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Modal representation of coalgebras over local BL-algebras

C. Nganteu[1], M. Kianpi[2] and A. Ogadoa[3]

[1,2,3] *University of Yaounde 1, Faculty of Science, Department of Mathematics, Laboratory of Algebra, P.O. Box 812, Yaounde, Republic of Cameroon*

nganteu2001@yahoo.fr, mkianpi@gmail.com, cogadoaamassayoga@yahoo.fr

Abstract

We consider the category $Coalg(\prod)$ of $\prod$-coalgebras where $\prod$ is the endofunctor on the category of local BL-algebras and BL-morphisms which assigns to each local BL-algebra its quotient by its unique maximal filter and we characterize homomorphisms and subcoalgebras in $Coalg(\prod)$. Moreover, we introduce local BL-frames based on local BL-algebras, and show that the category of local BL-frames is isomorphic to $Coalg(\prod)$.

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1 Introduction

Coalgebras were introduced by Aczel and Mendler [1] to model various type of transition systems. Up to now, coalgebras were studied over the category of sets and mappings (see for example [5, 11]), arbitrary categories (see for example [2, 7, 8]) or categories of topological spaces (see for example [9]), but not specially on algebraic structures. It has been shown that Kripke frames can be seen as coalgebras of the covariant powerset functor [16] and descriptive frames as coalgebras of the Vietoris functor, the topological analogue of the powerset functor, on Stone spaces [9] . These results provide a strong link between coalgebras and modal logic. The aim of this paper is to further investigate this connection for coalgebras over BL-algebras.

Coalgebras over the category $\mathcal{BL}$ of BL-algebras and BL-morphisms were introduced in [10] by the authors. They show that coalgebras of the MV-functor, which assigns each BL-algebra to its MV-center have very nice properties. In this short paper, we establish the link between modal logic and coalgebras over BL-algebras via a new type of logical frame, namely local BL-frame.

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The outline of the paper is as follows: In Section 2, we recollect some definitions and results which will be used throughout the paper. In Section 3, we state some facts about the category of local BL-algebras and introduce coalgebras of the functor $\prod$, which assigns each local BL-algebra to its quotient by its maximal filter. We characterize homomorphisms and subcoalgebras of $\prod$-coalgebras and show that the corresponding category is not complete. In the last part of the paper, we present local BL-frames and models and show that the categories of local BL-frames and $\prod$-coalgebras are isomorphic.

2 Preliminaries

BL-algebras were invented by P. Hájek [6] in order to provide an algebraic proof of the completeness theorem of basic logic (BL, for short) arising from the continuous triangular norms, familiar in the fuzzy logic framework. The language of propositional Hájek basic logic contains the binary connectives $\circ$ and $\Rightarrow$ and the constant $\bar{0}$. Axioms of BL are:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$

(A2) $(\varphi \circ \psi) \Rightarrow \varphi$

(A3) $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$

(A4) $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$

(A5a) $(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$

(A5b) $((\varphi \circ \psi) \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$

(A6) $((\varphi \Rightarrow \psi) \Rightarrow \omega) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega))$

(A7) $\bar{0} \Rightarrow \omega$.

We recall some definitions and basic results that can be found in [3, 6, 12, 16].

An algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *bounded commutative residuated lattice* if it satisfies the following conditions:

(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(BL2) $(L, *, 1)$ is a commutative monoid;

(BL3) $*$ is a left adjoint of $\rightarrow$, that is $x * z \leq y$ if and only if $z \leq x \rightarrow y$.

A BL-*algebra* is a bounded commutative residuated lattice which satisfies the following:

(BL4) $x \wedge y = x * (x \rightarrow y)$ (divisibilty);

(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

A BL-algebra L is called a *Gödel algebra* if $x^2 = x * x = x$ for every $x \in L$. In addition, L is called an *MV-algebra* if $\bar{\bar{x}} = x$ for all $x \in L$, where $\bar{x} = x \rightarrow 0$.

