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On block commutative groupoids

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"This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday."

Abstract

In this paper, we introduce the notion of a block commutativity in several groupoids, and show that the class of block commutative groupoids and the class of d/BCK-algebras are Smarandache disjoint. The block commutativity in linear/quadratic groupoids is investigated, and we prove that every group is a normal groupoid. Moreover, we discuss block n-commutative groupoids and block ranks.

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1 Introduction

The theory of groupoids [3, 4] has been introduced by some researchers. It has been combined with the theory of general algebraic structures [7, 10, 11]. One of the methods for the generalization of axioms is to employ special functions, i.e., by using of proper mappings, we may generalize axioms in mathematical structures.

The notion of BCK-algebras was formulated by K. Iséki. The motivation of this notion is based on both set theory and propositional calculus (see [6, 8, 12]). As a generalization of this notion, the notion of d-algebras has been developed by many researchers (see [5, 13, 14]).

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Allen et al. [1, 2] introduced several new families of Smarandache-type P-algebras and studied some of their properties in relation to the properties of previously defined Smarandache-types. Moreover, they introduced the notion of Smarandache disjointness, and showed that semigroups and d/BCK-algebras are Smarandache disjoint, and also that groups and d-algebras are Smarandache disjoint, while groups and semigroups are clearly not Smarandache disjoint.

The notion of the semigroup $(Bin(X), \Box)$ was introduced by Kim and Neggers [9]. Given binary operations "$*$" and "$\bullet$" on a set X, they defined a product binary operation "$\Box$" as follows: $x\Box y = (x * y) \bullet (y * x)$. This in turn yields a binary operation on $Bin(X)$, the set of all groupoids, defined on X turning $(Bin(X), \Box)$ into a semigroup with identity $(x * y = x)$, the left zero semigroup, and an analog of negative one in the right zero semigroup [9].

In this paper, given a groupoid $(X, *)$, we consider the $\Box$-product, i.e., $x\Box y = (x * y) * (y * x)$, and we investigate the role of $\Box$ in several groupoids related to commutativity, and show that the class of block commutative groupoids and the class of d/BCK-algebras are Smarandache disjoint. We discuss the block commutativity in linear/quadratic groupoids, and investigate (block) centers and normal groupoids. Especially, we prove that every group is a normal groupoid. Moreover, we discuss block n-commutative groupoids and block ranks.

2 Preliminaries

A *d-algebra* [14] is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A *BCK-algebra* [6, 8, 12] is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

A groupoid $(X, *)$ is said to be a *right zero semigroup* if $x * y = y$ for any $x, y \in X$, and a groupoid $(X, *)$ is said to be a *left zero semigroup* if $x * y = x$ for any $x, y \in X$. A groupoid $(X, *)$ is said to be a *rightoid* for $f : X \to X$ if $x * y = f(y)$ for any $x, y \in X$. Similarly, a groupoid $(X, *)$ is said to be a *leftoid* for $f : X \to X$ if $x * y = f(x)$ for any $x, y \in X$. Note that a right (left, resp.) zero semigroup is a special case of a rightoid (leftoid, resp.) (see [9]).

Given a non-empty set X, two groupoids $(X, *)$ and $(X, \bullet)$ are said to be *Smarandache disjoint* [1, 2] if X has both an $(X, *)$-structure and an $(X, \bullet)$-structure, then $|X| = 1$. The notion of "Smarandache disjoint" means that, given a groupoid $(X, *)$, if we combine another groupoid $(X, \bullet)$ to it, then it can only be a trivial groupoid.

Given a non-empty set X, we let $(Bin(X), \Box)$ denote the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given groupoids $(X, *)$ and $(X, \bullet)$ in $(Bin(X), \Box)$, we define a product "$\Box$" on these groupoids as follows:

$$(X, *) \Box (X, \bullet) = (X, \Box)$$

where

$$x \Box y = (x * y) \bullet (y * x)$$

for any $x, y \in X$.

Using this notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1. [9] *$(Bin(X), \Box)$ is a semigroup, i.e., the operation "$\Box$" as defined in general is associative. Furthermore, the left-zero-semigroup is the identity for this operation.*

3 Block commutative groupoids

A groupoid $(X, *)$ is said to be *block commutative* if $x\Box y = y\Box x$ for all $x, y \in X$, where $x\Box y := (x * y) * (y * x)$.

