On uni-soft p-semisimple ideals and uni-soft strong ideals in BCI-algebras

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"This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday."

Abstract

In this paper, the notions of uni-soft p-semisimple ideals and uni-soft strong ideals in BCI-algebras are introduced and related properties are investigated. Some conditions for a uni-soft ideal to be a uni-soft p-semisimple ideal are given. Also, a necessary and sufficient condition for a uni-soft ideal to be a uni-soft strong ideal is given. Finally, the relationships between uni-soft p-semisimple ideals and uni-soft strong ideals are investigated.

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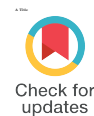
1 Introduction

Many complicated problems in economics, engineering, medical science, environments, etc have various uncertainties. For dealing with uncertainties, some kind of mathematical theories like theory of fuzzy sets [20], theory of intuitionistic fuzzy sets [2, 3], theory of vague set [7], rough sets [18], are given. As it was mentioned in [15, 21], these theories have their own difficulties. In 1999, Molodtsov [17] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties. After Molodtsovs work, Chen et al. [5] and Maji et al. [14] introduced the concepts of some different operations and application of soft sets such as subset, intersection, union and complement of soft sets. Many researchers have contributed towards the algebraic structure of the

soft set theory. Aktas and Cagman [1] introduced and investigated the basic notions of soft groups as a generalization of the idea of fuzzy groups. Feng et al. [6] worked on soft semirings, soft ideals, and idealistic soft semirings. Jun and park [13], studied the application of soft sets in ideal theory of BCK/BCI-algebras. In 2013, Jun [12] introduced the uni-soft set theory with application in BCK/BCI-algebras.

In this paper, we introduce the notions of uni-soft p-semisimple ideals and uni-soft strong ideals and investigate some related properties. We provide some conditions for a uni-soft ideal to be a uni-soft p-semisimple ideal. Also, we give a condition for a uni-soft ideal to be a uni-soft strong ideal. Finally, we investigate the relationships between uni-soft p-semisimple ideals and uni-soft strong ideals.

2   Preliminaries

In this section, we present some definitions and known results, which will be used in this paper. The reader is referred to the books [19, 16] for more details.

An algebraic structure $(X, *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions: for any $x, y, z \in X$,
(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $x * 0 = 0$,
(a3) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We often write X instead of $(X, *, 0)$ for a BCI-algebra in brevity. We can define a partial order $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra.

Any BCI-algebra X satisfies the following conditions: for any $x, y, z \in X$,
(b1) $x * x = 0$,
(b2) $(x * y) * z = (x * z) * y$,
(b3) $x \leq y$ implies $x * z \leq y * z$, and $z * y \leq z * x$,
(b4) $x * (x * (x * y)) = x * y$,
(b5) $0 * (x * y) = (0 * x) * (0 * y)$,
(b6) $x * (x * y) \leq y$,
(b7) $(x * z) * (y * z) \leq x * y$.

A non-empty subset S of a BCI-algebra X is called a subalgebra of X if $x * y \in A$ for all $x, y \in A$. A subset A of a BCI-algebra X is called an ideal of X if

$$0 \in A; \ and \ (\forall x \in X)(\forall y \in A)(x * y \in I \Rightarrow x \in A). \tag{1}$$

Every ideal A of a BCI-algebra X satisfies

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A). \tag{2}$$

An ideal A of a BCI-algebra X is called p-semisimple if

$$(\forall x, z \in X)(\forall y \in A)((x * z) * (y * z) \in A \Rightarrow x \in A). \tag{3}$$

Lemma 2.1. *Let A be an ideal of a BCI-algebra X. Then the following conditions are equivalent:*
(i) A is p-semisimple.
(ii) $0 * (0 * x) \in A$ implies $x \in A$ for all $x \in X$.

Proof. $(i) \Rightarrow (ii)$ Assume that A is a p-semisimple ideal of X. Taking $z = x$ and replacing y by 0 in (3) the result obviously holds.