The following holds in any BL-algebra L:

Lemma 2.1. [13] *For all $x, y, z \in L$*

(1) $x \leq y$ *if and only if* $x \to y = 1$;

(2) $x * y \leq x \wedge y$;

(3) $x \to (y \to z) = y \to (x \to z)$;

(4) *If* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$;

(5) $x \leq y \to (x * y)$; $x * (x \to y) \leq y$;

(6) $x * \bar{x} = 0$;

(7) $(x * y) \to z = x \to (y \to z)$;

(8) $1 \to x = x$; $x \to 1 = 1$; $x \to x = 1$; $x \leq y \to x$; $x \leq \overline{\overline{x}}$; $\overline{\overline{\overline{x}}} = \overline{x}$.

A *filter* of L is a non-empty subset F of L such that for all $x, y \in L$,

(F1) $x, y \in F$ implies $x * y \in F$;

(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A subset D of a BL-algebra L is called a *deductive system* if

(DS1) $1 \in D$;

(DS2) $x \in D$ and $x \to y \in D$ imply $y \in D$.

Deductive systems have been widely studied in BL-algebras namely to characterize fragments of Basic fuzzy logic (see [15]); it is obvious that for a non-empty subset F of L, F is a deductive system if and only if it is a filter.

Let L_1 and L_2 be two BL-algebras, a map $f : L_1 \longrightarrow L_2$ is called a *homomorphism of BL-algebras (BL-morphism)*, if $f(0) = 0$ and $f(x \propto y) = f(x) \propto f(y)$ for all $\propto \in \{*, \to\}$. We obviously have $f(1) = 1$ for any BL-homomorphism f and it is shown in [13] that for any BL-morphism f, $f(x \propto y) = f(x) \propto f(y)$ with $\propto \in \{\vee, \wedge\}$ and if $x \leq y$, then $f(x) \leq f(y)$.

For any deductive system F of a BL-algebra $L = (L, \wedge, \vee, *, \to, 0, 1)$, we can define a relation θ_F on L as follows: for all $x, y \in L$,

$$(x \theta_F y) \Longleftrightarrow ((x \to y) \wedge (y \to x) \in F).$$

It is well known that θ_F is a congruence on L (see, e.g. [6]) and since the class of BL-algebras is a variety, the quotient structure L/θ_F is also a BL-algebra for which for all $x, y \in L$, $[x \propto y] := [x] \propto [y]$ where $\propto \in \{\wedge, \vee, *, \to\}$, and $[x] := [x]_{\theta_F}$. A congruence θ on L is called *induced* by F if $[1]_\theta = F$. In addition, θ_F is clearly induced by F.

The class of BL-algebras equipped with BL-morphisms form a category. We will denote it by $\mathcal{BL}$. The one-element BL-algebra $\{0 = 1\}$ is called the *degenerate* BL-algebra (see [12], Remark 8), we will denote it by $\mathbf{G}_1$. The two-element non-degenerate BL-algebra $\{0, 1\}$ is called the *trivial BL-algebra*, we will denote it by $\mathbf{G}_2$. These two algebras are examples of BL-algebras which are both Gödel-algebras and MV-algebras.

Proposition 2.2. [10] *There are only two non-degenerate BL-algebras with three elements:*

(i) The chain $\{0, x, 1\}$*, with the operations* $*$ *and* $\to$ *defined by the following tables:*

$*$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

It is the unique Gödel-algebra with three elements and we will denote it by $\boldsymbol{G}_3$*.*

(ii) The chain $\{0, x, 1\}$*, with the operations* $*$ *and* $\to$ *defined by the following tables:*

$*$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

It is the unique MV-algebra with three elements and we will denote it by $\boldsymbol{M}_3$*.*

Remark 2.3. For any set X, define for $A \subseteq X$ and $B \subseteq X$, $A * B = A \cap B$ and $A \to B = A^C \cup B$. Then the structure $(P(X), \cap, \cup, *, \to, \emptyset, X)$ where $P(X)$ is the powerset of X is a BL-algebra called the *powerBL-algebra* of X.