Example 3.1. *Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := \lceil x \rceil + \lfloor y \rfloor$ for all $x, y \in \mathbf{R}$, e.g., $\lceil 2.14 \rceil = 3, \lfloor 3.56 \rfloor = 3$. Then $2.14 * 4 = \lceil 2.14 \rceil + \lfloor 4 \rfloor = 3 + 4 = 7$, but $4 * 2.14 = \lceil 4 \rceil + \lfloor 2.14 \rfloor = 4 + 2 = 6$. Hence $(\mathbf{R}, *)$ is not commutative. Consider $(\mathbf{R}, \Box) = (\mathbf{R}, *)\Box(\mathbf{R}, *)$, i.e., $x\Box y = (x * y) * (y * x)$ for all $x, y \in \mathbf{R}$. Then we have*

$$\begin{aligned}(x * y) * (y * x) &= (\lceil x \rceil + \lfloor y \rfloor) * (\lceil y \rceil + \lfloor x \rfloor) \\ &= \lceil \lceil x \rceil + \lfloor y \rfloor \rceil + \lfloor \lceil y \rceil + \lfloor x \rfloor \rfloor \\ &= \lceil x \rceil + \lfloor y \rfloor + \lceil y \rceil + \lfloor x \rfloor \\ &= (y * x) * (x * y).\end{aligned}$$

Hence $(\mathbf{R}, \Box)$ is commutative, i.e., $(\mathbf{R}, *)$ is block commutative.

The example above is an example of a block commutative groupoid which is not commutative. Obviously, if $(X, *)$ is commutative, then it is block commutative. For example, let $\mathbf{R}$ be the set of all real numbers and let $\alpha, \beta \in \mathbf{R}$. If we define $x * y := \alpha + \beta xy$ for all $x, y \in \mathbf{R}$, then $(\mathbf{R}, *)$ is both block commutative and commutative.

Example 3.2. *Let $X := \{a, b, c, d\}$ be a set. Define a map $\varphi : X \to X$ by*

$$\varphi(a) = \varphi(b) = a, \quad \varphi(c) = \varphi(d) = b.$$

If we define a binary operation "$*$" on X by $x * y := \varphi(x)$ for all $x, y \in X$, then the groupoid $(X, *)$ becomes as follows:

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	b	b	b	b
d	b	b	b	b

Given $x, y \in X$, we have

$$x\Box y = (x * y) * (y * x) = \varphi(x) * \varphi(y) = \varphi(\varphi(x)) = a,$$

and

$$y\Box x = (y * x) * (x * y) = \varphi(y) * \varphi(x) = \varphi(\varphi(y)) = a.$$

It shows that $(X, *)$ is block commutative. But $(X, *)$ is not commutative, since $c*b = b \neq a = b*c$.

Proposition 3.3. *Let $(X, *)$ be a rightoid for φ such that $\varphi(X) \subseteq A$ and $\varphi(A) = \{a\}$ for some $a \in A$. Then $(X, *)$ is block commutative.*

Proof. Given $x, y \in X$, we have

$$(x * y) * (y * x) = \varphi(y) * \varphi(x) = \varphi(\varphi(x)) = a.$$

Similarly, we obtain $(y * x) * (x * y) = a$. Hence $(X, *)$ is block commutative. □

Example 3.4. *Let* $\mathbf{R}$ *be the set of all real numbers. Define a map* $\varphi : \mathbf{R} \to \mathbf{R}$ *by*

$$\varphi(x) := \begin{cases} 5 & \text{if } x \geq 0, \\ |x| & \text{if } x < 0. \end{cases}$$

Then $\varphi(\mathbf{R}) = [0, \infty)$ *and* $\varphi(\varphi(\mathbf{R})) = \{5\}$. *If we define a binary operation "$*$" on* X *by* $x*y := \varphi(y)$ *for all* $x, y \in X$, *by applying Proposition 3.3,* $(X, *)$ *is block commutative.*

Proposition 3.5. *Let* $(X, *)$ *be a block commutative groupoid. If* $(X, *)$ *is selective, then it is commutative.*

Proof. Assume that there exist x, y in X such that $x * y \neq y * x$. Since $(X, *)$ is selective, we may let $x * y = y$ and $y * x = x$. It follows that

$$y * x = (x * y) * (y * x) = x \Box y = y \Box x = (y * x) * (x * y) = x * y,$$

a contradiction, since $(X, *)$ is block commutative. □

Theorem 3.6. *Let* $X := \mathbf{R}$ *be the set of all real numbers and let* $\alpha, \beta, \gamma \in X$. *Define a binary operation "$*$" on* X *by* $x * y := \alpha + \beta x + \gamma y$ *for all* $x, y \in X$. *If* $(X, *)$ *is block commutative, then* $\beta = \gamma$, *i.e.,* $x * y$ *is of the form* $x * y = \alpha + \beta(x + y)$ *for all* $x, y \in X$.