$(ii) \Rightarrow (i)$ Let A be an ideal of X satisfying (ii). Assume that $(x * z) * (y * z) \in A$ and $y \in A$ for some $x, z \in X$. Using $(b2)$, $(b5)$ and $(b7)$, we have

$$\begin{aligned}
(0 * (0 * x)) * ((x * z) * (y * z)) &= ((0 * (x * z)) * (0 * (y * z))) * (0 * x) \\
&\leq ((y * z) * (x * z)) * (0 * x) \\
&\leq (y * x) * (0 * x) \leq y.
\end{aligned}$$

Then, by (2), we get $(0 * (0 * x)) * ((x * z) * (y * z)) \in A$. Thus it follows from $(x * z) * (y * z) \in A$ that $0 * (0 * x) \in A$ and so by (ii), we conclude $x \in A$. Therefore, A is a p-semisimple ideal of X. $\square$

An element x of a BCI-algebra X is called a positive element if $0 * x = 0$. The set of all positive element of X is called the BCK-part of X and we will denote it by B. It is known that B is a subalgebra of X.

For an element x of a BCI-algebra X and a positive integer n we denote

$$0 * x^n = ((...(0 * \underbrace{x) * x) * x)...) * x}_{n-times}.$$

A BCI-algebra X is called nilpotent if for any $x \in X$ there is a positive integer n such that $0 * x^n = 0$.

In what follows, let U be an *initial universe* set and E be a set of *parameters*. The pair (U, E) is called a *soft universe* and the power set of U is denoted by $\mathcal{P}(U)$.

An pair $\mathcal{F}_A = (f_A, A)$ is called a *soft set over* U, where A is a nonempty subset of E and f_A is a function $f_A : E \to \mathcal{P}(U)$ with $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the *approximate function* of the soft set $\mathcal{F}_A$, and the set of all soft sets over U is denoted by $S(U)$.

Let $\mathcal{F}_A \in S(U)$. For any subset τ of U, the τ-exclusive set of $\mathcal{F}_A$, denoted by $e(\mathcal{F}_A, \tau)$, is defined by the set $e(\mathcal{F}_A, \tau) := \{x \in A \mid f_A(x) \in \tau\}$.

Definition 2.2. [12] *Let* $(U, E) = (U, X)$. *Given a subalgebra* A *of* E, *let* $\mathcal{F}_A \in S(U)$. *Then* $\mathcal{F}_A$ *is called a uni-soft algebra over* U *if the approximate function* f_A *of* $\mathcal{F}_A$ *satisfies*

$$(\forall x, y \in A)(f_A(x * y) \subseteq f_A(x) \cup f_A(y)).$$

Definition 2.3. [12] *Let* $(U, E) = (U, X)$. *Given a subalgebra* A *of* E, *let* $\mathcal{F}_A \in S(U)$. *Then* $\mathcal{F}_A$ *is called a uni-soft ideal over* U *if the approximate function* f_A *of* $\mathcal{F}_A$ *satisfies*

$$(\forall x \in A)(f_A(0) \subseteq f_A(x)); \tag{4}$$

$$(\forall x, y \in A)(f_A(x) \subseteq f_A(x * y) \cup f_A(y)). \tag{5}$$

Definition 2.4. [4] *Let* $(U, E) = (U, X)$. *Given a subalgebra* A *of* E, *let* $\mathcal{F}_A \in S(U)$. *A uni-soft ideal* $\mathcal{F}_A$ *over* U *is said to be closed if the approximate function* f_A *of* $\mathcal{F}_A$ *satisfies*

$$(\forall x \in A)(f_A(0 * x) \subseteq f_A(x)). \tag{6}$$

Lemma 2.5. [12] *Let $(U, E) = (U, X)$. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. If $\mathcal{F}_A$ is a uni-soft ideal over U, then the approximate function f_A of $\mathcal{F}_A$ satisfies the following condition:*

$$(\forall x, y, z \in A)(x * y \leq z \Rightarrow f_A(x) \subseteq f_A(y) \cup f_A(z)). \tag{7}$$

Lemma 2.6. [12] *Let $(U, E) = (U, X)$. Given a subalgebra A of E, let $\mathcal{F}_A \in S(U)$. Then the following are equivalent:*
(1) *$\mathcal{F}_A$ is a uni-soft ideal over U.*
(2) *The non-empty τ-exclusive set $\mathcal{F}_A$ is an ideal of A for any $\tau \subseteq U$.*

3   Uni-soft p-semisimple ideals

In what follows, let $E = X$ and X denote a BCI-algebra unless otherwise specified.