A *Kripke frame* is a pair (X, R) where X is a set and R is a binary relation on X. For $x \in X$, let $[x]_R = \{y \in X \mid xRy\}$ be the R-image of x. A *p-morphism* between two Kripke frames (X, R) and (Y, R') is a function $f : X \longrightarrow Y$ satisfying $f([x]_R) = [f(x)]_{R'}$ for each $x \in X$. Kripke frames and p-morphisms form a category denoted by $\mathcal{KF}r$.

A *Kripke model* is a tuple (W, R, ν), where (W, R) is a Kripke frame and $\nu : Prop \longrightarrow P(L)$ sends proposition letters to the set of states where they are true. A *modal algebra* is a structure $(L, \wedge, \vee, \neg, 0, 1, \Box)$ such that $(L, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and $\Box$ preserves 1 and $\wedge$.

Definition 2.4. *Let* $\mathcal{C}$ *be a category.*

(1) *A full subcategory* $\mathcal{D}$ *of* $\mathcal{C}$ *is called* isomorphism-closed *provided that every* $\mathcal{C}$*-object that is isomorphic to some* $\mathcal{D}$*-objects is itself a* $\mathcal{D}$*-object.*

(2) *A* coalgebra *for an endofunctor* $F : \mathcal{C} \longrightarrow \mathcal{C}$ *is a pair* (A, α) *where* A *is an object of* $\mathcal{C}$ *and* $\alpha : A \longrightarrow F(A)$ *is a* $\mathcal{C}$*-morphism.*

(3) *A* homomorphism *between two coalgebras* (A, α) *and* (B, β) *for* F *is a* $\mathcal{C}$*-morphism* $f : A \longrightarrow B$ *such that* $\beta \circ f = F(f) \circ \alpha$*.*

(4) *Coalgebras for* F *and their homomorphisms form a category denoted by* $\mathcal{C}oalg(F)$*.*

3 $\prod$-coalgebras

In this section, we present some properties of local BL-algebras which are BL-algebras with a unique maximal filter. We define a non-trivial endofunctor of the category of local BL-algebras and investigate the corresponding coalgebras.

Definition 3.1. Let L be a BL-algebra.

(1) A deductive system F of L is *proper* if $0 \notin F$.

(2) A deductive system M of L is called *maximal* if it is proper and not contained in any other proper deductive system.

(3) L is *local* if it has a unique maximal deductive system.

Theorem 3.2. [13] *Let L be a BL-algebra. Define*

$$D(L) = \{x \in L \mid x^n \neq 0 \text{ for all integers } n\}.$$

The following are equivalent:

(i) $D(L)$ is a deductive system of L;

(ii) L is local;

(iii) $D(L)$ is the unique maximal deductive system of L.

Example 3.3. (i) $D(\mathbf{G}_3) = \{x, 1\}$, $D(\mathbf{M}_3) = D(\mathbf{G}_2) = \{1\}$ are deductive systems. So by the above theorem, $\mathbf{G}_3$, $\mathbf{M}_3$ and $\mathbf{G}_2$ are local BL-algebras;

(ii) Consider $A = ([0;1], \wedge, \vee, *, \rightarrow, 0, 1)$ the BL-algebra such that for all $x, y \in L$, $x * y = x \cdot y$ and $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = \frac{y}{x}$ else. Then $D(A) =]0;1]$ is a deductive system of A. Thus A is a local BL-algebra.

(iii) [[15], Proposition 11] Any BL-algebra such that $MV(L) = \{0, 1\}$ is local.

(iv) [[15], Example 1] The chain $\{0, x, y, 1\}$, with the operations $*$ and $\rightarrow$ defined by the following tables

*	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

→	0	x	y	1
0	1	1	1	1
x	x	1	1	1
y	0	x	1	1
1	0	x	y	1

is a local BL-algebra such that $MV(L) = \{0, x, 1\}$.

(v) $\mathbf{G}_1$ is not local.

Proposition 3.4 ([4], Proposition 1.10)**.** *Let $f : L \longrightarrow L'$ be a BL-morphism. If M' is a maximal deductive system of L', then $f^{-1}(M')$ is a maximal deductive system of L.*

Lemma 3.5 ([4], Lemma 1.9)**.** *Let L be a nontrivial BL-algebra and M a proper deductive system of L. The following are equivalent:*

(i) M is maximal;

(ii) for any $x \in L$, $x \notin M \Leftrightarrow \overline{(x^n)} \in M$ for some integer n.

