqcap) = (F(\underline{D}_\sqcap), \cap, \vee, \odot, \to, F_\sqcap(\top \sqcap \top), D_\sqcap)$ are isomorphic residuated lattices via Φ.*
2. *The algebras $\mathcal{I}(\underline{D}) = (I(\underline{D}), \cap, \vee, \odot, \to, I(\bot), D)$ and $\mathcal{I}(\underline{D}_\sqcup) = (I(\underline{D}_\sqcup), \cap, \vee, \odot, \to, I_\sqcup(\bot), D_\sqcup)$ are isomorphic residuated lattices via Ψ.*

Proof. For (1), by (1) of Proposition 4.13 $\mathcal{F}(\underline{D}) = (F(\underline{D}), \cap, \vee, \odot, \to, F(\top), D)$ and $\mathcal{F}(\underline{D}_\sqcap) = (F(\underline{D}_\sqcap), \cap, \vee, \odot, \to, F_\sqcap(\top \sqcap \top), D_\sqcap)$ are residuated lattices; in addition from (1) of Lemma 4.14 we get that Φ is an isomorphism of residuated lattices.
The proof of (2) is similar to that of (1). □

Theorem 4.16 ([8], p.99)**.** *Let L be a pseudocomplemented semi-lattice, we defined the* ***skeleton*** *of L:*

$$Skel(L) := \{b^* : b \in L\}.$$

Then the ordering of L orders $SkelL$ into a Boolean lattice. For $a, b \in Skel(L)$; the meet $a \wedge b$, is in $Skel(L)$; the join in $Skel(L)$ is described as follows:

$$a \vee_{Skel} b = (a^* \wedge b^*)^*.$$

The complement of a in $Skel(L)$ is a^*.
We need to give a description of the elements of $Skel(\mathcal{F}(\underline{D}))$ and $Skel(\mathcal{I}(\underline{D}))$. Following the above Theorem 4.16 we known that $Skel(\mathcal{F}(\underline{D}))$ and $Skel(\mathcal{I}(\underline{D}))$ are Boolean algebras. It was shown in [14] that principal filters of $\underline{D}$ form a Boolean algebra isomorphic to $\underline{D}_\sqcap$, principal ideals of $\underline{D}$ form a Boolean algebra isomorphic to $\underline{D}_\sqcup$ and if $\underline{D}$ is a finite or a complete dBa, then all filters (resp. all ideals) of $\underline{D}$ form a Boolean algebra (see [14]). Now we suppose that $\underline{D}$ is not a finite nor a complete dBa; knowing from Theorem 4.16 that $Skel\mathcal{F}(\underline{D})$ and $Skel(\mathcal{I}(\underline{D}))$ are Boolean algebras, our aim here is to give an explicit characterization of the elements of these Boolean algebras.
First, for a principal filter $F(a)$ and principal ideal $I(a)$ we need to characterize $F(a)^*$ and $I(a)^\star$. Recall that $\mathcal{F}_p(\underline{D})$ (resp. $\mathcal{I}_p(\underline{D})$) is the set of principal filter(resp.ideals) of $\underline{D}$. It is known that $F(a \sqcap a) = F(a)$ and $I(a \sqcup a) = I(a)$ for any $a \in D$.

Lemma 4.17. *Let $\underline{D}$ be a dBa and $a \in D$, then:*

(1) (i) $I(a)^{\star} = I(\lrcorner a)$, (ii) $F(a)^{*} = F(\neg a)$, (iii) $F(a)^{**} = F(a)$, (iv) $I(a)^{\star\star} = I(a)$.

(2) (i) $I(a)^{\star} \cap I(b)^{\star} = (I(a) \vee I(b))^{\star} = I(a \sqcup b)^{\star}$, (ii) $(I(a) \cap I(b))^{\star} = I(a)^{\star} \vee I(b)^{\star} = I(a \wedge b)^{\star}$.

(3) (i) $F(a)^{*} \cap F(b)^{*} = (F(a) \vee F(b))^{*}$, (ii) $(F(a) \cap F(b))^{*} = F(a)^{*} \vee F(b)^{*}$.