Definition 3.1. *Let A be a subalgebra of X and $\mathcal{F}_A \in S(U)$. Then $\mathcal{F}_A$ is called a uni-soft p-semisimple ideal over U if the approximate function f_A of $\mathcal{F}_A$ satisfies the following conditions:*

$$(\forall x \in A)(f_A(0) \subseteq f_A(x)); \tag{8}$$

$$(\forall x, y, z \in A)(f_A(x) \subseteq f_A((x * z) * (y * z)) \cup f_A(y)). \tag{9}$$

Example 3.2. *Let $U = \mathbb{Z}$ and $X = \mathbb{Q} - \{0\}$ be a BCI-algebra with a binary operation "$\div$" (usual division). Consider the subalgebra $A = \{2^n \mid n \in \mathbb{Z}\}$ of X and let $\mathcal{F}_A \in S(U)$ be such that its approximation function f_A is defined as follows:*

$$f_A : E \to \mathcal{P}(U), \quad f_A(x) = \begin{cases} 4\mathbb{Z} & \text{if } x = 2^n, n \geq 0 \\ 2\mathbb{Z} & \text{if } x = 2^n, n < 0 \\ \emptyset & \text{otherwise} \end{cases}$$

Easy calculations show that $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U.

Theorem 3.3. *Every uni-soft p-semisimple ideal is a uni-soft ideal.*

Proof. Let $\mathcal{F}_A$ be a uni-soft p-semisimple ideal over U, where A is a subalgebra of X. By taking $z = 0$ in (9) and using (a2), we get

$$f_A(x) \subseteq f_A((x * 0) * (y * 0)) \cup f_A(y) \subseteq f_A(x * y) \cup f_A(y)$$

for any $x, y \in A$. Therefore, $\mathcal{F}_A$ is a uni-soft ideal over U. $\qquad\square$

The following example shows that the converse of Theorem 3.3 is not true.

Example 3.4. *Let $U = \mathbb{N}$ and $X = \{0, 1, a, b\})$ be a BCI-algebra with a binary operation "$*$" given by the following table:*

$*$	0	1	a	b
0	0	0	b	a
1	1	0	b	a
a	a	a	0	b
b	b	b	a	0

Consider the subalgebra $A = \{0, 1\}$ of X and let $\mathcal{F}_A \in S(U)$ be such that its approximation function f_A is defined as follows:

$$f_A : E \to \mathcal{P}(U), \quad f_A(x) = \begin{cases} \{1\} & \text{if } x = 0 \\ \{1, 2\} & \text{if } x = 1 \\ \mathbb{N} & \text{if } x = a \\ \mathbb{N} & \text{if } x = b \end{cases}$$

Easy calculations show that $\mathcal{F}_A$ is a uni-soft ideal over U. But it is not a uni-soft p-semisimple ideal, since

$$f_A(1) = \{1, 2\} \nsubseteq f_A((1 * 1) * (0 * 1)) \cup f_A(0) = f_A(0) = \{1\}.$$

Proposition 3.5. *Let B be the BCK-part of X and $\mathcal{F}_B \in S(U)$. Then $\mathcal{F}_B$ is a uni-soft p-semisimple ideal over U.*

Proof. First by using $(b1)$, $(b2)$ and $(b7)$, we get

$$(y * z) * y = (y * y) * z = 0 * z = 0 \text{ and}$$
$$(x * (y * z)) * ((x * (y * z) * z) = (x * (y * z)) * ((x * z) * (y * z)) \leq (x * z) * x = 0 * z = 0,$$

for all $x, y, z \in B$. Hence, by using Lemma 2.5, we get

$$f_B(y * z) \subseteq f_B(y); \quad \text{and} \quad f_B(x * (y * z)) \subseteq f_B((x * z) * (y * z)),$$

for any $x, y, z \in B$. Thus, from (5), we obtain

$$f_B(x) \subseteq f_B(x * (y * z)) \cup f_B(y * z) \subseteq f_B((x * z) * (y * z)) \cup f_B(y),$$

for all $x, y, z \in X$. Therefore, $\mathcal{F}_B$ is a uni-soft p-semisimple over U. $\square$