Lemma 3.6. *Let f be a BL-morphism between two local BL-algebras L and L' whose maximal deductive systems are M and M', respectively. If f is surjective, then $f(M) = M'$.*

Lemma 3.7. *Let L be a BL-algebra and F be a deductive system of L. Then θ_F is the unique congruence on L induced by F.*

Proof. Let θ be a congruence on L induced by F. We have to show that $\theta_F = \theta$. Let $(x, y) \in \theta_F$. Then $x \to y \in [1]_\theta$ and $y \to x \in [1]_\theta$. So by compatibility,

$$(x * (x \to y), x * 1) \in \theta \text{ and } (y * (y \to x), y * 1) \in \theta.$$

Hence by BL4 we obtain $(x \wedge y, x) \in \theta$ and $(y \wedge x, y) \in \theta$. Since θ is symmetric and $\wedge$ is commutative, it follows that $(x, x \wedge y) \in \theta$ and $(x \wedge y, y) \in \theta$. By transitivity, we have $(x, y) \in \theta$. Conversely, let $(x, y) \in \theta$. Then $(x \to y, y \to y) \in \theta$ and $(y \to x, y \to y) \in \theta$. So $(x \to y, 1) \in \theta$ and $(y \to x, 1) \in \theta$. It follows that $x \to y \in F$ and $y \to x \in F$ and therefore, $(x, y) \in \theta_F$. □

In the sequel we will denote L/θ_F by L/F and $[x]_{\theta_F}$ by $[x]_F$.

Let M be the maximal deductive system of a local BL-algebra L. Then by ([4], Proposition 1.13), since M is the unique maximal deductive system which contains M, L/M is a local BL-algebra. Therefore, we have:

Lemma 3.8. *Let M be the maximal deductive system of a local BL-algebra L. Then L/M is a local BL-algebra and $D(L/M) = \{M\}$.*

Proof. We have $M^n = [1]_M^n = [1]_M \neq [0]_M$, which means that $M \in D(L/M)$. Let $[x]_M \in D(L/M)$. Then $[x^n]_M = [x]_M^n \neq [0]_M$, for all integer n. It follows that $x^n \to 0 \notin M$, for all integer n. Thus by Lemma 3.5, $x \in M$; That is, $[x]_M = M$. □

Local BL-algebras and BL-morphisms form a category which will be denoted by $l\mathcal{BL}$.

Proposition 3.9. *$l\mathcal{BL}$ is an isomorphism-closed subcategory of $\mathcal{BL}$.*

Proof. Let $f : L \longrightarrow G$ be an isomorphism between a BL-algebra L and a local BL-algebra G, whose inverse is g. Then by Proposition 3.4, $f^{-1}(M')$ is a maximal filter of L, where M' is the unique maximal filter of G. Moreover, let H be another maximal filter of L. Then $g^{-1}(H) = M'$ and so $H = g(M') = f^{-1}(M')$. Thus L is a local BL-algebra. □

Remark 3.10. Let L and L' be two local BL-algebras, M and M' their respectives maximal filters. Then $M \times L'$ and $L \times M'$ are maximal filters of $L \times L'$. Thus, $L \times L'$ is not a local BL-algebra. It follows that $l\mathcal{BL}$ has no (co)products and therefore $l\mathcal{BL}$ is not complete, nor cocomplete.

Proposition 3.11. *Consider the correspondence $\prod : l\mathcal{BL} \longrightarrow l\mathcal{BL}$ such that $\prod(L) = L/M$ for any local BL-algebra L whose unique maximal filter is M and $\prod(f) : L/M \longrightarrow L/M'$ such that*

$$\prod(f)([x]_M) = [f(x)]_{M'}.$$

Then $\prod$ is a covariant endofunctor on $l\mathcal{BL}$.

