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k-nilpotent groups based on hypergroups

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Abstract

Fundamental relations are one of the main tools in connection relation between hyperstructures theory and structures theory. In this paper, we introduce a general fundamental relation on any hypergroup in such a way that all fundamental relations are a special case of this relation. Also, this study considered the notation of the relation on the derivation of k-nilpotent groups from any hypergroups.

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1 Introduction

Let G be a group and $k, n \in \mathbb{N}$. Then the lower k-central series of G is defined by $G = \gamma_0^k(G) \supseteq \gamma_1^k(G) \supseteq \ldots$, where

$$\gamma_{n+1}^k(G) = \langle \{[x, y] \mid x \in \gamma_n^k(G), y \in G^k = \{g^k \mid g \in G\}\} \rangle.$$

A group G is called a k-nilpotent group, if for some $n \in \mathbb{N}$ we have $\gamma_n^k(G) = \{1\}$, in particular for $k = 1$ it is a nilpotent group.

The hyperstructure theory was first introduced, by Marty at the 8th congress of Scandinavian Mathematicians in 1934 [7]. Marty introduced the concept of hypergroups as a generalization of groups and used it in different contexts like algebraic functions, rational fractions, and non-commutative groups. In classical algebraic structures, the synthetic result of two elements is an element, while in the hyper algebraic system, the synthetic result of two elements is a set of elements, therefore it can be said that the notion of hyperstructures is a generalization of classical

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algebraic structures, from this point of view. Hyperstructures have many applications to several sectors of both pure and applied sciences as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, relation algebras, C–algebras, artificial intelligence, probabilities, chemistry, physics, especially in atomic physics and in the harmonic analysis [3, 4].

In this paper, we introduce a strongly regular relation as extended fundamental relation on any hypergroup in such a way that some fundamental relations are a special case of this relation. The motivation of this relation is obtained from the connection between hypergroups and groups. This study introduces the concept of relation–part and investigates some properties of relation–part. Indeed, we apply the extended fundamental relation to constructing of relation–part. The main result of this paper is the derivation of the class of k–nilpotent groups from hypergroup.

2 Preliminaries

In this section, we review some definitions and results from [6], which we need in what follows.

Let H be a non-empty set and $P^*(H)$ be the family of all non-empty subsets of H. Every function $\cdot_i : H \times H \longrightarrow P^*(H)$ where $i \in \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$ is called a *hyperoperation.* For all x, y of H, $\cdot_i(x, y)$ is called a *hyperproduct* of x, y. An algebraic system $(H, \cdot_1, \cdot_2, \dots, \cdot_n)$ is called a *hyperstructure* and binary structure $(H, \cdot)$ endowed with only hyperoperation is called a *hypergroupoid.* For every two non-empty subsets A and B of $H, A \cdot B$ means $\bigcup\limits_{a\in A, b\in B} a \cdot b$. Recall that a *hypergroupoid* $(H, \cdot)$ is called a *semihypergroup* if for any $x, y, z \in H, (x \cdot y) \cdot z = x \cdot (y \cdot z)$ and a semihypergroup $(H, \cdot)$ is called a *hypergroup* if satisfies in the *reproduction axiom*, i. e. for any $x \in H, x \cdot H = H \cdot x = H$. A semihypergroup $(H, \cdot)$ is called a *polygroup*, provided that (i) it has a scalar identity e (i.e., $e \cdot x = x \cdot e = \{x\}$, for every $x \in H$), (ii) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$, where -1 is a unitary operation on H(it follows that every element x of H has a unique inverse x^{-1} in H i.e $e \in (x \cdot x^{-1}) \cap (x^{-1} \cdot x), e^{-1} = e, (x^{-1})^{-1} = x$) and we will denote it by $(H, \cdot, e, ^{-1})$. A non-empty subset K of H is said to be a sub-polygroup of H, if for any $x, y \in K, x \cdot y^{-1} \subseteq K$ and is denoted by $K \leq H$. Let X be a non-empty subset of a polygroup H define the sub-polygroup generated by X, $\langle X\rangle$ to be the intersection of all sub-polygroups of H which contain X.

In every hypergroup H, a *commutator* of $x, y \in H$ is denoted by $[x, y] = \{h \in H \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset\}$ and $H = L_0(H) \supseteq L_1(H) \supseteq \cdots$ is called a *lower series* of H, where for all $n \in \mathbb{N}^*$, $L_{n+1}(H) = \{h \in [x, y] \mid x \in L_n(H), y \in H\}$. Also, $H = \Gamma_0(H) \supseteq \Gamma_1(H) \supseteq \cdots$ is called a *derived series* of H, where for all $n \in \mathbb{N}^*$, $\Gamma_{n+1}(H) = \{h \in [x, y] \mid x, y \in \Gamma_n(H)\}$.

The polygroup $(H, \cdot, e, ^{-1})$ is called a *nilpotent polygroup*, if for some integer $n \in \mathbb{N}, l_n(H) \cdot w_H = w_H$, where $l_{n+1}(H) = \langle\{h \in [x, y] \mid x \in l_n(H), y \in H\}\rangle$ and $l_0(H) = H$ (if there exists the smallest integer c in such a way that $l_c(H) \cdot w_H = w_H$, then c is called the *nilpotency class* for H). Also for all $n \in \mathbb{N}$, we have $H' = H^{(1)} = \langle\Gamma_1(H)\rangle$ and $H^{(n+1)} = (H^{(n)})'$.