Theorem 3.6. *Let A be a subalgebra of X and $\mathcal{F}_A \in S(U)$. Then the following are equivalent:*
(i) $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U.
(ii) $\mathcal{F}_A$ is a uni-soft ideal over U and the approximate function f_A of $\mathcal{F}_A$ satisfies the following condition:

$$(\forall x, y \in A)(f_A(x * y) \subseteq f_A((x * z) * (y * z))). \tag{10}$$

Proof. Assume that $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U. Then, by Theorem 3.3, $\mathcal{F}_A$ is a uni-soft ideal over U. To prove the condition (10), first, from $(b6)$, $(b3)$ and $(b2)$, we get

$$\begin{aligned}
((x * y) * z) * (0 * z)) * ((x * z) * (y * z)) &= ((x * z) * y) * (0 * z)) * ((x * z) * (y * z)) \\
&\leq (((x * z) * ((x * z) * (y * z))) * y) * (0 * z) \\
&\leq ((y * z) * y) * (0 * z) = ((y * y) * z) * (0 * z) \\
&= (0 * z) * (0 * z) = 0,
\end{aligned}$$

for all $x, y, z \in X$. Hence, by Lemma 2.5, we obtain

$$f_A(((x * y) * z) * (0 * z)) \subseteq f_A((x * z) * (y * z)). \tag{11}$$

Replacing x by $x * y$ and taking $y = 0$ in the condition (9), we have

$$\begin{aligned}
f_A(x * y) &\subseteq f_A(((x * y) * z) * (0 * z)) \cup f_A(0) \\
&\subseteq f_A(((x * y) * z) * (0 * z)) && \text{by (8)} \\
&\subseteq f_A((x * z) * (y * z)), && \text{by (11)}
\end{aligned}$$

for all $x, y, z \in X$. Therefore, the condition (10) holds.

Conversely, assume that $\mathcal{F}_A$ is a uni-soft ideal over U and the approximate function f_A of $\mathcal{F}_A$ satisfies the condition (10). Then using (5), we have

$$f_A(x) \subseteq f_A(x * y) \cup f_A(y) \subseteq f_A((x * z) * (y * z)) \cup f_A(y)$$

for all $x, y, z \in X$. Therefore, $\mathcal{F}_A$ is a uni-soft p-semisimple over U. □

Theorem 3.7. [19] *A BCI-algebras X is p-semisimple if the it satisfies*

$$(\forall x \in X)(0 * x = 0 \Rightarrow x = 0). \tag{12}$$

Theorem 3.8. *Let X be a p-semisimple BCI-algebra X. Then every uni-soft ideal is a uni-soft p-semisimple ideal.*

Proof. Assume that $\mathcal{F}_A$ is a uni-soft ideal of U, where A is a subalgebra of X. We have

$$
\begin{aligned}
& 0 * ((x * y) * ((x * z) * (y * z))) \\
&= (0 * (x * y)) * (0 * ((x * z) * (y * z))) && \text{by (b5)} \\
&= (0 * (0 * ((x * z) * (y * z)))) * (x * y) && \text{by (b2)} \\
&\leq ((x * z) * (y * z)) * (x * y) \leq (x * y) * (x * y) = 0, && \text{by (b5) and (b6)}
\end{aligned}
$$

for any $x, y, z \in X$. Hence, $0 * ((x * y) * ((x * z) * (y * z))) = 0$. Thus, by Theorem 3.7, we get $(x * y) * ((x * z) * (y * z)) = 0$, that is, $x * y \leq (x * z) * (y * z)$. Hence, by Lemma 2.5, we obtain $f_A(x * y) \subseteq f_A((x * z) * (y * z))$, for all $x, y, z \in X$. Therefore, $\mathcal{F}_A$ is a uni-soft p-semisimple ideal of U. □

The following theorem provide some conditions for a uni-soft ideal to be a uni-soft p-semisimple ideal.