Proof. Given $x, y \in X$, we have

$$\begin{aligned} x \Box y &= (x * y) * (y * x) \\ &= \alpha + \beta(x * y) + \gamma(y * x) \\ &= \alpha + \beta(\alpha + \beta x + \gamma y) + \gamma(\alpha + \beta y + \gamma x) \\ &= \alpha(1 + \beta + \gamma) + (\beta^2 + \gamma^2)x + 2\beta\gamma y. \end{aligned}$$

Similarly, we obtain $y \Box x = \alpha(1 + \beta + \gamma) + (\beta^2 + \gamma^2)y + 2\beta\gamma x$. Since $(X, *)$ is block commutative, we obtain $\beta^2 + \gamma^2 = 2\beta\gamma$, and hence $(\beta - \gamma)^2 = 0$, proving that $\beta = \gamma$. □

Corollary 3.7. *Let* $X := \mathbf{R}$ *be the set of all real numbers and let* $\alpha, \beta, \gamma \in X$. *Define a binary operation "$*$" on* X *by* $x * y := \alpha + \beta x + \gamma y$ *for all* $x, y \in X$. *Then every block commutative groupoid is commutative.*

Proof. Straightforward. □

Theorem 3.8. *Let* $X := \mathbf{R}$ *be the set of all real numbers and let* $\alpha, \beta \in X$. *Define a binary operation "$*$" on* X *by* $x * y := \alpha x^2 + \beta y^2$ *for all* $x, y \in X$. *Then*

(i) *if* $\alpha + \beta = 0$, *then* $(X, *)$ *is block commutative, but not commutative,*

(ii) *if* $\alpha + \beta \neq 0$, *then* $(X, *)$ *is block commutative if and only if* $\alpha = \beta$.

Proof. Given $x, y \in X$, we have

$$\begin{aligned} x\Box y &= (x*y)*(y*x) \\ &= \alpha(x*y)^2 + \beta(y*x)^2 \\ &= \alpha[\alpha x^2 + \beta y^2]^2 + \beta[\alpha y^2 + \beta x^2]^2 \\ &= (\alpha^3+\beta^3)x^4 + \alpha\beta(\alpha+\beta)y^4 + 2\alpha\beta(\alpha+\beta)x^2y^2. \end{aligned} \tag{1}$$

Similarly, we have

$$y\Box x = (y*x)*(x*y) = (\alpha^3+\beta^3)y^4 + \alpha\beta(\alpha+\beta)x^4 + 2\alpha\beta(\alpha+\beta)x^2y^2 \tag{2}$$

Case (i). If $\alpha+\beta = 0$, then $x*y = \alpha(x^2-y^2)$ for all $x, y \in X$. Hence $y*x = -x*y$ and hence

$$x\Box y = (x*y)*(y*x) = \alpha[(x*y)^2 - (y*x)^2] = \alpha[(x*y)^2 - \{-(x*y)\}^2] = 0.$$

Similarly, we have $y\Box x = (y*x)*(x*y) = 0$. Hence $(X,*)$ is block commutative, but not commutative.

Case (ii). Let $\alpha+\beta \neq 0$. Assume $(X,*)$ is block commutative. By (1) and (2), we obtain $\alpha^3+\beta^3 = \alpha\beta(\alpha+\beta)$, and hence $(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) = \alpha\beta(\alpha+\beta)$. Since $\alpha+\beta \neq 0$, we obtain $\alpha^2-\alpha\beta+\beta^2 = \alpha\beta$, proving that $\alpha = \beta$. If we assume $\alpha = \beta$, then $x*y = \alpha(x^2+y^2)$, which proves that $(X,*)$ is both block commutative and commutative. □

Proposition 3.9. *Let $(X,*,0)$ be a d-algebra. If $(X,*)$ is block commutative, then $|X| = 1$.*

Proof. Given $x \in X$, since $(X,*,0)$ is a d-algebra, we have $0*x = 0$. Since $(X,*)$ is block commutative, we obtain

$$0 = 0*(x*0) = (0*x)*(x*0) = (x*0)*(0*x) = (x*0)*0, \quad \text{i.e., } 0 = (x*0)*0.$$

Since $0*(x*0) = 0$ and $(X,*,0)$ is a d-algebra, we obtain $x*0 = 0$. In addition, from $0*x = 0$, we obtain $x = 0$ for all $x \in X$. Hence $|X| = 1$. □