Let $(H, \cdot)$ be a hypergroup and ρ be an equivalence relation on H. Letting $\frac{H}{\rho} = \{\rho(g) \mid g \in H\}$, be the set of all equivalence classes of H with respect ρ. Define a hyper operation $*$ by $\rho(a) * \rho(b) = \{\rho(c) \mid c \in \rho(a) \cdot \rho(b)\}$. In [5] it was proved that $(\frac{H}{\rho}, *)$ is a hypergroup if and only if ρ is a regular equivalence relation. Moreover, $(\frac{H}{\rho}, *)$ is a group if and only if ρ is a strongly regular equivalence relation([5]). Let $\mathcal{U}(H)$ denote the set of all finite product of elements of H. Define relation β on H by $a\beta b \Longleftrightarrow \exists\ u \in \mathcal{U}(H)$ such that $\{a, b\} \subseteq u$. In [5] it was proved that β^* is the *transitive closure* of β (the smallest transitive relation such that contains β), and $(\frac{H}{\beta^*}, *)$ is called the *fundamental* *group* of $(H, \cdot)$. In [5] it was rewritten the definition of β^* on H as follows:

$$a\beta^* b \Longleftrightarrow \exists z_1 = a, z_2, ..., z_{n+1} = b \in H, u_1, u_2, \cdots, u_n \in \mathcal{U} \text{ s.t } \{z_i, z_{i+1}\} \subseteq u_i, \forall\, 1 \le i \le n.$$

Also, Freni, introduced a strongly regular relation γ on hypergroup H as follows: $\gamma_1 = \{(x, x) \mid x \in H\}$ and for all $n \ge 2, (x, y) \in \gamma_n$ if and only if there exist $z_1, z_2, \ldots, z_n \in H$, $\sigma \in S_n$ such that $x \in \prod_{i=1}^{n} z_i$, $y \in \prod_{i=1}^{n} z_{\sigma(i)}$ and $\gamma = \bigcup_{n\ge 1} \gamma_n$, in addition, it was proved that G/γ^* is an abelian group [5].

Davvaz et. al introduced the relation $\nu_n = \bigcup_{m\ge 1} \nu_{m,n}$, where $\nu_{1,n} = \gamma_1$ and for every $m > 1, \nu_{m,n}$ is defined by, $(a, b) \in \nu_{m,n} \Leftrightarrow\ \exists\ u = \prod_{i=1}^{m} z_i \in \mathcal{U},\ \exists \sigma \in S_m$ such that $\sigma(i) = i$ if $z_i \notin L_n(H)$ and $a \in u, b \in u_\sigma$, in addition, it was proved that G/γ^* is a nilpotent group. Also $\tau_n = \bigcup_{m\ge 1} \tau_{m,n}$, where $\tau_{1,n} = \{(x, x) \mid x \in H\}$ and for every $m > 1, \tau_{m,n}$ is defined by, $(a, b) \in \tau_{m,n} \Leftrightarrow\ \exists\ u = \prod_{i=1}^{m} z_i$, $\exists \sigma \in S_m : \sigma(i) = i$ if $z_i \notin \Gamma_n(H)$ and $a \in u, b \in u_\sigma$, in addition, it was proved that G/γ^* is a solvable group [1, 2].

The map $f : H_1 \to H_2$ is called a homomorphism of hypergroups if for all $x, y \in H_1$, we have $f(x \cdot y) = f(x) \cdot f(y)$. A homomorphism f is called an isomorphism if f is a one-to-one and onto a map, also we define $Aut(H) = \{f : H \to H \mid f \text{ is an isomorphism on hypergroup } H\}$. Let $\varphi : H \longrightarrow H/\beta^*$ by $\varphi(x) = \beta^*(x)$ be the *canonical* homomorphism. Then $w_H = \{x \in H \mid \varphi(x) = 1\}$ is called *heart* of H.

3 Relation-part in hypergroups

In this section, we introduce a fundamental relation on hypergroups such that it is a generalization of fundamental relations such as β^* and γ^*. Also, the concept of relation-part in hypergroups is defined and is obtained some relation-part with respect to this extended fundamental relation on hypergroups.

Definition 3.1. *Let H be a hypergroup and $K \subseteq H$. Define $R_{1,K} = \{(x, x) \mid x \in H\}$ and for all $2 \le n \in \mathbb{N}$:*

$$(x, y) \in R_{n,K} \Leftrightarrow \exists\ (z_1, \ldots, z_n) \in H^n, u = \prod_{i=1}^{n} z_i,\ \exists\ \sigma \in S_n, \quad \text{such that } x \in u, y \in u_\sigma$$

and for all $1 \le i \le n, z_i \in K$ implies that $\sigma(i) = i$, where $u_\sigma = \prod_{i=1}^{n} z_{\sigma(i)}$.

Obviously, $R_K = \bigcup_{n\ge 1} R_{n,K}$ is a reflexive and symmetric relation. Let R_K^* be the transitive closure of R_K (the smallest transitive relation in such a way that contains R_K), then we have the following results.

Example 3.2. *Let $H = \{a, b, c\}$. Consider the hypergroup $(H, \cdot)$ as follows:*

$\cdot$	a	b	c
a	$\{b, c\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{c\}$
c	$\{a\}$	$\{b, c\}$	$\{b, c\}$

.