Theorem 3.9. *Let A be a subalgebra of X and $\mathcal{F}_A \in S(U)$ be a uni-soft ideal over U. Then the following are equivalent:*
(i) $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U.
(ii) the approximate function f_A of $\mathcal{F}_A$ satisfies the following condition:

$$(\forall x \in A)(f_A(x) \subseteq f_A(0 * (0 * x))). \tag{13}$$

(iii) the approximate function f_A of $\mathcal{F}_A$ satisfies the following condition:

$$(\forall x, y \in X)(f_A(x) \subseteq f_A(y * (y * x))). \tag{14}$$

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathcal{F}_A \in S(U)$ is a uni-soft p-semisimple ideal over U. By taking $y = 0$ and replacing z by x in (10) of Theorem 3.6, we get

$$f_A(x) \subseteq f_A((x * x) * (0 * x)) \subseteq f_A(0 * (0 * x))),$$

for all $x \in X$. Therefore, $\mathcal{F}_A$ satisfies the condition (13).

$(ii) \Rightarrow (iii)$ Assume that the approximate function f_A of $\mathcal{F}_A$ satisfies the condition (13). Using (b1), (b2) and (b7), we have

$$0 * (0 * x) = (y * y) * ((y * y) * x) = (y * y) * ((y * x) * y) \leq y * (y * x),$$

for all $x, y \in A$. Then by Lemma 2.5 and Theorem 3.5, we get

$$f_A(x) \subseteq f_A(0 * (0 * x)) \subseteq f_A(y * (y * x)),$$

for all $x, y \in X$. Therefore, the approximate function f_A of $\mathcal{F}_A$ satisfies the condition (14).

$(iii) \Rightarrow (i)$ Assume that the approximate function f_A of $\mathcal{F}_A$ satisfies (14). Replacing y by $x * z$ in (14) and using $(b2)$ and Lemma 2.5, we get

$$
\begin{aligned}
f_A(x) \subseteq f_A((x * z) * ((x * z) * x)) &= f_A((x * z) * ((x * x) * z)) = f_A((x * z) * (0 * z)) \\
&\subseteq f_A(((x * z) * ((0 * z) * ((y * z) * (0 * z)))) \cup f_A((y * z) * (0 * z)) \\
&\subseteq f_A((x * z) * (y * z)) \cup f_A(y),
\end{aligned}
$$

for all $x, y, z \in X$. Therefore, $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U. $\qquad \square$

Theorem 3.10. *Let A be a subset of X and $\mathcal{F}_A \in S(U)$. Then the following are equivalent:*
(i) $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U.
(ii) The non-empty τ-exclusive set of $\mathcal{F}_A$ is a p-semisimple ideal of A for any $\tau \subseteq U$.

Proof. Assume that $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U. Then $\mathcal{F}_A$ is a uni-soft ideal by Theorem 3.3. Hence, by Lemma 2.6, $e(\mathcal{F}_A, \tau)$ is an ideal of A for all $\tau \subseteq U$. Let $\tau \subseteq U$ and $x \in A$ be such that $0 * (0 * x) \in e(\mathcal{F}_A, \tau)$. Thus $f_A(0 * (0 * x)) \in \tau$. On the other hand, by Theorem 3.9(ii), we have $f_A(x) \subseteq f_A(0 * (0 * x))$. Thus $f_A(x) \in \tau$ and so $x \in e(\mathcal{F}_A, \tau)$. Hence, by Lemma 2.1, $e(\mathcal{F}_A, \tau)$ is a p-semisimple ideal of A. Therefore, the τ-exclusive set is a p-semisimple ideal of A.

Conversely, assume that the non-empty τ-exclusive set of $\mathcal{F}_A$ is a p-semisimple ideal of A for any $\tau \subseteq U$. Thus, by Lemma 2.6, $\mathcal{F}_A$ is a uni-soft ideal over U. Let $x \in A$. We take $\tau = f_A(0 * (0 * x))$. Then $0 * (0 * x) \in e(\mathcal{F}_A, \tau)$, and so by Theorem 3.5, we get $x \in e(\mathcal{F}_A, \tau)$. Hence $f_A(x) \subseteq \tau = f_A(0 * (0 * x))$, that is $f_A(x) \subseteq f_A(0 * (0 * x))$. Therefore, $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U by Theorem 3.9(ii). $\qquad \square$