(4) $\mathcal{I}_p(\underline{D}) \subseteq Skel\mathcal{I}(\underline{D})$.

(5) $\mathcal{F}_p(\underline{D}) \subseteq Skel\mathcal{F}(\underline{D})$.

Proof. (1) it is enough to show that for any $a \in D$, $I(a)^{\star} = I(\lrcorner a)$ and $F(a)^{*} = F(\neg a)$.
From (6) of Proposition 2.8 we get $I(a) \cap I(\lrcorner a) = I(a \wedge \lrcorner a) = I(\bot)$. So $I(\lrcorner a) \subseteq I(a)^{\star}$. Let J be an ideal of $\underline{D}$ such that $I(a) \cap J = I(\bot)$. We show that $J \subseteq I(\lrcorner a)$. Let $j \in J$, we need to show that $j \in I(\lrcorner a)$. We have by (vi) of Proposition 2.5 $j \wedge a = (j \sqcup j) \wedge (a \sqcup a) \sqsubseteq j \sqcup j, a \sqcup a$; so $j \wedge a \in I(a) \cap J = I(\bot)$, therefore, $j \wedge a \sqsubseteq \bot \sqcup \bot$ and $j \wedge a = \bot \sqcup \bot$. So $(j \sqcup j) \wedge (a \sqcup a) = \bot \sqcup \bot$; since $\bot \sqcup \bot$ is a zero of the Boolean algebra $\underline{D}_{\sqcup}$, we have that $\lrcorner(a \sqcup a)$ is the complement of $a \sqcup a$ and using the fact that $\lrcorner(a \sqcup a)$ is the greatest element with the property $(a \sqcup a) \wedge x = \bot \sqcup \bot$, we conclude that $j \sqcup j \sqsubseteq \lrcorner(a \sqcup a)$ and $j \sqsubseteq \lrcorner a$ (due to $\lrcorner(a \sqcup a) = \lrcorner a$ by axiom (4b)). Hence $j \in I(\lrcorner a)$ and $I(a)^{*} = I(\lrcorner a)$. Thus (1)(i) holds. (1)(ii) is proved dually. For (1)(iv) we get $I(a)^{**} = (I(a)^{\star})^{*} = I(\lrcorner\lrcorner a) = I(a \sqcup a) = I(a)$.
The proof of (1)(iii) is similar to that of (1)(iv).

For (2) and (3), using (1),(2),(5) and (6) of Proposition 2.8 together with (1), we obtain the results.

The proof of (4) and (5) are the consequences of (1). □

Proposition 4.18. *Let $\underline{D}$ be a dBa, then the following statements hold:*
(1) $\mathcal{I}_p(\underline{D})$ is a Boolean algebra which is a subalgebra of $Skel(I(\underline{D}))$.
(2) $\mathcal{F}_p(\underline{D})$ is a Boolean algebra which is a subalgebra of $Skel(F(\underline{D}))$.

Proof. It follows from the Lemma 4.17. □

5 Annihilators and co-annihilators on double Boolean algebras.

The concept of annihilator was introduced for lattices by M. Mandelker in [13] as a generalization of the concept of pseudocomplement and latter extended to the class of distributive lattices by Cornish in [4]. In this section, the notions of annihilators and co-annihilators are introduced in double Boolean algebras. Some properties of annihilators and co-annihilators are studied.

Definition 5.1. *For any subset S of a dBa $\underline{D}$, define S^{+} and S^{-} as follow:*

$$S^{+} = \{x \in D : s \vee x = \top \sqcap \top \textit{ for all } s \in S\}, \quad S^{-} = \{x \in D : x \wedge s = \bot \sqcup \bot, \textit{ for all } s \in D\}.$$

Here S^{+} is called the **co-annihilator** of S and S^{-} the **annihilator** of S. For $S = \{x\}$ we denote simply $(x)^{+}$ for $(\{x\})^{+}$ and $(x)^{-}$ for $(\{x\})^{-}$. Clearly:

$$D^{+} = F(\top),\ F(\top)^{+} = D,\ D^{-} = I(\bot),\ \text{ and } I(\bot)^{-} = D.$$