Corollary 3.10. *The class of d/BCK-algebras and the class of block commutative groupoids are Smarandache disjoint.*

Proof. It follows immediately from Proposition 3.9. □

4 Centers and normal subgroupoids

The concepts of a center and a block center of groupoids can be derived from the center of a group. Let $(X,*)$ be a groupoid. We define two notions, i.e., a *center* and a *block center* of $(X,*)$ as follows:

$$Z(X,*) := \{x \in X \mid x*y = y*x, \forall y \in X\},$$

and

$$BZ(X,*) := \{x \in X \mid (x*y)*(y*x) = (y*x)*(x*y), \forall y \in X\}.$$

Moreover, a non-empty subset W of a groupoid $(X,*)$ is said to be a *commutative subgroupoid* of X if $x*y = y*x$ for all $x, y \in W$. If $(X,*)$ is a commutative groupoid, then $X = Z(X,*) = BZ(X,*)$.

Example 4.1. *Let $(\mathbf{R}, *)$ be a groupoid as in Example 3.1. Then $(\mathbf{R}, *)$ is not a commutative groupoid. We claim that $Z(\mathbf{R}, *) = \emptyset$. Assume that $n \in \mathbf{R}$ satisfies the condition $n * y = y * n$ for all $y \in \mathbf{R}$.*

Case 1. *$n \in Z$, i.e., n is an integer. We take $y := 2.14$. Then $n * 2.14 = \lceil n \rceil + \lfloor 2.14 \rfloor = n + 2$ and $2.14 * n = \lceil 2.14 \rceil + \lfloor n \rfloor = 3 + n$, a contradiction.*

Case 2. *$n \notin Z$, say $n = 3.78$. We take $y := 1$. Then $3.78 * 1 = \lceil 3.78 \rceil + \lfloor 1 \rfloor = 5$ and $1 * 3.78 = \lceil 1 \rceil + \lfloor 3.78 \rfloor = 4$, a contradiction.*

We claim that $BZ(\mathbf{R}, *) = \mathbf{R}$, since $(\mathbf{R}, *)$ is block commutative.

Proposition 4.2. *Let $(X, *)$ be a groupoid and $(Y, *)$ be a subgroupoid of $(X, *)$. Then*

(i) *$Y \cap BZ(X, *) \subseteq BZ(Y, *)$,*

(ii) *$Y \cap Z(X, *) \subseteq Z(Y, *)$.*

Proof. (i) If $x \in Y \cap BZ(X, *)$, then $x \in Y$ and $(x * z) * (z * x) = (z * x) * (x * z)$ for all $z \in X$. Since $Y \subseteq X$, we obtain $x \in BZ(Y, *)$.
(ii) The proof is similar to (i). □

Proposition 4.3. *Let $(X, *)$ be a groupoid and $(W, *)$ be a commutative subgroupoid of $(X, *)$ with $X * X \subseteq W$. Then $(X, *)$ is block commutative.*

Proof. Given $x, y \in X$, since $X * X \subseteq W$, we have $x * y, y * x \in W$. In addition, from $(W, *)$ is commutative, we have $x * y = y * x$, and hence $(x * y) * (y * x) = (y * x) * (x * y)$. Therefore, $(X, *)$ is block commutative. □

Theorem 4.4. *Let $(X, *)$ be a groupoid and $(W, *)$ be a commutative subgroupoid of $(X, *)$. Assume if $x, y \notin W$, then $x * y \in W$. Then $(X, *)$ is block commutative.*

Proof. Given $x, y \in X$, we have 3 cases as follows.

Case (i). $x, y \in W$. Since $(W, *)$ is a commutative subgroupoid of $(X, *)$, $x * y = y * x \in W$. It follows that $x \square y = (x * y) * (y * x) = (y * x) * (x * y) = y \square x$.

Case (ii). $x, y \notin W$. By assumption, we have $x * y, y * x \in W$. Since $(W, *)$ is commutative, we have $x * y = y * x$, and hence $x \square y = y \square x$.

Case (iii). $x \in W$ and $y \notin W$.

Subcase (iii-1). $x * y \in W$. It is similar to Case (i).