Theorem 3.11. *Let A be a subset of X and $\mathcal{F}_A \in S(U)$ be a uni-soft ideal over U. Then the following are equivalent:*
(i) The non-empty τ-exclusive set of $\mathcal{F}_A$ is a p-semisimple ideal of A for any $\tau \subseteq U$.
(ii) The BCK-part of X is contained in the τ-exclusive set of $\mathcal{F}_A$ for any $\tau \subseteq U$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in B$. Then $0 * x = 0$. Assume that $e(\mathcal{F}_A, \tau)$ be a non-empty τ-exclusive set of $\mathcal{F}_A$ for some $\tau \subseteq U$. Then there is $y \in A$ such that $y \in e(\mathcal{F}_A, \tau)$. Thus $f_A(y) \in \tau$. By Theorem 3.10(i), $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U. Hence $f_A(x) \subseteq f_A(0 * (0 * x)) = f_A(0) \subseteq f_A(y) \subseteq \tau$. Thus $f_A(x) \subseteq \tau$ and so $x \in e(\mathcal{F}_A, \tau)$. Therefore, B is contained in the τ-exclusive set of $\mathcal{F}_A$ for any $\tau \subseteq U$.

$(ii) \Rightarrow (i)$ Assume that $e(\mathcal{F}_A, \tau)$ is a non-empty τ-exclusive set of $\mathcal{F}_A$ for some $\tau \subseteq U$. Let $x \in e(\mathcal{F}_A, \tau)$. Then $f_A(x) \subseteq \tau$. We note that $x * (0 * (0 * x)) \in B$. Thus, by (ii), we have $x * (0 * (0 * x)) \in e(\mathcal{F}_A, \tau)$. Hence, by Lemma 2.6, we get $x \in e(\mathcal{F}_A, \tau)$. Therefore, the τ-exclusive set is a p-semisimple ideal of A. $\qquad \square$

4 Uni-soft strong ideals

Definition 4.1. *Let A be a subalgebra of X and $\mathcal{F}_A \in S(U)$. Then a uni-soft ideal $\mathcal{F}_A$ over U is called a uni-soft strong ideal if the approximate function f_A of $\mathcal{F}_A$ satisfies the following condition:*

$$(\forall x, y \in A)(f_A(x) \subseteq f_A(y * x) \cup f_A(y). \tag{15}$$

Example 4.2. *Let $U = \mathbb{Z}$ and $E = X = \{0, 1, a, b\}$ be a BCI-algebra in which the binary operation "$*$" is given by the following table:*

$*$	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

Consider the subalgebra $A = \{0, a, b\}$ of X and let $\mathcal{F}_A \in S(U)$ be such that its approximation function f_A is defined as follows:

$$f_A : E \to \mathcal{P}(U), \quad f_A(x) = \begin{cases} 4\mathbb{Z} & \text{if } x = 0 \\ \phi & \text{if } x = 1 \\ 2\mathbb{Z} & \text{if } x = a \\ 2\mathbb{Z} & \text{if } x = b \end{cases}$$

Easy calculations show that $\mathcal{F}_A$ is a uni-soft strong ideal over U.

The following theorem provide a condition for a uni-soft ideal to be a uni-soft stong ideal.

Theorem 4.3. *Let A be a subalgebra of X and $\mathcal{F}_A \in S(U)$ be a uni-soft ideal over U. Then the following are equivalent:*
(i) $\mathcal{F}_A$ is a uni-soft strong ideal over U.
(ii) The approximate function f_A of $\mathcal{F}_A$ satisfies the following condition:

$$(\forall x \in A)(f_A(x) \subseteq f_A(0 * x)). \tag{16}$$

Proof. Let $\mathcal{F}_A$ be a uni-soft strong ideal over U. Let $x \in X$. Then by taking $y = 0$ in (15) and using (6), we get $f_A(x) \subseteq f_A(0 * x)$.

Conversely, assume that the condition (16) holds. Then we have

$$\begin{aligned} f_A(x) \subseteq f_A(0 * x) &\subseteq f_A((0 * x) * (y * x)) \cup f_A(y * x) & \text{by (5)} \\ &\subseteq f_A(0 * y) \cup f_A(y * x) & \text{by (b7) and Lemma 2.5} \\ &\subseteq f_A(0 * (0 * y) \cup f_A(y * x) & \text{by (16)} \\ &\subseteq f_A(y) \cup f_A(y * x), & \text{by (b6) and Lemma 2.5} \end{aligned}$$

for all $x, y \in X$. Therefore, $\mathcal{F}_A$ is a uni-soft strong ideal over U. $\square$

A uni-soft p-semisimple ideal may not be a uni-soft strong ideal, as shown in the following example.