Lemma 5.2. *Let $\underline{D}$ be a dBa. For any non-empty subset S of D the following statements hold:*

1. *S^{+} (resp. S^{-}) is a filter (resp. an ideal) of $\underline{D}$.*

2. *(i) $S^{+} = (Filter(S))^{+}$; (ii) $S^{-} = (Ideal(S))^{-}$.*

3. (i) $Filter(S) \cap S^+ = F(\top)$; (ii) $Ideal(S) \cap S^- = I(\bot)$.

Proof. (1) Let $x, y \in S^+$, then for any $s \in S, x \vee s = y \vee s = \top \sqcap \top$. In addition, using distributivity of $\vee$ and $\sqcap$ we get that for any $s \in S, s \vee (x \sqcap y) = (x \vee s) \sqcap (y \vee s) = \top \sqcap \top$, so $x \sqcap y \in S^+$. Assume that $x \sqsubseteq y$ and $x \in S^+$, let $t \in S$, then $t \vee x = \top \sqcap \top$, as $x \vee t \sqsubseteq y \vee t \sqsubseteq \top \sqcap \top$, we get $y \vee t = \top \sqcap \top$. Hence $y \in S^+$. Thus S^+ is a filter of $\underline{D}$. A similar argument show that S^- is an ideal of $\underline{D}$ and (1) holds.

(2) We will show that $S^+ = (Filter(S))^+$. Let $x \in S^+$, then for all $s \in S, s \vee x = \top \sqcap \top$. We will show that $x \in (Filter(S))^+$. Let $t \in Filter(S)$, then by (d) of Proposition 2.7 there exist $u_1, \dots, u_n \in S$ such that $u_1 \sqcap \dots \sqcap u_n \sqsubseteq t$. Furthermore, using (i) of Proposition 2.5 we get $x \vee (u_1 \sqcap \dots \sqcap u_n) = (u_1 \vee x) \sqcap \dots \sqcap (u_n \vee x) = \top \sqcap \top \sqsubseteq t \vee x \sqsubseteq \top \sqcap \top$ (due to $u_i \vee x = \top \sqcap \top$); so $x \vee t = \top \sqcap \top$ and $x \in (Filter(S)^+$, therefore $S^+ \subseteq (Filter(S))^+$. Conversely, let $x \in Filter(S)^+$, then for any $s \in Filter(S)$, $x \vee s = \top \sqcap \top$, as $S \subseteq Filter(S)$ we deduce that $x \in S^+$; therefore $Filter(S))^+ \subseteq S^+$. Thus $S^+ = Filter(S)^+$. A similar arguments show that (2)(ii) holds.

(3) Let $x \in Filter(S) \cap S^+$, then there exist $u_1, \dots, u_n \in S$ such that $u_1 \sqcap \dots \sqcap u_n \sqsubseteq x$ and $x \in S^+$; hence $u_i \vee x = \top \sqcap \top, i = 1, \dots, n$. Hence $x \vee x = \top \sqcap \top$ and $x \in F(\top \sqcap \top)$. Thus $Filter(S) \cap S^+ = F(\top)$ and (3)(i) holds. A similar argument shows that (3)(ii) holds. □

Proposition 5.3. *For any $S, T \subseteq D$.*

(1) (i) $S \subseteq T \implies T^+ \subseteq S^+$. (ii) $S \subseteq T \implies T^- \subseteq S^-$.

(2) (i) $S \subseteq S^{++}$. (ii) $S \subseteq S^{--}$.

(3) (i) $S^+ = S^{+++}$, (ii) $S^{---} = S^-$.

Proof. (1) It is easy to verify.

(2)(i) Let $x \in S$, we show that $x \in S^{++}$. Let $s \in S^+$, then for any $t \in S$, $t \vee s = \top \sqcap \top$, since $x \in S$ we have $x \vee s = \top \sqcap \top$; therefore $x \in S^{++}$ and $S \subseteq S^{++}$. Dually we have the proof of (2)(ii).