If $K = \{a, b\}$*, then* $R_K = R_{1,K} \cup \{(b, c), (c, b)\} = R^*_K = \beta^*$*.*

Theorem 3.3. *Let* H *be a hypergroup and* $K \subseteq H$*. Then* R^*_K *is a strongly regular relation on* H*.*

Proof. Let $(x, y) \in R_K$ and $z \in H$. Then there exist $(z_1, \ldots, z_n) \in H^n, u = \prod_{i=1}^{n} z_i$ and $\sigma \in S_n$ in such a way that $x \in u, y \in u_\sigma$ and for all $1 \leq i \leq n, z_i \in K$ implies $\sigma(i) = i$. So we have $x \cdot z \subseteq u \cdot z$, $y \cdot z \subseteq u_\sigma \cdot z$ and $z_i \in K$ implies that $\sigma(i) = i$. Consider $z_{n+1} = z$, $\alpha(i) = \sigma(i)$, where $i \in \{1, \ldots, n\}$ and $\alpha(n+1) = n+1$. Thus $x \cdot z \subseteq \prod_{i=1}^{n+1} z_i = v$ and $y \cdot z \subseteq v_\alpha$ such that $z_i \in K$ implies $\alpha(i) = i$. It follows that $(x \cdot z)\ \overline{\overline{R^*_K}}\ (y \cdot z)$. In a similar way, we have $(z \cdot x)\ \overline{\overline{R^*_K}}\ (z \cdot y)$. Hence R^*_K is a strongly regular relation on H. $\square$

Theorem 3.4. *Let* H *be a hypergroup. Then* $\dfrac{H}{R^*_K}$ *is a group.*

Example 3.5. *Let* H *be a hypergroup and* $n \in \mathbb{N}$*. Then*

(i) *if* H *is a commutative hypergroup, then* $R_K = \beta = \gamma = \tau_n = \nu_n$*;*

(ii) *if* $K = H$*, then* $R_K = \beta$*;*

(iii) *if* $K = \emptyset$*, then* $R_K = \gamma$*;*

(iv) *if* $K = H \setminus \Gamma_n(H)$*, then* $R_K = \tau_n$*;*

(v) *if* $K = H \setminus L_n(H)$*, then* $R_K = \nu_n$*;*

(vi) *if* $K = \{x \in H \mid x \cdot h = h \cdot x, \forall h \in H\}$*, then* H/R^*_K *is an abelian group.*

Definition 3.6. *Let* H *be a hypergroup,* $A \subseteq H$ *and* R *be a strongly regular relation on* H*. We say that* A *is an* R*-complete part of* H *(or simply* R*-part) if for every* $x \in A$ *and* $y \in H$*,* $(x, y) \in R$ *implies that* $y \in A$ *and it will be denoted by* $A \sqsubseteq_R H$*. Clearly* $H \sqsubseteq_R H$*.*

Consider the hypergroup as in Example 3.2 and $R = \beta$. Then $\{a\} \sqsubseteq_\beta H$.

Theorem 3.7. *Let* H *be a hypergroup, and* R *be a strongly regular relation on* H*. If* $A, B \sqsubseteq H$*, and* $A\overline{\overline{R}}B$*, then* $A = B$*.*

Theorem 3.8. *Let* H *be a hypergroup and* $\emptyset \neq M \subseteq H$ *and* $K \subseteq H$*. Then* M *is called an* R_K*-part of* H *if for any* $n \in \mathbb{N}$*,* $z_1, \ldots, z_n \in H$ *and* $\sigma \in S_n$ *such that* $z_i \in K$ *implies that* $\sigma(i) = i$*, then* $\prod_{i=1}^{n} z_i \cap M \neq \emptyset$ *implies* $\prod_{i=1}^{n} z_{\sigma(i)} \subseteq M$*.*

Corollary 3.9. *Let* H *be a hypergroup and* $A \subseteq H$*. Then*

(i) $A \sqsubseteq_{R_H} H$ *if and only if for all* $u \in \mathcal{U}$*,* $u \cap A \neq \emptyset$ *implies* $u \subseteq A$*.*

(ii) $A \sqsubseteq_{R_\emptyset} H$ *if and only if for all* $n \in \mathbb{N}$ *and* $u = \prod_{i=1}^{n} z_i \in \mathcal{U}$*,* $u \cap A \neq \emptyset$ *implies for all* $\sigma \in S_n$*,* $u_\sigma \subseteq A$*.*

Theorem 3.10. *Let H be a hypergroup, $A \subseteq H$ and R_1, R_2 be strongly regular relations on H. Then*