Proof. By Lemma 3.7 and the fact that θ_M is a congruence, $\prod(L)$ is well defined. Moreover, let $L \xrightarrow{f} L'$ and $L' \xrightarrow{g} L''$ be two BL-morphisms. Let $x \in L$. We have

$$\prod(g) \circ \prod(f)([x]_M) = \prod(g)([f(x)]_{M'}) = [g \circ f(x)]_{M'} = \prod(g \circ f)([x]_M)$$

and also

$$\prod(id_L)([x]_M) = [x]_M = id_{\prod(L)}([x]_M).$$

□

Let $Coalg(\prod)$ be the category of $\prod$-coalgebras and $\prod$-homomorphisms. Let (L,α) be a $\prod$-coalgebra. For any x, y in a BL-algebra L, we denote $x \overset{\alpha}{\to} y$ by $\alpha(x) = [y]_M$. Then one can observe that $\prod$-coalgebras mimic non-deterministic transition systems.

Let (L,α) and (L',α') be two $\prod$-coalgebras. A BL-morphism $f : L \longrightarrow L'$ *weakly reflects* transition systems if for all $x \in L$ and $y \in L'$, $f(x) \overset{\alpha'}{\to} y$ implies $x \overset{\alpha}{\to} t$, with $f(t) \in [y]_M$, $t \in L$.

Proposition 3.12. *Let (L,α) and (L',α') be two $\prod$-coalgebras, and $f : L \longrightarrow L'$ a BL-morphism. The following are equivalent:*

(i) f is a $\prod$-homomorphism;

(ii) for all $x \in L$, $\alpha'(f(x)) = [f(z)]_{M'}$, whenever $\alpha(x) = [z]_M$;

(iii) f preserves and weakly reflects transitions.

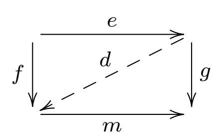
Proof. $(i) \Leftrightarrow (ii)$ Straightforward.

$(ii) \Rightarrow (iii)$ Suppose for all $x \in L$, $\alpha'(f(x)) = [f(z)]_{M'}$, whenever $\alpha(x) = [z]_M$. Let $x, y \in L$ such that $x \overset{\alpha}{\to} y$. Then $\alpha(x) = [y]_M$. So by hypothesis, $\alpha'(f(x)) = [f(y)]_{M'}$ implying $f(x) \overset{\alpha'}{\to} f(y)$. So f preserves transitions. Moreover, let $x \in L$ and $y \in L'$ such that $f(x) \overset{\alpha'}{\to} y$. Then $\alpha'(f(x)) = [y]_{M'}$. Let $z \in L$ such that $\alpha(x) = [z]_M$. Then $x \overset{\alpha}{\to} z$ and by hypothesis, $[f(z)]_{M'} = \alpha'(f(x)) = [y]_{M'}$, i.e., so $f(z) \in [y]_{M'}$. Thus, f weakly preserves transitions.

$(iii) \Rightarrow (ii)$ Let $x \in L$, such that $\alpha(x) = [z]_M$. Then $x \overset{\alpha}{\to} z$, which implies by hypothesis that $f(x) \overset{\alpha'}{\to} f(z)$, i.e. $\alpha'(f(x)) = [f(z)]_{M'}$.

□

Definition 3.13. *[3]* A monomorphism m is called *strong* in a category $\mathcal{C}$ if for every epimorphism e and every commutative square



there exists a diagonal d such that $g = m \circ d$ and $f = d \circ e$.

Proposition 3.14. *Let (L',α') be a $\prod$-coalgebra. A local BL-subalgebra L of L' is a $\prod$-subcoalgebra of (L',α') iff there exists a strong mono $L \overset{m}{\longrightarrow} L'$, verifying the following property: for all $x \in L$, there exists $z \in L$ such that $m(x) \overset{\alpha'}{\to} m(z)$.*

Proof. Suppose that (L,α) is a $\prod$-subcoalgebra of (L',α'), and m the corresponding strong mono. Let $x \in L$. Since m is a $\prod$-homomorphism, it follows from Proposition 3.12 that $\alpha' \circ m(x) = [m(z)]_{M'}$, where $\alpha(x) = [z]_M$. So $m(x) \overset{\alpha'}{\to} m(z)$, $z \in L$.

