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On neutro-d-subalgebras

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Abstract

This paper introduces the novel concept of neutro-d-algebra. In neutro-d-algebra, the outcome of any given two elements under an underlying operation (neutrosophication procedure) has three cases, such as: Appurtenance, non-appurtenance, or indeterminate and for an axiom: Equal, non-equal, or indeterminate. This study investigates the neutro-d-algebra and shows that neutro-d-algebra is different from d-algebra. The notation of neutro-d-algebra generates a new concept of neutro poset. Finally, we introduce a notation of neutro d-homomorphism and extend the structure of neutro-d-algebras.

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1 Introduction

Neutrosophy, as a newly-born science, is a branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It can be defined as the incidence of the application of a law, an operation, an axiom, an idea, a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible. Neutrosophic sets and systems international journal (which is in Scopus and Web of Science) is a tool for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics, neutrosophic measure, neutrosophic integral, and so on, studies that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc. Recently, Smarandache [9], generalized the classical algebraic structures to neutro-algebraic structures (neutro-algebras)

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and anti-algebraic structures (anti-algebras) and he proved that the neutro-algebra is a generalization of partial algebra [9]. He considered $\langle A\rangle$ as an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, he split the non-empty space and worked onto three regions two opposite ones corresponding to $\langle A\rangle$ and $\langle antiA\rangle$, and one corresponding to neutral (indeterminate) $\langle neutA\rangle$ (also denoted $\langle neutroA\rangle$) between the opposites, regions that may or may not be disjoint - depending on the application, but they are exhaustive (their union equals the whole space). A neutro-algebra is an algebra that has at least one neutro-operation (operation that is well-defined (also called inner-defined) for some elements, indeterminate for others, and outer-defined for the others or one neutro-axiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A partial algebra is an algebra that has at least one partial operation (well-defined for some elements, and indeterminate for other elements), and all its axioms are classical (i.e., the axioms are true for all elements). Through a theorem he proved that neutro-algebra is a generalization of partial algebra, and examples of neutro-algebras that are not partial algebras were given. Also, the neutro-function and neutro-operation were introduced [9]. Unlike the abstract algebraic structures, from pure mathematics, constructed on a given perfect space (set), where the axioms (laws, rules, theorems, results etc.) are totally (100%) true for all spaces elements, our world and reality consist of approximations, imperfections, vagueness, and partialities. Most of mathematical models are too rigid to completely describe the imperfect reality. Many axioms are actually neutro-axioms (i.e. axioms that are true for some spaces elements, indeterminate for other spaces elements, and false for other spaces elements). This makes that neutro-algebras be important in the real world.

Regarding these points, we now introduce the concept of neutro-d-algebras based on axioms of d-subalgebras, but having a different outcome. In the system of d-subalgebras, the operation is totally well-defined for any two given elements, but in neutro-d-subalgebras its outcome may be well-defined, outer-defined, or indeterminate. Any d-subalgebra is a system that considers that all its axioms are true, but we weaken the conditions that the axioms are not necessarily totally true, but also partially false, and partially indeterminate. So, one of our main motivations is a weak coverage of the classical axioms of d-subalgebras. This causes new partially ordered relations on a non-empty set, such as neutro posets. In final, we define the notation of d-homomorphisms and extend the structures of neutro-d-subalgebras.

2 Preliminaries

In this section, we recall some definitions and results from [9], which are needed throughout the paper.

Let $n \in \mathbb{N}$. Then an n-ary operation $\circ : X^n \to Y$ is called a neutro-operation if it has $x \in X^n$ for which $\circ(x)$ is well-defined (degree of truth (T)), $x \in X^n$ for which $\circ(x)$ is indeterminate (degree of indeterminacy (I)), and $x \in X^n$ for which $\circ(x)$ is outer-defined (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (total, or classical) operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary anti-operation. Again, in this definition neutro stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general. In this regards, for any given set X, we classify n-ary operation on X^n by (i); (**classical**) **operation** is an operation well-defined for all set's elements, (ii); **neutro operation** is an operation partially well-defined, partially indeterminate, and partially outer-defined on the given set and (iii); **anti-operation** is an operation outer-defined for all set's elements.