(i) *If $R_1 \subseteq R_2$ and A is an R_2-part of H, then A is an R_1-part;*

(ii) *If A is an R_1-part or R_2-part, then A is an $(R_1 \cap R_2)$-part;*

(iii) *If A is an R_1-part and R_2-part, then A is an $(R_1 \cup R_2)$-part.*

Theorem 3.11. *Let H be a hypergroup and R be a strongly regular relation on H.*

(i) *If $K_1, K_2 \sqsubseteq_R H$, then $K_1 \cap K_2 \sqsubseteq^R H$.*

(ii) *If $K_1, K_2 \sqsubseteq_R H$, then $K_1 \cup K_2 \sqsubseteq^R H$.*

(ii) *For all $a \in H$, $R(a) \sqsubseteq_R H$.*

Definition 3.12. *Let H be a hypergroup and $A \subseteq H$. The intersection of all R-complete parts of H which contains A is called an R-closure of A in H and it will be denoted by $C_R(A)$. Consider $T_1(A) = A$ and for every $n \in \mathbb{N}$,*

$$T_{n+1}(A) = \{x \in H \mid \exists\ y \in T_n(A)\ s.t\ (x, y) \in R\} \text{ and } T(A) = \bigcup_{n \geq 1} T_n(A).$$

Example 3.13. *Let $H = \{e, a, b\}$. Consider the polygroup $(H, \cdot)$ as follows:*

$\cdot$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{b, a\}$	$\{e, a\}$

.

For $R = H \times H$, $A = \{e, a\}$ and for all $n \geq 1$, we have $T_n(A) = H$.

From now on, we consider R is a strongly regular relation on hypergroup H and $\sqsubseteq_R$ will by $\sqsubseteq$.

Theorem 3.14. *Let H be a hypergroup and $\emptyset \neq A \subseteq H$. Then*

(i) *$C_R(A) = T(A)$;*

(ii) *$C_R(A) = \bigcup_{a \in A} C_R(a)$.*

Proof. (i) Let $x \in T(A)$ and $y \in H$. Then $(x, y) \in R$ implies that there exists $n \in \mathbb{N}$ such that $x \in T_n(A)$. So we have $y \in T_{n+1}(A)$. In addition, if $A \subseteq B$ and $B \sqsubseteq H$, then by induction we show that $T(A) \subseteq B$. Clearly $T_1(A) = A \subseteq B$. Suppose $T_n(A) \subseteq B$. Thus for every $x \in T_{n+1}(A)$, there exists $y \in T_n(A)$ such that $(x, y) \in R$. Since $B \sqsubseteq H$, we get $x \in B$.

(ii) By induction, we have $T_n(A) \subseteq \bigcup_{a \in A} T_n(a)$. Hence $C_R(A) = \bigcup_{a \in A} C_R(a)$. □

Lemma 3.15. *Let H be a hypergroup, $x \in H$ and $n \in \mathbb{N}$. Then*

(i) *$T_n(T_2(x)) = T_{n+1}(x)$;*

(ii) *for all $x, y \in H$ and $n \in \mathbb{N}$, $x \in T_n(y)$ if and only if $y \in T_n(x)$.*

Proof. By definition, $T_1(T_2(x)) = T_2(x)$. If $T_{n-1}(T_2(x)) = T_n(x)$, then by induction,

$$\begin{aligned} T_n(T_2(x)) &= \{x \mid \exists\, y \in T_{n-1}(T_2(x)) \text{ and } (x,y) \in R\} \\ &= \{x \mid \exists\, y \in T_n(x) \text{ and } (x,y) \in R\} = T_{n+1}(x) \end{aligned}$$

(ii) It is clear that $x \in T_1(y) \Leftrightarrow y \in T_1(x)$. Let for every $x, y \in H$, $x \in T_{n-1}(y)$ if and only if $y \in T_{n-1}(x)$. If $x \in T_n(y)$, then there exists $z \in T_{n-1}(y)$ such that $(x,z) \in R$. Using hypotheses of induction, we conclude that $y \in T_{n-1}(z)$. Moreover, $x \in T_1(x)$, and $(x,z) \in R$ implies $z \in T_2(x)$. Hence $y \in T_{n-1}(z) \subseteq T_{n-1}(T_2(x)) = T_n(x)$. □

Theorem 3.16. *Let H be a hypergroup and $S = \{(x,y) \mid x \in T(y)\}$. Then $S = R$.*

Proof. Let $x, y \in H$. Since $(x,y) \in S$ we have $x \in T(y)$, then there exists $n \in \mathbb{N}$ such that $x \in T_n(y)$. Thus, $z_1 \in T_{n-1}(y)$ such that $(x, z_1) \in R$ Hence, there is $z_2 \in T_{n-2}(y)$ such that $(z_1, z_2) \in R$. Then there exists $z_{n-1} \in T_1(y) = \{y\}$ such that $(z_{n-2}, z_{n-1}) \in R$. So $(x,y) \in R$.

Conversely

$$(x,y) \in R \Rightarrow y \in T_1(y) \text{ and } (x,y) \in R \Rightarrow x \in T_2(y) \Rightarrow (x,y) \in S.$$

□

Theorem 3.17. *Let H be a hypergroup and $\emptyset \neq A \subseteq H$. Then $C_R(A) = \bigcup_{a \in A} R(a)$.*

Proof. Let $x \in H$. Then

$$\begin{aligned} x \in C_R(A) &\Leftrightarrow \exists\, a \in A \text{ such that } x \in C_R(a) \\ &\Leftrightarrow \exists\, a \in A \text{ such that } (a,x) \in T \\ &\Leftrightarrow \exists\, a \in A \text{ such that } (a,x) \in R \\ &\Leftrightarrow \exists\, a \in A : x \in R(a) \\ &\Leftrightarrow x \in \bigcup_{a \in A} R(a). \end{aligned}$$

□

Let R be a strongly regular relation on a hypergroup H and $\pi : H \to H/R$ by $\pi(x) = R(x)$ be the canonical homomorphism and $w_R = \{x \in H \mid \pi(x) = 1\}$. Then w_R is called an R-heart of H.