Conversely, assume that there is a strong mono $m : L \longrightarrow L'$ such that for all $x \in L$, there exists $z \in L$ such that $m(x) \overset{\alpha'}{\to} m(z)$. Define $\alpha : L \longrightarrow \prod(L)$ by $\alpha(x) = [z]_M$, where $m(x) \overset{\alpha'}{\to} m(z)$. Let $x, x' \in L$ such that $\alpha(x) = [z]_M$ and $\alpha(x') = [z']_M$. If $x = x'$, then $\alpha' \circ m(x) = \alpha' \circ m(x')$. So by Proposition 3.12 (ii), we obtain $[m(z)]_{M'} = [m(z')]_{M'}$. Hence

$$(m(z) \to m(z')) \wedge (m(z') \to m(z)) \in M'.$$

Thus

$$(z \to z') \wedge (z' \to z) \in m^{-1}(M').$$

It follows from Proposition 3.4 that $(z \to z') \wedge (z' \to z) \in M$. So $[z]_M = [z']_M$. Thus α is well defined. Moreover, since α' and m are BL-morphisms, we have

$$\alpha' \circ m(0) = \alpha'(0) = [1]_M = [m(1)]_M \,.$$

Hence $m(0) \stackrel{\alpha'}{\to} m(1)$, implying $\alpha(0) = [1]_M$. On another hand, let $x, y \in L$ such that $\alpha(x \propto y) = [t]_M$, $\alpha(x) = [u]_M$ and $\alpha(y) = [v]_M$ where $\propto \in \{*, \to\}$. Then we have $m(x \propto y) \stackrel{\alpha'}{\to} m(t)$, i.e. $\alpha' \circ m(x \propto y) = [m(t)]_{M'}$. Since $\alpha' \circ m$ is a BL-morphism, we have

$$\alpha' \circ m(x) \propto \alpha' \circ m(y) = [m(t)]_{M'} \,,$$

i.e.

$$[m(u)]_{M'} \propto [m(v)]_{M'} = [m(t)]_{M'} \,.$$

Thus $m([u]_M \propto [v]_M) = m([t]_M)$. Since m is a mono, $[u]_M \propto [v]_M = [t]_M$ and so $\alpha(x \propto y) = \alpha(x) \propto \alpha(y)$. Therefore, α is a BL-morphism. It follows that (L, α) is a $\prod$-subcoalgebra of (L', α'). □

It follows from Remark 3.10 that $l\mathcal{BL}$ has no products and then bisimulations cannot be defined on $\prod$-coalgebras. Moreover, since limits and colimits in the categories of coalgebras are carried by limits and colimits in the base categories, we obtain the following result:

Proposition 3.15. *$Coalg(\prod)$ is not complete, nor cocomplete.*

4 Local BL-frames as $\prod$-coalgebras

Throughout this section, we fix a set $Prop$ of proposition letters.

Definition 4.1. (1) A *local* BL-*frame* is a structure (L, θ_M) where L is a local BL-algebra and M is the maximal filter of L;

(2) A *local* BL-*model* is a structure (L, θ_M, ν) where (L, θ_M) is a local BL-frame and $\nu : Prop \longrightarrow \prod(L)$ is a compatible valuation, that is for all $x, y \in L$, we have

(i) $\nu^{-1}(\{[x]_M * [y]_M\}) = \nu^{-1}(\{[x]_M\}) \cap \nu^{-1}(\{[y]_M\})$;

(ii) $\nu^{-1}(\{[x]_M \to [y]_M\}) = \nu^{-1}(\{[x]_M\})^C \cup \nu^{-1}(\{[y]_M\})$;

(iii) $\nu^{-1}(\{[0]_M\}) = \emptyset$.

Local BL-frames (models) and BL-morphisms form a category which will be denoted by $\mathcal{F}r(lBL)$ ($\mathcal{M}od(lBL)$).

Remark 4.2. It is well known that the normal modal logic S_5 is characterized by the class of reflexive, symmetric, and transitive Kripke frames, that is, the frames for S_5 are exactly that Kripke frames in which the accessibility relation is an equivalence relation. Therefore S_5 is sound and complete in the class of local BL-frames.