Subcase (iii-2). $x * y \notin W$. If $y * x \in W$, then $x * y = y * x \in W$. Since $(W, *)$ is a commutative subgroupoid of $(X, *)$, we get $x * y \in W$, a contradiction. Hence this case does not happen. Let $y * x \notin W$. Since $x * y \notin W$, by assumption, we obtain $(x * y) * (y * x), (y * x) * (x * y) \in W$. It follows from $(W, *)$ is a commutative groupoid that $x \square y = (x * y) * (y * x) = (y * x) * (x * y) = y \square x$. This completes the proof. □

Example 4.5. *In Example 3.1, if we let $W := \mathbf{Z}$, then $(\mathbf{Z}, *)$ is a commutative subgroupoid of $(\mathbf{R}, *)$, since $m * n = \lceil m \rceil + \lfloor n \rfloor = m + n = n * m$ for all $m, n \in \mathbf{Z}$. We see that $\mathbf{R} * \mathbf{R} \subseteq \mathbf{Z}$. Hence, by Proposition 4.2, we show that $(\mathbf{R}, *)$ is block commutative.*

A groupoid $(X, *)$ is said to be *normal* if $x * X = X * x$ for all $x \in X$.

Example 4.6. (a) *Let $X := [0, \infty)$ be a set and "$+$" be the usual addition on X. Then $(X, +)$ is a semigroup. For any $x \in X$, we have $x + X = [x, \infty) = X + x$. Hence $(X, +)$ is a normal groupoid, but not a group.*

(b) *Let $X := [0, \infty)$ be a set and $x \bullet y := \min\{x, y\}$ on X. Then $x \bullet X = [0, \infty) = X \bullet x$ for all $x \in X$. Hence $(X, \bullet)$ is a normal groupoid, but not a group.*

Example 4.7. *Consider Example 3.1. We show that $y*\mathbf{R} = \mathbf{R}*y$ for all $y \in \mathbf{R}$. Given $x, y \in \mathbf{R}$, for any $x*y \in X*y$, we need to find an element $u \in \mathbf{R}$ such that $y*u = x*y$. We take u in $\mathbf{R}$ satisfying*

$$\lfloor u \rfloor := \begin{cases} \lceil x \rceil - 1 & \text{if } y \notin \mathbf{Z}, \\ \lceil x \rceil & \text{if } y \in \mathbf{Z}. \end{cases}$$

If $y \notin \mathbf{Z}$, then there exists $n \in \mathbf{Z}$ and $\alpha \in \mathbf{R}$ such that $y = n+\alpha$. Hence $x*y = \lceil x \rceil + \lfloor y \rfloor = \lceil x \rceil + n$ and

$$y*u = \lceil y \rceil + \lfloor u \rfloor = n+1+\lfloor u \rfloor = n+1+(\lceil x \rceil - 1) = \lceil x \rceil + n,$$

which proves that $x*y = y*u$. If $y \in \mathbf{Z}$, say $y = n$ for some $n \in \mathbf{Z}$, then $x*y = \lceil x \rceil + \lfloor y \rfloor = \lceil x \rceil + n$ and $y*u = \lceil y \rceil + \lfloor u \rfloor = n + \lfloor u \rfloor = n + \lceil x \rceil$. Hence $x*y = y*u$. Therefore, $y*\mathbf{R} = \mathbf{R}*y$ for all $y \in \mathbf{R}$, proving that $(\mathbf{R}, *)$ is a normal groupoid.

Theorem 4.8. *Every group is a normal groupoid.*

Proof. Let $(X, *, e)$ be a group and $x, y \in X$. We take $u := x*y*x^{-1}$. Then $u*x = (x*y*x^{-1})*x = x*y$ for all $x \in X$. This proves that $x*X = X*x$ for all $x \in X$, proving the theorem. $\square$

Theorem 4.9. *The class of d/BCK-algebras and the class of normal groupoids are Smarandache disjoint.*

Proof. Let $(X, *, 0)$ be both a d/BCK-algebra and a normal groupoid. Then $0*X = X*0$. It follows that $x*0 = 0*x = 0$ for all $x \in X$. Since $(X, *, 0)$ is a d/BCK-algebra, we obtain $x = 0$. Hence $|X| = 1$. $\square$

Let $(X, *)$ be a groupoid. A subset B of X is said to be a *block subset* of X if $x, y \in B$, then $x\Box y = y\Box x$, i.e., $(x*y)*(y*x) = (y*x)*(x*y)$.

Proposition 4.10. *Let $(X, *)$ be a leftoid for φ and $a \in X$. If we define*

$$B_a := \{x \in X | \varphi(x) = a\},$$

then B_a is a block subset of X.