Proposition 3.18. *Let H be a hypergroup and $\emptyset \neq A \subseteq H$. Then*

(i) $w_R = \pi^{-1}(1_{H/R})$ *is a sub-hypergroup of* H*;*

(ii) $\pi^{-1}\pi(A) = w_R \cdot A = A \cdot w_R$.

Proof. (i) It is concluded immediately.

(ii) Suppose that $x \in \pi^{-1}\pi(A)$, then there is $a \in A$ such that $\pi(x) = \pi(a)$ and by the reproduction axim there exists $u \in H$ such that $x \in a \cdot u$ so $R(x) = R(a) \cdot R(u)$. It follows $R(u) = 1_{H/R}$, thus $u \in w_R$ and $x \in A \cdot w_R$. Conversely, if $x \in A \cdot w_R$ then there are $a \in A$, $w \in w_R$ such that $x \in a \cdot w$ so $\pi(x) = R(x) = R(a) = \pi(a)$ and so $x \in \pi^{-1}\pi(A)$. In a similar way, we can prove $w_R \cdot A = \pi^{-1}\pi(A)$. □

Theorem 3.19. *Let H be a hypergroup and $\emptyset \neq A \subseteq H$. Then $\pi^{-1}\pi(A) = C_R(A)$.*

Proof. Let $x \in R$. Then $x \in \pi^{-1}\pi(A)$ if and only if there is $a \in A$ such that $\pi(x) = \pi(a)$ if and only if there exists $a \in A$ such that $R(x) = R(a)$ if and only if there exists $a \in A$ such that $(x, a) \in R = T$ if and only if there exists $a \in A$, $x \in C_R(a) \subseteq C_R(A)$ if and only if $x \in C_R(A)$. $\square$

Corollary 3.20. *Let H be a hypergroup and $\emptyset \neq A \subseteq H$. Then*

(*i*) $C_R(A) = \pi^{-1}\pi(A) = w_R \cdot A = A \cdot w_R$;

(*ii*) *if $w \in w_R$ then $C_R(w) = w_R$.*

Corollary 3.21. *Let H be a polygroup and $\emptyset \neq A, B \subseteq H$. If A is an R-part of H, then*

(*i*) *A is a complete part of H;*

(*ii*) *for every $x \in H$, we have $x \cdot x^{-1} \cdot A = A$;*

(*iii*) *A^{-1} is a complete part of H;*

(*iv*) *for all $x \in P$, we have $x \cdot A$ and $A \cdot x$ are complete parts of H;*

(*v*) *$A \cdot B$ and $B \cdot A$ are complete parts of H;*

(*vi*) *if for every $i \in I$, A_i is an R-part, then $\bigcup\limits_{i \in I} A_i$ and $\bigcap\limits_{i \in I} A_i$ are R-parts of H;*

(*ix*) *if H is a commutative polygroup and $N \leq H$ is an R-part, then $N \trianglelefteq H$.*

Theorem 3.22. *Let H be a hypergroup and $\emptyset \neq M, K \subseteq H$. Then the following statements are equivalent:*

(*i*) *M is an R_K-part of H;*

(*ii*) *if $x \in M$ and $(x, y) \in R_K$, then $y \in M$;*

(*iii*) *if $x \in M$ and $(x, y) \in R_K^*$, then $y \in M$;*

Theorem 3.23. *Let H be a hypergroup and $K \subseteq H$. Then the following conditions are equivalent:*

(*i*) *for all $a \in H$, $R_K(a)$ is an R_K-part of H;*

(*ii*) $R_K = R_K^*$.

Remark 3.24. *Consider $K = H$, then $R_H = \beta$ and every β-part is called the complete part of H.*

Definition 3.25. *Let H be a hypergroup and $A \subseteq H$. The intersection of all R_K-parts which contain A is called an R_K-closure of A in H and it will be denoted by $C(A)$. Consider $K_1(A) = A$ and for every $n \in \mathbb{N}$, $K_{n+1}(A) = \{x \in H \mid \exists\ u = \prod\limits_{i=1}^{m} z_i, x \in u$ and $\exists\ \sigma \in S_m$ such that $\sigma(i) = i$ if $z_i \in K$ and $u_\sigma \cap K_n(A) \neq \emptyset$ and $K(A) = \bigcup\limits_{n \in \mathbb{N}} K_n(A)$.*

Theorem 3.26. *Let H be a hypergroup and $A \subseteq H$. Then*

(*i*) $C(A) = K(A)$;

(ii) $C(A) = \bigcup_{a \in A} C(a)$.

Theorem 3.27. *Let H be a hypergroup. Then for all $x, y \in H$ and for all $n \in \mathbb{N}$:*

(i) $K_n(K_2(x)) = K_{n+1}(x)$;

(ii) $x \in K_n(y)$ *if and only if* $y \in K_n(x)$.

Let H be a hypergroup and $T = \{(x, y) \mid x \in K(y)\}$. Then T is an equivalence relation and we have the following results.