Moreover, we have (i); a (**classical**) **axiom** defined on a non-empty set is an axiom that is totally true (i.e. true for all set's elements), (ii); **neutro-axiom (or neutrosophic axiom)** defined on a non-empty set is an axiom that is true for some elements (degree of true = T), indeterminate for other elements (degree of indeterminacy = I), and false for the other elements (degree of falsehood = F), where T, I, F are in $[0,1]$ and (T, I, F) is different from $(1,0,0)$ i.e., different from totally true axiom, or classical axiom and (T, I, F) is different from $(0,0,1)$ i.e., different from totally false axiom, or anti-axiom. (iii); an **anti-axiom** of type $\mathcal{C}$ defined on a non-empty set is an axiom that is false for all set's elements.

Based on the above definitions, there is a classification of algebras as follows. Let X be a non-empty set and $\mathcal{O}$ be a family of binary operations on X. Then $(A, \mathcal{O})$ is called

(i) a (**classical**) **algebra** of type $\mathcal{C}$, if $\mathcal{O}$ is the set of all total operations (i.e. well-defined for all set's elements) and $(A, \mathcal{O})$ is satisfied by (classical) axioms of type $\mathcal{C}$ (true for all set's elements);

(ii) a **neutro-algebra (or neutro-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one neutro-operation (or neutro-function), or $(A, \mathcal{O})$ is satisfied by at least one neutro-axiom of type $\mathcal{C}$ that is referred to the sets (partial-, neutro-, or total-) operations or axioms;

(iii) an **anti-algebra (or anti-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one anti-operation or $(A, \mathcal{O})$ is satisfied by at least one anti-axiom of type $\mathcal{C}$.

Definition 2.1. [3] *Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a BCK-algebra if it satisfies the following conditions:*

$(BCI$-1) $((x * y) * (x * z)) * (z * y) = 0$,
$(BCI$-2) $(x * (x * y)) * y = 0$,
$(BCI$-3) $x * x = 0$,
$(BCI$-4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,
$(BCK$-5) $0 * x = 0$.

Definition 2.2. [5] *Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a d-algebra if it satisfies the following conditions:*

$(D$-1) $x * x = 0$,
$(D$-2) $0 * x = 0$,
$(D$-3) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 2.3. [2] *Let X be a non-empty set, $0 \in X$ be a constant, and "$*$" be a binary operation on X. An algebra $(X, *, 0)$ of type $(2,0)$ is said to be a neutro-BCK-algebra, if it satisfies at least one of the following neutro-axioms (while the others are classical BCK-axioms):*

$(NBCI$-1) *there exist* $x, y, z \in X$ *such that* $((x*y)*(x*z))*(z*y) = 0$ *and there exist* $x, y, z \in X$ *such that* $((x * y) * (x * z)) * (z * y) \neq 0$ *or indeterminate;*

$(NBCI$-2) *there exist* $x, y \in X$ *such that* $(x * (x * y)) * y = 0$ *and there exist* $x, y \in X$ *such that* $(x * (x * y)) * y \neq 0$ *or indeterminate;*

$(NBCI$-3) *there exists* $x \in X$ *such that* $x * x = 0$ *and there exists* $x \in X$ *such that* $x * x \neq 0$ *or indeterminate;*

$(NBCI$-4) *there exist* $x, y \in X$, *such that if* $x*y = y*x = 0$, *then* $x = y$ *and there exist* $x, y \in X$, *such that if* $x * y = y * x = 0$, *then* $x \neq y$*;*

($NBCK$-5) there exists $x \in X$ such that $0 * x = 0$ and there exists $x \in X$ such that $0 * x \neq 0$ or indeterminate.