Proof. If $x, y \in B_a$, then $\varphi(x) = a = \varphi(y)$ and hence

$$(x*y)*(y*x) = \varphi(x)*\varphi(y) = a*a = \varphi(a),$$

and

$$(y*x)*(x*y) = \varphi(y)*\varphi(x) = a*a = \varphi(a),$$

proving the proposition. $\square$

5 Block n-commutative groupoids

Let $(X, *)$ be a groupoid and $x, y \in X$. We define

$$\begin{aligned} E_1^*(x,y) &:= x*y \\ E_2^*(x,y) &:= E_1^*(x,y)*E_1^*(y,x) \\ E_3^*(x,y) &:= E_2^*(x,y)*E_2^*(y,x) \\ &\ldots\ldots\ldots\ldots \\ E_{n+1}^*(x,y) &:= E_n^*(x,y)*E_n^*(y,x). \end{aligned}$$

If we assume $E_n^*(x,y) = E_n^*(y,x)$, then

$$E_{n+1}^*(x,y) = E_n^*(x,y) * E_n^*(y,x) = E_n^*(y,x) * E_n^*(x,y) = E_{n+1}^*(y,x).$$

Hence we have $E_k^*(x,y) = E_k^*(y,x)$ for all $k \geq n$.

A groupoid $(X,*)$ is said to be

- *commutative* if $E_1^*(x,y) = E_1^*(y,x)$ for all $x,y \in X$,
- *block commutative* if $E_2^*(x,y) = E_2^*(y,x)$ for all $x,y \in X$,
- *block n-commutative* if $E_n^*(x,y) = E_n^*(y,x)$ for all $x,y \in X$.

A groupoid $(X,*)$ is said to have a *block rank* m, and denote it by $brank(X,*) = m$, if

(i) $E_m^*(x,y) = E_m^*(y,x)$ for all $x,y \in X$,

(ii) for any $n \leq m-1$, there exist $x,y \in X$ such that $E_n^*(x,y) \neq E_n^*(y,x)$.

The groupoid $(\mathbf{R},*)$ in Example 3.1 has the block rank 2.

Example 5.1. *Let $X := \{1,2,3,4,5\}$ be a set. Define a binary operation "$*$" on X by $x*y := \varphi(x)$ for all $x,y \in X$, where $\varphi : X \to X$ is a map defined by*

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}.$$

Then $E_2^*(3,5) = \varphi(\varphi(3)) = \varphi(2) = 1 \neq 2 = \varphi(3) = \varphi(\varphi(5))$, and hence $E_2^*(3,5) \neq E_2^*(5,3)$. Now, we have $E_3^*(x,y) = E_2^*(x,y) * E_2^*(y,x) = \varphi(\varphi(x)) * \varphi(\varphi(y)) = \varphi(\varphi(\varphi(x))) = 1 = E_3^*(y,x)$. Hence $brank(X,*) = 3$.

We construct a groupoid $(X,*)$ having $brank(X,*) = 3$ by the following method. Given two elements $x,y \in X$, we define a map $\varphi : X \to X$ by $\varphi(X) \subseteq A, \varphi(A) \subseteq B, \varphi(B) = \{b\}$ for some $b \in B$, where A, B are subsets of X and $\varphi(\varphi(x)) \neq \varphi(\varphi(y))$. Then

$$E_2^*(x,y) = (x*y)*(y*x) = \varphi(x)*\varphi(y) = \varphi(\varphi(x)) \neq \varphi(\varphi(y)) = \varphi(x)*\varphi(y) = (y*x)*(x*y) = E_2^*(y,x).$$

Since $\varphi(B) = \{b\}$, we obtain $E_3^*(x,y) = E_3^*(y,x)$. This shows that $(X,*)$ is a groupoid having $brank(X,*) = 3$. In Example 5.1, if we let $A := \{1,2,3,4\}$ and $B := \{1,2\}$, then the mapping φ satisfies all conditions, and hence $(X,*)$ has the block rank 3.

In this manner we may construct leftoids $(X,*,\varphi)$ having the block rank n as follows:

Proposition 5.2. *If we define a binary operation "$*$" on X by $x*y := \varphi(x)$ for all $x,y \in X$, where $\varphi : X \to X$ by $\varphi(X) \subseteq A_1, \varphi(A_1) \subseteq A_2, \cdots, \varphi(A_{n-1}) \subseteq A_n, \varphi(A_n) = \{a_n\}$ for some $a_n \in A_n$ and $\overbrace{\varphi(\varphi(\cdots((\varphi}^{n-1}(x))\cdots) \neq \overbrace{\varphi(\varphi(\cdots((\varphi}^{n-1}(y))\cdots)$, then $(X,*)$ has the block rank n.*