Lemma 3.28. *Let H be a hypergroup. Then $R_K^* = T$.*

Proof. Let $(x, y) \in R_K$. Then there exist $u = \prod_{i=1}^{m} z_i$ and $\sigma \in S_m$ such that $z_i \in K$ implies $\sigma(i) = i$, $x \in u$ and $y \in u_\sigma$. It follows that $x \in u$ and $y \in u_\sigma \cap K_1(y)$, thus $x \in K_2(y)$. Then $(x, y) \in T$ and $R_K^* \subseteq T^* = T$.

Conversely, if $(x, y) \in T$, then $x \in K(y) = \bigcup_{n \geq 1} K_n(y)$ and so there is $n \in \mathbb{N}$ such that $x \in K_{n+1}(y)$ and by definition there are $u_1 = \prod_{i=1}^{n_1} z_{1_i}$ and $\sigma_1 \in S_{n_1}$ with $\sigma_1(1_i) = 1_i$ if $z_{1_i} \in K$, such that $x \in u_1$ and $u_{1_{\sigma_1}} \cap K_n(y) \neq \emptyset$. Hence there exists $x_1 \in u_{1_{\sigma_1}} \cap K_n(y)$ and so $(x, x_1) \in R_K$. Now, $x_1 \in K_n(y)$ and by definition there are $u_2 = \prod_{i=1}^{n_2} z_{2_i}$, $\sigma_2 \in S_{n_2}$ with $\sigma_2(2_i) = 2_i$ if $z_{2_i} \in K$ and $x_1 \in u_2$, $x_2 \in u_{2_{\sigma_2}} \cap K_{n-1}(y) \neq \emptyset$. So $x_1 \in u_2$ and $x_2 \in u_{2_{\sigma_2}}$. It implies that $(x_1, x_2) \in R_K$ and by induction, there are u_n and σ_n and x_n such that $x_n \in u_{n_{\sigma_n}} \cap K_1(y)$, $(x_{n-1}, x_n) \in R_K$. In addition, $x_n = y$ and $(x, y) \in R_K^*$, so $T \subseteq R_K^*$. □

Let R be a strongly regular relation on a hypergroup H and $\pi : H \to H/R$ be the canonical homomorphism. Then $w_K = \pi^{-1}(1_{H/R})$ is a sub-hypergroup of H. If $R = R_K^*$, then $\pi^{-1}(1_{H/R})$ is called an R_K-heart.

Proposition 3.29. *Let H be a hypergroup and $A \subseteq H$. Then $\pi^{-1}\pi(A) = w_K \cdot A = A \cdot w_K$.*

Theorem 3.30. *Let H be a hypergroup and $A \subseteq H$. Then $\pi^{-1}\pi(A) = C(A)$.*

Corollary 3.31. *Let H be a hypergroup and $\emptyset \neq A \subseteq H$. Then*

(i) $C(A) = \pi^{-1}\pi(A) = w_K \cdot A = A \cdot w_K$;

(ii) *if* $w \in w_K$, *then* $C(w) = w_K$.

Corollary 3.32. *Let H be a polygroup, $x \in H$ and $A, B \subseteq H$. If A is an R_K-part of H, then*

(i) *A is a complete part of H;*

(ii) $x \cdot x^{-1} \cdot A = A$;

(iii) *A^{-1} is a complete part of H;*

(iv) *$x \cdot A$ and $A \cdot x$ are complete parts of H;*

(v) *$A \cdot B$ and $B \cdot A$ are complete parts of H;*

(vi) *if for every $i \in I$, A_i is an R_K-part, then $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ are complete parts of H;*

(vii) *if H is a commutative polygroup and $N \leq H$ is an R_K-part of H, then $N \trianglelefteq H$.*

Theorem 3.33. *Let H be a polygroup. Then $R(w_K) = \{(x, y) \mid x \cdot y^{-1} \cap w_K \neq \emptyset\} = R_K^*$.*

Proof. Let $x, y \in H$. Then we have

$$(x, y) \in R_K^* \text{ if and only if } R_K^*(x) = R_K^*(y) \text{ if and only if } R_K^*(x) \cdot R_K^*(y)^{-1} = 1$$
$$\text{if and only if } x \cdot y^{-1} \cap w_K \neq \emptyset.$$

□

Example 3.34. *Consider the polygroup $H = \{1, 2, 3, 4, 5, 6, 7\}$ as follows:*

$\cdot$	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
2	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
3	$\{3\}$	$\{3\}$	$\{1,2\}$	$\{7\}$	$\{6\}$	$\{5\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{6\}$	$\{1,2\}$	$\{7\}$	$\{3\}$	$\{5\}$
5	$\{5\}$	$\{5\}$	$\{7\}$	$\{6\}$	$\{1,2\}$	$\{4\}$	$\{3\}$
6	$\{6\}$	$\{6\}$	$\{4\}$	$\{5\}$	$\{3\}$	$\{7\}$	$\{1,2\}$
7	$\{7\}$	$\{7\}$	$\{5\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{6\}$

If $K = \{1, 6\}$ and $A = \{7\}$, then it is easy to see that $K(A) = \{1, 2, 6, 7\}$. Also, $w_K = \{1, 2, 6, 7\}$. If $A = \{3\}$, then $K(A) = \{3, 4, 5\}$ and by Corollary 3.31, $C(3) = 3 \cdot \{1, 2, 6, 7\} = \{3, 4, 5\}$.