Each above neutroAxiom has a degree of equality (T), degree of non-equality (F), and degree of indeterminacy (I), where $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

If $(X, *, 0)$ is a neutro algebra and satisfies the conditions ($NBCI$-1) to ($NBCI$-4) and ($NBCK$-5), then we call it is a neutro-BCK-algebra of Type 5 (i.e. it satisfies 5 neutro-axioms).

Definition 2.4. [2] *Let X be a non-empty set and R be a binary relation on X. Then R is called a*

- (i) *neutro-reflexive if there exists $x \in X$ such that $(x, x) \in R$ (degree of truth T), and there exists $x \in X$ such that (x, x) is indeterminate (degree of indeterminacy I), and there exists $x \in X$ such that $(x, x) \notin R$ (degree of falsehood F);*

- (ii) *neutro-antisymmetric if there exist $x, y \in X$ such that $(y, x) \in R$ and $(x, y) \in R$ imply $x = y$ (degree of truth T), and there exist $x, y \in X$ such that (x, y) or (y, x) are indeterminate in R (degree of indeterminacy I), and there exist $x, y \in X$ such that $(x, y) \in R$ and $(y, x) \in R$ imply $x \neq y$ (degree of falsehood F);*

- (iii) *neutro-transitive if there exist $x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$ imply $(x, z) \in R$ (degree of truth T), and there exist $x, y, z \in X$ such that (x, y) or (y, z) are indeterminate in R (degree of indeterminacy I), and there exist $x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$, but $(x, z) \notin R$ (degree of falsehood F). In all above neutro-axioms $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.*

- (iv) *neutro-partially ordered binary relation if the relation satisfies at least one of the above neutro-axioms neutro-reflexivity, neutro-antisymmetry, neutro-transitivity, while the others (if any) are among the classical axioms reflexivity, antisymmetry, transitivity.*

If R is a neutro-partially ordered relation on X, then we call (X, R) by neutro-poset. We denote, the related diagram to neutro-poset (X, R) by neutro-Hass diagram.

3 Neutro d-algebras

In this section, we define neutro-d-algebras and investigate their properties.

Definition 3.1. *Let X be a non-empty set, $0 \in X$ be a constant and "$*$" be a binary operation on X. An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a neutro-d-algebra if it satisfies at least one of the following neutro-axioms (while the others are classical D-axioms):*

(ND-1) there exists $x \in X$ such that $x * x = 0$ and there exists $x \in X$ such that $x * x \neq 0$ or indeterminate;

(ND-2) there exists $x \in X$ such that $0 * x = 0$ and there exists $x \in X$ such that $0 * x \neq 0$ or indeterminate;

(ND-3) there exist $x, y \in X$, such that if $x * y = y * x = 0$, then $x = y$ and there exist $x, y \in X$, such that if $x * y = y * x = 0$, then $x \neq y$.

Each above neutro-axiom has a degree of equality (T), degree of non-equality (F), and degree of indeterminacy (I), where $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$. If $(X, *, 0)$ is a neutro-algebra and satisfies the conditions (ND-1) to (ND-3), then sometimes we call it as a neutro-d-algebra of Type 3 (i.e. it satisfies 3 neutro-axioms).

Example 3.2. *(i)* $(\mathbb{Z}, *, 0)$ *is a neutro-d-algebra, but* $(\mathbb{Z}, -, 0)$ *is not a neutro-d-algebra, where "$-$" is the ordinary substraction of real numbers, "$\cdot$" is the ordinary product of real numbers and for* $x, y \in \mathbb{Z}, x * y = x - y + xy$.

(ii) $(\mathbb{Z}, *, 1)$ *is a neutro-d-algebra, but* $(\mathbb{Z}, -, 1)$ *is not a neutro-d-algebra, where "$-$" is the ordinary substraction of real numbers, "$\cdot$" is the ordinary product of real numbers and for* $x, y \in \mathbb{Z}, x * y = x - y + xy$.

(iii) Let X be a non-empty set. Then the system of algebraics $(P(X), \cap, \emptyset)$, $(P(X), \cap, X)$, $(P(X), \cup, \emptyset)$, $(P(X), \setminus, \emptyset)$, $(P(X), \setminus, X)$ *and* $(P(X), \cup, X)$ *are not neutro-d-algebras, where* $P(X) = \{Y \mid Y \subseteq X\}$.