Example 5.3. *Let $X := \mathbf{R}$ be the set of all real numbers and $0 < a_n < a_{n-1} < \cdots < a_2 < a_1$. Define a map $\varphi : X \to X$ by $\varphi(X) \subseteq [-a_1, a_1], \varphi([-a_1,a_1]) \subseteq [-a_2,a_2], \cdots, \varphi([-a_{n-1},a_{n-1}]) \subseteq [-a_n,a_n]$ and $\varphi([-a_n,a_n]) = \{0\}$. Define a binary operation $x*y := \varphi(x)$ for all $x,y \in X$. If $\overbrace{\varphi(\varphi(\cdots((\varphi}^{n-1}(x))\cdots)(x) \neq \overbrace{\varphi(\varphi(\cdots((\varphi}^{n-1}(y))\cdots)(y)$ for some $x \neq y$ in $[-a_{n-1},a_{n-1}]$, then $(X,*)$ has the block rank n.*

Theorem 5.4. *Let $\varphi : (X, *) \to (Y, \bullet)$ be an epimorphism of groupoids. If $(X, *)$ has the block rank m, then there exists $n \le m$ such that $(Y, \bullet)$ has the block rank n.*

Proof. Let $(X, *)$ have the block rank m. Then $E^*_m(x, y) = E^*_m(y, x)$ for all $x, y \in X$. Since φ is an epimorphim of groupoids, we have $E^\bullet_m(x, y) = E^\bullet_m(y, x)$ for all $x, y \in X$. It follows that $E^\bullet_m(u, v) = E^\bullet_m(v, u)$ for all $u, v \in Y$. If $(Y, \bullet)$ is a commutative groupoid, then $(Y, \bullet)$ has the block rank 1. Otherwise, $E^\bullet_1(u, v) \neq E^\bullet_1$ for some $u, v \in Y$. In this case, we consider a groupoid $(Y, \Box^\bullet_2)$, where $x\Box^\bullet_2 y = (x \bullet y) \bullet (y \bullet x)$. If $(Y, \Box^\bullet_2)$ is commutative, then $(Y, \bullet)$ has the block rank 2. Otherwise, $E^\bullet_2(u, v) \neq E^\bullet_2$ for some $u, v \in Y$. In this case, we consider a groupoid $(Y, \Box^\bullet_3)$, where $x\Box^\bullet_3 y = (x\Box^\bullet_2 y) \bullet (y\Box^\bullet_2 x)$. We continue this process until $n \le m$, proving that $(Y, \bullet)$ has the block rank n. $\square$

Note that if $m = 1$ in Theorem 5.4, then $n = 1$ as well, i.e., if $\varphi : (X, *) \to (Y, \bullet)$ is an epimorphism of groupoids and if $(X, *)$ is commutative, then $(Y, \bullet)$ is also commutative.

Let $(X, *)$ be a groupoid and $x, y \in X$. If $E^*_n(x, y) = E^*_n(y, x)$ for some $n \in \mathrm{N}$, then

$$E^*_{n+1}(x, y) = E^*_n(x, y) * E^*_n(y, x) = E^*_n(y, x) * E^*_n(x, y) = E^*_{n+1}(y, x).$$

Hence $E^*_{n+k}(x, y) = E^*_{n+k}(y, x)$ for all $k \ge 1$.

Lemma 5.5. *Let $(X, *)$ and $(Y, \bullet)$ be groupoids. Define a binary operation "$\triangledown$" on $X \times Y$ by $(x, y)\triangledown(u, v) := (x * u, y \bullet v)$ for all $(x, y), (u, v) \in X \times Y$. Then, for any $(x, y), (u, v) \in X \times Y$,*

$$E^\triangledown_k((x, y), (u, v)) = (E^*_k(x, u), E^\bullet_k(y, v))$$

for all $k \in N$.

Proof. Given $(x, y), (u, v) \in X \times Y$, we have

$$E^\triangledown_1((x, y), (u, v)) = (x, y)\triangledown(u, v) = (x * u, y \bullet v) = (E^*_1(x, u), E^\bullet_1(y, v)),$$

and hence

$$\begin{aligned} E^\triangledown_2((x, y), (u, v)) &= E^\triangledown_1((x, y), (u, v))\triangledown E^\triangledown_1((u, v), (x, y)) \\ &= (x * u, y \bullet v)\triangledown(u * x, v \bullet y) \\ &= ((x * u) * (u * x), (y \bullet v) \bullet (v \bullet y)) \\ &= (E^*_2(x, u), E^\bullet_2(y, v)). \end{aligned}$$