Example 3.35. *Let $H = \{a, b, c\}$ be a hypergroup as follows:*

$\cdot$	a	b	c
a	$\{b,c\}$	a	a
b	a	b	c
c	a	$\{b,c\}$	$\{b,c\}$

If $K = \{a, b\}$, then $w_K = \{b, c\} = K(b) = K(c)$, $K(a) = \{a\}$, $C(a) = a \cdot w_K = \{a\}$ and $T = \Delta \cup \{(b, c), (c, b)\}$.

Example 3.36. *Let $H = \{e, a, b, c, d, f, g\}$ and $(H, \circ)$ be a polygroup as follows:*

$\circ$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	b	$\{e,g\}$	f	c	d	a
b	b	$\{e,g\}$	a	d	f	c	b
c	c	d	f	$\{e,g\}$	a	b	c
d	d	f	c	b	$\{e,g\}$	a	d
f	f	c	d	a	b	$\{e,g\}$	f
g	g	a	b	c	d	f	e

If $K = H$ and $A = \{a, c, d\}$, then $w_K = \{e, g\}$, $C(A) = A \circ w_K = \{a, c, d\} \circ \{e, g\} = \{a, c, d\}$ and $T = \Delta \cup \{(e, g), (g, e)\}$.

3.1 Transitivity conditions of R_K

In this subsection, we show that for any hypergroup H and for all $K \subseteq H$, R_K is a transitive relation.

Definition 3.37. *Let H be a hypergroup and $x \in H$. Then*

(i) $\mathcal{U}_n(x) = \{u \in \mathcal{U}(H) \mid x \in u = \prod_{i=1}^{n} z_i,\ z_i \in H\}$,

(ii) $\mathcal{U}(x) = \{u \in \mathcal{U}(H) \mid \exists n \in \mathbb{N}\ s.t,\ u \in \mathcal{U}_n(x)\}$,

(iii) $P(x) = \bigcup\{u_\sigma \mid \exists n \in \mathbb{N},\ \sigma \in S_n,\ u = \prod_{i=1}^{n} z_i \in \mathcal{U}_n(x),\ \sigma(i) = i,\ if\ z_i \in K\}$.

Example 3.38. *Consider the hypergroup H which is defined in* (3.35) *and $K = H$. Then $\mathcal{U}_2(a) = \{ab, ba, ca, ac\}$ and $P(x) = \{b, c\}$.*

Lemma 3.39. *Let H be a hypergroup and $x \in H$. Then $P(x) = \{y \in H \mid (x, y) \in R_K\}$.*

Proof. Let $y \in H$ and $(x, y) \in R_K$. Then there exist $u = \prod_{i=1}^{n} z_i \in \mathcal{U}_n(x)$ and $\sigma \in S_n$ such that $\sigma(i) = i$ if $z_i \in K$ and $y \in u_\sigma$ so $y \in P(x)$. □

Theorem 3.40. *Let H be a hypergroup. Then the following conditions are equivalent:*

(i) $R_K = R_K^*$;

(ii) *for all $x \in H$, $R_K^*(x) = P(x)$;*

(iii) *for all $x \in H$, $P(x)$ is an R_K-part of H.*

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in H$. Then

$$y \in R_K^*(x) \Leftrightarrow (x, y) \in R_K^* = R_K \Leftrightarrow y \in P(x).$$

$(ii) \Rightarrow (iii)$ Let $u = \prod_{i=1}^{n} z_i \in \mathcal{U}(H)$ and $a \in u \cap P(x)$, then $(a, x) \in R_K$. For every $\sigma \in S_n$ such that $\sigma(i) = i$ if $z_i \in K$ and $y \in u_\sigma$ we have $(a, y) \in R_K$ so $(x, y) \in R_K^*$. Thus $y \in R_K^*(x) = P(x)$ and thus $u_\sigma \subseteq P(x)$.

$(iii) \Rightarrow (i)$ Let $(x, y) \in R_K^*$. Then $x \in P(x)$, $(x, y) \in R_K^*$ and by Lemma 3.22, we conclude that $y \in P(x)$. Thus there exist $u = \prod_{i=1}^{n} z_i \in \mathcal{U}(x)$ and $\sigma \in S_n$ such that $\sigma(i) = i$ if $z_i \in K$ and $y \in u_\sigma$ so $(x, y) \in R_K$. □

Theorem 3.41. *Let H be a hypergroup with an identity e (for all $h \in H, e \cdot h = h \cdot e = h$). Then $R_K = R_K^*$.*

Proof. Let $z \in H$ and $u = \prod_{i=1}^{m} z_i \in \mathcal{U}(H)$ such that $x \in u \cap P(z)$. If $x = z$, then for every $\sigma \in S_m$ such that $\sigma(i) = i$ if $z_i \in K$ and for every $y \in u_\sigma$ we have $x = z$, $(x, y) \in R_K$ and so $y \in P(z)$. It follows that $u_\sigma \subseteq P(z)$.