(3)(i) By (2)(i) we have $S \subseteq S^{++}$, and using (1)(i) we get $S^{+++} \subseteq S^+$. Let $x \in S^+$, we show that $x \in S^{+++}$; that is for any $t \in S^{++}$, $s \vee x = \top \sqcap \top$. Let $t \in S^{++}$, then for any $z \in S^+$, $x \vee t = \top \sqcap \top$, as $x \in S^+$ we deduce that $x \vee t = \top \sqcap \top$, so $x \in S^{+++}$. Hence $S^+ = S^{+++}$. Dually we have the proof of (3)(ii). □

Definition 5.4. [6, Definition 14, p.8] *A* ***closure operator*** *ϕ on D is a map assigning a closure $\phi(X) \subseteq D$ to each subset $X \subseteq D$ under the following conditions:*

(1) $X \subseteq Y \implies \phi(X) \subseteq \phi(Y)$, *(monotony)*

(2) $X \subseteq \phi(X)$, *(extensity)*

(3) $\phi \circ \phi(X) = \phi(X)$. *(idempotency).*

From Proposition 5.3 we get the following corollary.

Corollary 5.5. *If $\underline{D}$ is a dBa, then $^{++}$ and $^{--}$ are closure operators on D.*

Proof. Using (1)(i), (2)(i) and (3)(i) of Proposition 5.3, it is easy to check that $^{++}$ is a closure operator. A similar argument shows that $^{--}$ is a closure operator. □

In case of filters or ideals on dBa, we have more results.

Proposition 5.6. *Let $\underline{D}$ be a dBa, let $F, G \in F(\underline{D})$ and $I, J \in I(\underline{D})$, then the following statements hold:*

1. (i) $(F \vee G)^+ = F^+ \cap G^+$, (ii) $(I \vee J)^- = I^- \cap J^-$.
2. (i) $F^+ = D$ *iff* $F = F(\top)$. (ii) $I^- = D$ *iff* $I = I(\bot)$.
3. (i) $F \cap G = F(\top)$ *iff* $F \subseteq G^+$. (ii) $I \cap J = I(\bot)$ *iff* $I \subseteq J^-$.

Proof. (1) (i) Since $F, G \subseteq F \vee G$, we get by (2) of Lemma 5.2 that $(F \vee G)^+ \subseteq F^+ \cap G^+$. Let $x \in F^+ \cap G^+$. We will show that $x \in (F \vee G)^+$. Let $t \in F \vee G$, then there exist $u \in F, v \in G$ such that $u \sqcap v \sqsubseteq t$; as $x \in F^+ \cap G^+$, we get $u \vee x = \top \sqcap \top$ and $x \vee v = \top \sqcap \top$. In addition, using distributivity of $\vee$ over $\sqcap$ we get $x \vee (u \sqcap v) = (x \vee u) \sqcap (x \vee v) = \top \sqcap \top \sqsubseteq x \vee t \sqsubseteq \top \sqcap \top$; so $x \in (F \vee G)^+$. Thus $F^+ \cap G^+ = (F \vee G)^+$. A similar argument shows that (1)(ii) holds.

(2) (i) Assume that $F^+ = D$. We show that $F = F(\top)$. Let $x \in F$, then $x \in D = F^+$, so $x \vee x = x \sqcap x = \top \sqcap \top$, hence $x \in F(\top)$. Therefore $F = F(\top)$. Conversely, assume that $F = F(\top)$. Then $F^+ = F(\top)^+ = D$. A similar argument shows that (ii) holds.