The validity of modal formulas at a world x in a local BL-model (L, θ_M, ν) is defined recursively as:

$\mathcal{M}, x \models p$ iff $x \in \nu(p)$

$\mathcal{M}, x \models \neg\varphi$ iff not $\mathcal{M}, x \models \varphi$

$\mathcal{M}, x \models \varphi \wedge \psi$ iff $\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \varphi \vee \psi$ iff $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \varphi \rightarrow \psi$ iff not $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \Box\varphi$ iff for every $y \in [x]_M$, $\mathcal{M}, y \models \varphi$

$\mathcal{M}, x \models \Diamond\varphi$ iff there exists $y \in [x]_M$, $\mathcal{M}, y \models \varphi$

The *truth set* of a formula φ in a model $\mathcal{M}$ is the set $[[\varphi]]^{\mathcal{M}} = \{x \in L/\mathcal{M}, x \models \varphi\}$. For any subset K of L, we define the operators $\lhd$ and $\widetilde{\Box}$ by:

$$\lhd K = L \setminus K \text{ and } \widetilde{\Box} K = \{x \in L/\, [x]_M \subseteq K\}.$$

By checking the semantics clause above, we have the following result:

Lemma 4.3. *For any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$,*

(i) $[[p]]^{\mathcal{M}} = \nu(p)$;

(ii) $[[\neg\varphi]]^{\mathcal{M}} = \lhd\, [[\varphi]]^{\mathcal{M}}$;

(iii) $[[\varphi \wedge \psi]]^{\mathcal{M}} = [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$;

(iv) $[[\Box\varphi]]^{\mathcal{M}} = \widetilde{\Box}\, [[\varphi]]^{\mathcal{M}}$.

The following result shows how to construct modal algebras with any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$:

Theorem 4.4. *For any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$, define the set*

$$\tau(\mathcal{M}) = \{[[\varphi]]^{\mathcal{M}}, \varphi \in Prop\}.$$

Then the structure $(\tau(\mathcal{M}), \cap, \cup, \lhd, \emptyset, L, \widetilde{\Box})$ is a modal algebra.

Proof. Using Lemma 4.3, it is easily checked that $(\tau(\mathcal{M}), \cap, \cup, \lhd, \emptyset, L)$ is a Boolean algebra and that $\widetilde{\Box} L = L$. We only show that $\widetilde{\Box}$ preserves intersections. Let $\varphi, \psi \in Prop$. We have

$$\widetilde{\Box}([[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}) = \{x \in L \mid [x]_M \subseteq [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}\} \subseteq \widetilde{\Box}\, [[\varphi]]^{\mathcal{M}} \cap \widetilde{\Box}\, [[\psi]]^{\mathcal{M}}.$$

Conversely, let $x \in \widetilde{\Box}\, [[\varphi]]^{\mathcal{M}} \cap \widetilde{\Box}\, [[\psi]]^{\mathcal{M}}$. Then $[x]_M \subseteq [[\varphi]]^{\mathcal{M}}$ and $[x]_M \subseteq [[\psi]]^{\mathcal{M}}$. Thus for all $y \in [x]_M$, we have $\mathcal{M}, y \models \varphi$ and $\mathcal{M}, y \models \psi$. So $\mathcal{M}, y \models \varphi \wedge \psi$. By Lemma 4.3 we obtain $y \in [[\varphi \wedge \psi]]^{\mathcal{M}} = [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$. It follows that $[x]_M \subseteq [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$ and so $x \in \widetilde{\Box}([[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}})$. □

For each BL-algebra L, let $\underline{L}$ denote the carrier.