Let $x \neq z$ and $\sigma \in S_m$ such that $\sigma(i) = i$ if $z_i \in K$. Then $(x, z) \in R_K \setminus \Delta$ and so there exist $w_i \in H$, $i = 1, \ldots, n$ and $\alpha \in S_n$ such that $\alpha(i) = i$ if $w_i \in K$, $z \in \prod_{i=1}^{n} w_i$ and $x \in \prod_{i=1}^{n} w_{\alpha(i)}$. By the reproduction axiom, there exist $a, b \in H$ such that $z \in x \cdot b$ and $z_{m+1} = e \in a \cdot z$. Hence

$$z \in x \cdot b \subseteq \prod_{i=1}^{m} z_i \cdot b = \prod_{i=1}^{m} z_i \cdot z_{m+1} \cdot b \subseteq \prod_{i=1}^{m} z_i \cdot a \cdot z \cdot b \subseteq \prod_{i=1}^{m} z_i \cdot a \cdot \prod_{i=1}^{n} w_i \cdot b = v \in \mathcal{U}_{m+n+2}(z).$$

Moreover, if $\sigma(m+1) = m+1$, then

$$\prod_{i=1}^{m} z_{\sigma(i)} = \prod_{i=1}^{m} z_{\sigma(i)} \cdot z_{m+1} \subseteq \prod_{i=1}^{m} z_{\sigma(i)} \cdot a \cdot z \subseteq \prod_{i=1}^{m} z_{\sigma(i)} \cdot a \cdot x \cdot b$$
$$\subseteq \prod_{i=1}^{m} z_{\sigma(i)} \cdot a \cdot \prod_{i=1}^{n} w_{\alpha(i)} \cdot b \subseteq P(z).$$

So there exists $\delta \in S_{m+n+2}$ such that $v_\delta \subseteq P(z)$. In addition, since $\prod_{i=1}^{m} z_{\sigma(i)} \subseteq v_\delta$ and $z \in v$, we get $P(z)$ is an R_K-part of H and by Theorem 3.40, the proof is complete. □

4 k-nilpotent groups derived from hypergroups

In this section, we consider a hypergroup H, for any $K \subseteq H$, apply the relation R_K and show that H/R_K^* is a k-nilpotent group.

Definition 4.1. *Let $k \in \mathbb{N}$ and H be a hypergroup. Define $L_0^k(H) = H$ and for every $n \geq 0$, $L_{n+1}^k(H) = \{h \in [x,y] \mid x \in L_n^k(H), y \in H^k = \bigcup_{h \in H} h^k\}$. Clearly, for all $n \in \mathbb{N}$, $L_{n+1}^k(H) \subseteq L_n^k(H)$.*

Lemma 4.2. *Let H be a hypergroup. Then for every $n \in \mathbb{N}$, $L_n^k(H/R_K^*) = \{\overline{h} = R_K^*(h) \mid h \in L_n^k(H)\}$.*

Theorem 4.3. *Let H be a hypergroup and $K = H \setminus L_n^k(H)$. Then $G = H/R_K^*$ is a k-nilpotent group.*

Proof. Let $h \in L_{n+1}^k(H)$. Then there exists $x \in L_n^k(H)$ and $y \in H^k$ such that $h \in [x,y]$, so $x \cdot y \cap h \cdot y \cdot x \neq \emptyset$ and $x, h \in L_n^k(H)$. By definition of R_K^*, we have $\overline{x} \cdot \overline{y} = \overline{h} \cdot \overline{y} \cdot \overline{x} = \overline{x} \cdot \overline{y} \cdot \overline{h}$, hence $\overline{h} = 1$ and so $L_{n+1}^k(G) = \{1\}$. For $i = 0$, $\gamma_i^k(G) \subseteq L_i^k(G)$. Let $a \in \gamma_{i+1}^k(G)$. Without loss generality, suppose that $a = [x,y]$, where $x \in \gamma_i^k(G)$ and $y \in G^k$. By the hypothesis of induction we conclude that $x \in L_i^k(G)$, thus $a = [x,y] \in L_{i+1}^k(G)$. Now we have $\gamma_{n+1}^k(G) \subseteq L_{n+1}^k(G) = \{1\}$ and so G is a k-nilpotent group of class at most $n+1$.

□

Example 4.4. *Consider the hypergroup which is defined in Example 3.35. Let $k \in \mathbb{N}$, then*

$$H^k = \begin{cases} H, & \text{if } k \text{ is odd}, \\ \{b, c\}, & \text{if } k \text{ is even}, \end{cases}$$

and for every $n \in \mathbb{N}$, $L_n^k(H) = \{b, c\}$. If $K = \{b, c\}$, then $\overline{b} = \overline{c}$ and $G = \{\overline{a}, \overline{b}\}$ is a k-nilpotent group and we have the following tables:

$\circ$	$\overline{a}$	$\overline{b}$
$\overline{a}$	$\overline{b}$	$\overline{a}$
$\overline{b}$	$\overline{a}$	$\overline{b}$

5 Conclusions

(*i*) The current paper introduced a fundamental relation as a generalization fundamental relations on hypergroups in such a way that in particular is a generalization of β^*, γ^*, and τ^*.

(*ii*) The concept of relation-part of hypergroups is introduced and is shown that the heart of every hypergroup is a relation-part of hypergroup.

(*iii*) By using the concept of relation-part and fundamental relation on hypergroups, we obtain some relation-part of hypergroup.

(*iv*) With respect to the concept of relation the concept of k–nilpotent groups are obtained.

(*v*) It is proved that this relation is on a hypergroup with an identity that is transitive.

We hope that these results are helpful for furthers studies in hypergroup. In our future studies, we hope to obtain more results regarding polygroups, groups, and their applications.

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