(3) Assume that $F \cap G = F(\top)$. Let $x \in F$, we show that $x \in G^+$. Let $y \in G$, then $x \vee y \in F(x \vee y) = F(x) \cap F(y) \subseteq F \cap G = F(\top)$; so $x \vee y = \top \sqcap \top$, hence $F \subseteq G^+$. Conversely, assume that $F \subseteq G^+$. Let $x \in F \cap G$, then $x \vee x = \top \sqcap \top$, hence $x \in F(\top)$. Thus $F \cap G = F(\top)$. □

Corollary 5.7. *For any $x, y \in D$, the following conditions hold.*

1. (i) $x \sqsubseteq y \implies (x)^+ \subseteq (y)^+$, (ii) $x \sqsubseteq y \implies (y)^- \subseteq (x)^-$.
2. (i) $(x \sqcap y)^+ = (x)^+ \cap (y)^+$, (ii) $(x \sqcup y)^- = (x)^- \cap (y)^-$.
3. (i) $(x)^+ = D$ *iff* $x \in F(\top)$, (ii) $(x)^- = D$ *iff* $x \in I(\bot)$.

Proof. We give the proof of (j)(i), $j = 1, 2, 3$ and the proof of (j)(ii) is obtained dually.

(1) (i) Assume that $x \sqsubseteq y$, we show that $(x)^+ \subseteq (y)^+$. Let $t \in (x)^+$, then $t \vee x = \top \sqcap \top$. We show that $t \in (y)^+$. Since $x \sqsubseteq y$, we get $t \vee x \sqsubseteq y \vee t \sqsubseteq \top \sqcap \top$, hence $y \vee t = \top \sqcap \top$ and $t \in (y)^+$.

(2) (i) From (1), as $x \sqcap y \sqsubseteq x, y$, we get $(x \sqcap y)^+ \subseteq (x)^+ \cap (y)^+$. Let $t \in (x)^+ \cap (y)^+$, then $t \vee x = t \vee y = \top \sqcap \top$. As $x \sqcap y \sqsubseteq x \vee y$, we get $t \vee (x \sqcap y) \sqsubseteq (x \vee y) \vee t \sqsubseteq \top \sqcap \top$, in addition, $t \vee (x \sqcap y) = (t \vee x) \sqcap (t \vee y) = \top \sqcap \top$; so $t \vee (x \sqcap y) = \top \sqcap \top$, therefore $t \in (x \sqcap y)^+$ and $(x)^+ \cap (y)^+ \subseteq (x \sqcap y)^+$. Thus (2) (i) holds.
(3) (i) Assume that $(x)^+ = D$, then $x \vee x = \top \sqcap \top$ and $x \in F(\top)$. Conversely, assume that $x \in F(\top)$, then $\top \sqcap \top \sqsubseteq x$ and for any $t \in D, \top \sqcap \top \sqsubseteq t \vee x \sqsubseteq \top \sqcap \top$, so $D = (x)^+$. □

Definition 5.8. *A filter F of $\underline{D}$ is called a* **direct factor filter** *of D if there exists a proper filter G such that $F \cap G = F(\top)$ and $F \vee G = D$. Dually is defined a* **direct factor ideal** *of $\underline{D}$.*

Theorem 5.9. (1) *Each $(x)^+$, $x \in D$, is a direct factor filter of D if and only if $(x)^+ \vee (x)^{++} = D$.*
(2) *Each $(x)^-, x \in D$, is a direct factor ideal of D if and only if $(x)^- \vee (x)^{--} = D$.*

Proof. (1) Assume that $(x)^+, x \in D$, is a direct factor of D. Then there exists a proper filter G such that $(x)^+ \cap G = F(\top)$ and $(x)^+ \vee G = D$. Since $(x)^+ \cap G = F(\top)$, we get by (3)(i) of Proposition 5.6, $G \subseteq (x)^{++}$. Hence $D = (x)^+ \vee G \subseteq (x)^+ \vee (x)^{++} \subseteq D$. Conversely, assume that the condition holds, that is $x \in D$ and, $(x)^{++} \vee (x)^+ = D$. We show that $(x)^+$ is a direct factor of D. We have always $(x)^+ \cap (x)^{++} = F(\top)$ by (2)(i) and (3)(i) of Lemma 5.2. Furthermore we have $(x)^+ \vee (x)^{++} = D$. Therefore $(x)^+$ is a direct factor of D. The proof of (2) is obtained dually. □

The concept of annihilator ideal and co-annihilator filter are now introduced in the following.