Example 4.4. *Consider the uni-soft p-semisimple ideal $\mathcal{F}_A$ as in Example 3.2. $\mathcal{F}_A$ is not a uni-soft strong, since for $n < 0$, $f_A(2^n) = 2\mathbb{Z} \nsubseteq f_A(1 \div 2^n) = f_A(2^{-n}) = 4\mathbb{Z}$.*

The following theorem gives the relationship between uni-soft strong ideals and uni-soft p-semisimple ideals.

Theorem 4.5. *Let A be a subalgebra of X and $\mathcal{F}_A \in S(U)$ be a uni-soft ideal over U. Then the following are equivalent:*
(i) $\mathcal{F}_A$ is a uni-soft strong ideal over U.
(ii) $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U and closed.

Proof. Let $\mathcal{F}_A$ be a uni-soft strong ideal over U. Then, by using (16) twice, we get $f_A(x) \subseteq f_A(0 * x) \subseteq f_A(0 * (0 * x))$ for any $x \in X$. Therefore, $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U by Theorem 3.5. To prove the closedness of $\mathcal{F}_A$, let $x \in X$. It follows from (16), (b6) and Lemma 2.5 that $f_A(0 * x) \subseteq f_A(0 * (0 * x)) \subseteq f_A(x)$ for all $x \in X$. Therefore, $\mathcal{F}_A$ is closed.

Conversely, assume that $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U and closed. Then, by Theorem 3.9(ii), we have $f_A(x) \subseteq f_A(0 * (0 * x))$ for any $x \in X$. On the other hand, by closedness of $\mathcal{F}_A$, we get $f_A(0 * (0 * x)) \subseteq f_A(0 * x)$. Therefore, $f_A(x) \subseteq f_A(0 * x)$, and so $\mathcal{F}_A$ is a uni-soft strong ideal over U. $\qquad\square$

Theorem 4.6. *Let X be a nilpotent BCI-algebra and A be a subalgebra of X. If $\mathcal{F}_A \in S(U)$ is a uni-soft ideal over U, then the following are equivalent:*
(i) $\mathcal{F}_A$ is a uni-soft strong ideal over U.
(ii) $\mathcal{F}_A$ is a uni-soft p-semisimple ideal over U.

Proof. $(i) \Rightarrow (ii)$ By Theorem 4.5, the result holds.

$(ii) \Rightarrow (i)$ Let $x \in X$. Then, since X is nilpotent, there is a positive integer n such that $0 * x^n = 0$. By Theorem 3.9(ii), we get $f_A(x) \subseteq f_A(0 * (0 * x))$. Thus, we have

$$
\begin{aligned}
f_A(x) = f_A((0 * (0 * x)) &= f_A((0 * x^n) * (0 * x)) \\
&= f_A(0 * x^{n-1}) * x) * (0 * x)) \\
&\subseteq f_A(0 * x^{n-1}) && \text{by (a1) and Lemma 2.5} \\
&\subseteq f_A((0 * x^{n-1}) * (0 * x)) \cup f_A(0 * x) && \text{by (5)} \\
&\subseteq f_A(0 * x^{n-2}) \cup f_A(0 * x) && \text{by (a1) and Lemma 2.5} \\
&\ \vdots && \text{by continuing this process} \\
&\subseteq f_A(0 * x) \cup f_A(0 * x) \subseteq f_A(0 * x),
\end{aligned}
$$

for all $x, y \in X$. Hence, $f_A(x) \subseteq f_A(0 * x)$ for all $x \in X$. Therefore, $\mathcal{F}_A$ is a uni-soft strong ideal over U. $\qquad\square$

5 Conclusions

In this paper, we introduced the concepts of uni-soft p-semisimple ideal and uni-soft strong ideal in a BCI-algebra and investigated some related properties. We gave some conditions for a uni-soft ideal to be a uni-soft p-semisimple ideal. Also, we gave a necessary and sufficient condition for a uni-soft ideal to be a uni-soft strong ideal. Finally, we investigated the relationships between uni-soft p-semisimple ideals and uni-soft strong ideals. In our future studies, we will introduce the concept of uni-soft implicative ideal and uni-soft positive implicative ideal in BCI-algebras and investigate related properties.

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