It follows that

$$\begin{aligned} E^\triangledown_3((x, y), (u, v)) &= E^\triangledown_2((x, y), (u, v))\triangledown E^\triangledown_2((u, v), (x, y)) \\ &= (E^*_2(x, u), E^\bullet_2(y, v))\triangledown(E^*_2(u, x), E^\bullet_2(v, y)) \\ &= (E^*_2(x, u) * E^*_2(u, x), E^\bullet_2(y, v) \bullet E^\bullet_2(v, y)) \\ &= (E^*_3(x, u), E^\bullet_3(y, v)). \end{aligned}$$

In this fashion, using the induction, we obtain the conclusion. $\square$

Theorem 5.6. *Let groupoids $(X, *)$ and $(Y, \bullet)$ have the block rank m and n, respectively. Define a binary operation "$\triangledown$" on $X \times Y$ by $(x, y)\triangledown(u, v) := (x * u, y \bullet v)$ for all $(x, y), (u, v) \in X \times Y$. Then $(X \times Y, \triangledown)$ has the block rank k where $k = \max\{m, n\}$.*

Proof. Since the groupoids $(X,*)$ and $(Y,\bullet)$ have the block rank m and n, respectively, we have $E^*_m(x,y) = E^*_m(y,x)$, $E^*_{m'}(x,y) \neq E^*_{m'}(y,x)$ where $m' \leq m-1$, and $E^\bullet_n(x,y) = E^\bullet_n(y,x)$, $E^\bullet_{n'}(x,y) \neq E^\bullet_{n'}(y,x)$ where $n' \leq n-1$. Without loss of generality, we let $m > n$. Then, since $(X,*)$ has the block rank m, by Lemma 5.5, we obtain

$$E^\triangledown_m((x,y),(u,v)) = (E^*_m(x,u), E^\bullet_m(y,v)) = (E^*_m(u,x), E^\bullet_m(v,y)) = E^\triangledown_m((u,v),(x,y)),$$

and

$$E_l((x,y),(u,v)) = (E^*_l(x,u), E_l(y,v)) \neq (E^*_l(u,x), E^\bullet_l(v,y)) = E_l((u,v),(x,y)),$$

for all $l \leq m-1$. This proves the theorem. □

A groupoid $(X,*)$ is said to have a *block rank* ∞ if, for any $n \in \mathrm{N}$, $E^*_m(x,y) \neq E^*_m(y,x)$ for some $x,y \in X$.

Example 5.7. *Let* $\mathbf{R}$ *be the set of all real numbers and* $x,y \in \mathbf{R}$*. Define a binary operation "$*$" on* $\mathbf{R}$ *by* $x * y := e^x$ *for all* $x,y \in \mathbf{R}$*. We see that* $E^*_1(x,y) = x*y = e^x$,

$$E^*_2(x,y) = E^*_1(x,y) * E^*_1(y,x) = e^x * e^y = e^{e^x} = e^{E^*_1(x,y)},$$

and

$$E^*_3(x,y) = E^*_2(x,y) * E^*_2(y,x) = e^{E^*_2(x,y)}.$$

In this fashion, we obtain $E^*_{n+1}(x,y) = e^{E^*_n(x,y)}$ *for any* $n \in N$*. We claim that* $e^x = e^y$ *if and only if* $x = y$ *for all* $x,y \in \mathbf{R}$*. Using the claim, we obtain the following:*

$$\begin{aligned} E^*_{n+1}(x,y) = E^*_{n+1}(y,x) &\Longleftrightarrow E^*_n(x,y) = E^*_n(y,x) \\ &\Longleftrightarrow E^*_{n-1}(y,x) * E^*_{n-1}(x,y) \\ &\vdots \\ &\Longleftrightarrow E^*_2(x,y) = E^*_2(y,x) \\ &\Longleftrightarrow E^*_1(x,y) = E^*_1(y,x) \\ &\Longleftrightarrow x = y \end{aligned}$$

Hence $(X,*)$ *has a block rank* ∞*.*

6 Conclusion

Block commutativity for groupoids is an important generalization of commutativity for groupoids. What counts here is that although groupoids may not be commutative, the degree of "non-commutativity" can be measured by an index called the block-rank which is defined for arbitrary groupoids. The degree of non-commutativity is also measured by this index which measures the minimum number of steps needed to achieve commutativity for arbitrary groupoids. If the groupoid is a group, then one is dealing with "commutators" at different levels and one is operating in more familiar territory. Again this shows that it is possible to embed theories of various kinds into a more comprehensive theory of groupoids of which this paper is another example.

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