In what follows, we give a link between local BL-frames and well known Kripke frames:

Proposition 4.5. *Let $\mathcal{F}r(lBL)^*$ be the category of local BL-frames with surjective morphisms. Then the correspondance $U : \mathcal{F}r(lBL)^* \longrightarrow \mathcal{KF}r$ which sends every (L, θ_M) to $(\underline{L}, \theta_M))$ and acts on morphisms as identity is a faithful functor.*

Proof. For any local BL-frame (L, θ_M), $U((L, \theta_M)) = (\underline{L}, \theta_M)$ is clearly a Kripke frame. Let $f : (L, \theta_M) \longrightarrow (L', \theta_{M'})$ be a surjective morphism. In order to show that U is well defined, we have to show that f is a p-morphism. Let $x \in L$ and $y \in f([x]_M)$. Then $y = f(z)$ with $z \in [x]_M$. So

$$(z \to x) \wedge (x \to z) \in M.$$

Thus

$$f((z \to x) \wedge (x \to z)) \in f(M).$$

It follows from Lemma 3.6 that

$$(y \to f(x)) \wedge (f(x) \to y) \in M'.$$

So $y \in [f(x)]_{M'}$ and we have $f([x]_M) \subseteq [f(x)]_{M'}$. Moreover, let $y \in [f(x)]_{M'}$. Since f is surjective, there exists $z \in L$ such that $y = f(z)$ and we have

$$(f(z) \to f(x)) \wedge (f(x) \to f(z)) \in M',$$

that is

$$f((z \to x) \wedge (x \to z)) \in M'$$

so that

$$(z \to x) \wedge (x \to z) \in f^{-1}(M') = M.$$

Thus $z \in [x]_M$. Therefore $y \in f([x]_M)$. Hence $f([x]_M) = [f(x)]_M$. So U is well defined. The functoriality and the faithfulness of U are straightforward.

□

We present now the result which allows to see local BL-frames as $\prod$-coalgebras:

Theorem 4.6. *$\mathcal{F}r(lBL)$ is isomorphic to $Coalg(\prod)$.*

Proof. Consider the correspondance $\mathbb{F}$ which assigns to each local BL-frame (L, θ_M) the pair $(L, L \xrightarrow{\alpha_L} L/M)$ such that $\alpha(x) = [x]_M$ for all $x \in L$ and to each BL-morphism $f : L \longrightarrow L'$, $\mathbb{F}(f) = f$. Let (L, θ_M) be a local BL-frame. Since θ_M is a congruence, α is a BL-morphism and so $(L, L \xrightarrow{\alpha_L} L/M)$ is a $\prod$-coalgebra. Moreover, let $(L, \theta_M) \xrightarrow{f} (L', \theta_{M'})$ be a BL-morphism. For all $x \in L$,

$$\alpha' \circ f(x) = [f(x)]_{M'} = \prod(f)([x]_M) = \prod(f) \circ \alpha(x).$$

So f is a $\prod$-homomorphism between $(L, L \xrightarrow{\alpha_L} L/M)$ and $(L', L' \xrightarrow{\alpha_L} L'/M')$. Hence, $\mathbb{F}$ is well defined. By spelling out the definitions, one shows that $\mathbb{F}$ preserves composition and identity. Thus $\mathbb{F} : \mathcal{F}r(lBL) \longrightarrow Coalg(\prod)$ is a covariant functor.

Moreover The correspondance $\mathbb{G}$ which assigns to each $\prod$-coalgebra $(L, L \xrightarrow{\alpha_L} L/M)$ the local BL-frame (L, θ_M) and which acts as identity on homomorphisms is functorial. Finally, Lemma 3.7 allows to prove that the two functors above statisfy the identities $\mathbb{F} \circ \mathbb{G} = id_{Coalg(\prod)}$ and $\mathbb{G} \circ \mathbb{F} = id_{\mathcal{F}r(lBL)}$. So $\mathcal{F}r(lBL)$ and $Coalg(\prod)$ are isomorphic. □

5 Conclusion

One of the main interests of the study of coalgebras is the development of coalgebraic logical foundations over base categories, as a way of reasoning in a quantitive way about transition systems. There is a strong link between coalgebras and modal logic. In this paper, we investigate this relation in the framework of BL-algebras. After the characterization of $\prod$-homomorphisms and $\prod$-subcoalgebras, where $\prod$ is the endofunctor on the category of local BL-algebras and BL-morphisms which assigns to each local BL-algebra its quotient by its unique maximal filter, we introduced local BL-frames based on local BL-algebras, and shown that the category of local BL-frames is isomorphic to the category of $\prod$-coalgebras.

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