Definition 5.10. *Let $\underline{D}$ be a dBa. A filter F of $\underline{D}$ is called a* ***co-annihilator filter*** *if $F = F^{++}$ or equivalently $F = S^{+}$ for some $S \subseteq D, S \neq \emptyset$. Dually is defined the* ***annihilator ideal*** *of $\underline{D}$.*

Lemma 5.11. *Let $\underline{D}$ be a dBa, F a filter and I an ideal of $\underline{D}$, then*

(i) $F^{*} = \{x \in D : F(x) \cap F(a) = F(\top), \forall a \in F\}$ *and F^{*} is a pseudocomplement of F.*

(ii) $I^{\star} = \{x \in D : I(x) \cap I(a) = I(\bot \sqcup \bot), \forall a \in I\}$ *and $I^{\star}$ is a pseudocomplement of I.*

Proof. For (i), set $L := \{x \in D : F(x) \cap F(a) = F(\top), \forall a \in F\}$. By (2)(i) of Lemma 4.9, F^{*} is a pseudo-complement of F and F^{*} is a filter. It remains to show that $F^{*} = L$. Let $x \in L$, then for any $a \in F, F(x) \cap F(a) = F(\top)$. We show that $x \in F^{*}$. Let $j \in F$, then by assumption, $F(x) \cap F(j) = F(\top)$, using (5) of Proposition 2.8 we have $F(x) \cap F(j) = F(x \vee j)$, so $F(x \vee j) = F(\top)$ and $x \vee j \in F(\top)$; therefore $\top \sqcap \top \sqsubseteq x \vee j \sqsubseteq \top \sqcap \top$. So $x \vee j = \top \sqcap \top$ and $x \in F^{*}$. Hence $L \subseteq F^{*}$. Now we show that $F^{*} \subseteq L$. Let $x \in F^{*}$ and $a \in F$, then $a \vee x = \top \sqcap \top$. By (5) of Proposition 2.8, we have $F(a) \cap F(x) = F(a \vee x) = F(\top \sqcap \top)$; so $x \in L$ and $F^{*} \subseteq L$. Thus $F^{*} = L$ and (i) holds. A similar argument show that (ii) holds. $\square$

Theorem 5.12. *Let $\underline{D}$ be a dBa. The following statements hold:*
(1) *$\mathcal{CA}(\underline{D})$ is a Boolean algebra.* (2) *$\mathcal{A}(\underline{D})$ is a Boolean algebra.*

Proof. One can show that $F^{+} = F^{*}$ and $I^{-} = I^{\star}$ for any $F \in F(\underline{D})$ and any $I \in I(\underline{D})$ and conclude with Theorem 4.16 $\square$

6 Conclusion

In order to give a complete description of the lattice $F(\underline{D})$ (resp. $I(\underline{D})$) of filters (resp. ideals) of an arbitrary dBa $\underline{D}$. In this paper, first we have studied some particular filters of an arbitrary dBA, namely, primary filters(resp. ideals). We have introduced dense and co-dense elements on dBas and show that dense (resp. co-dense) elements form a filter (resp. an ideal). We have also characterized trivial dBas using dense set and co-dense set. We have studied some properties of the lattice $F(\underline{D})$ (resp. $I(\underline{D})$ of filters (resp. ideals) of an arbitrary dBa $\underline{D}$, and we have obtained that these lattices are endowed with the structures of distributive, algebraic, pseudocomplemented lattices, Brouwerian algebras, Heyting algebras, Gödel algebras and possess also the structure of residuated lattice. We end this paper by introducing the notions of annihilators and co-annihilators on dBas and study related properties. We have shown that co - annihilators(resp. annihilators) of $F(\underline{D})$ (resp. $I(\underline{D})$) are exactly pseudocomplements and form a Boolean algebras.

Our future work is to study the relationship between filters(ideals) and congruences on arbitrary dBa and to characterize the sub-directly irreducible dBas and the indecomposable dBas.

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