Theorem 3.3. *Let* $(X, *, 0)$ *be a neutro-d-algebra and* $0 * 0 = 0$. *Then there exist* $x, y, z, w \in X$ *such that* $(x * y) * (z * w) = 0$.

Proof. Since $(X, *, 0)$ is a neutro d-algebra, we get there exists $a \in X$ such that $a * a = 0$ and there exists $b \in X$ such that $b * b \neq 0$ or indeterminate and there exists $c \in X$ such that $0 * c = 0$ and there exists $d \in X$ such that $0 * d \neq 0$ or indeterminate. Thus there exist $x, y, z, w \in X$ such that $(x * y) * (z * w) = 0 * 0 = 0$, when $x = 0, y = c, z = w = a$. □

Theorem 3.4. *Let X be a non-empty set. Then the following statements are equivalent:*

(i) $(X, *, 0)$ *is a neutro-BCK-algebra,*

(ii) $(X, *, 0)$ *is a neutro-d-algebra and the conditions (NBCI–1), (NBCI–2) hold.*

Proof. It is clear by definition. □

Let X be a non-empty set and $* : X \times X \to X$ be a map. Then $(X, *)$ is called a neutro groupoid if there exist $x, y \in X$ such that $x * y \in X$ and there exist $x, y \in X$ such that $x * y \notin X$ or indeterminate. For instance, $(\mathbb{N}, *)$ is a neutro groupoid, where for $x < y$, define $x - y$ is indeterminate.

Theorem 3.5. *Let* $(X, *, 0)$ *be a neutro d-algebra and* $(Y, *)$ *be a neutro groupoid, where* $Y \subset X$. *Then* $(Y, *)$ *is a neutro-d-algebra.*

Proof. Since $(Y, *)$ is a neutro groupoid, there exists $x, y \in X$ such that $x * y \in X$ and there exists $x, y \in X$ such that $x * y \notin X$ or indeterminate. Thus there exists $x \in X$ such that $x * x = 0 \in Y$ and there exists $x \in X$ such that $x * x \neq 0$ or indeterminate. It follows that $0 \in Y$ and so there exists $x \in X$ such that $0 * x = x \in Y$ and there exists $x \in X$ such that $0 * x \neq x$ or indeterminate. Since $0 \in Y$ and $Y \subset X$, we get that there exist $x, y \in X$, such that if $x * y = y * x = 0$, then $x = y$ and there exist $x, y \in X$, such that if $x * y = y * x = 0$, then $x \neq y$. Therefore, $(Y, *)$ is a neutro-d-algebra. □

Example 3.6. *Consider the neutro-d-algebra* $(\mathbb{Z}, *, 0)$ *in Example 3.2. Let* $Y = \{2, 3\} \subseteq \mathbb{Z}$. *Clearly* $(Y, *)$ *is not a neutro groupoid and is not a neutro d-algebra.*

A neutro groupoid $(X, *, 0)$, is called a (right)left neutro monoid if there exists $x \in X$ such that $x * 0 = 0$ and there exists $x \in X$ such that $x * 0 \neq 0$ or indeterminate, there exists $x \in X$ such that $0 * x = 0$ and there exists $x \in X$ such that $0 * x \neq 0$ or indeterminate. A neutro-groupoid $(X, *, 0)$, is called a neutro-monoid if it is right and left neutro-monoid. For instance, $(\mathbb{Z}, *, 1)$ in Example 3.2, is a neutro-monoid.

Corollary 3.7. *Let* $(X, *, 0)$ *be a neutro-d-algebra. Then* $(X, *, 0)$ *is a left neutro-monoid.*

Theorem 3.8. *Let $(G, *, e)$ be a group. If there exists $x \in G$ such that $x * x \neq e$, then $(G, *, e)$ is a neutro-d-algebra.*

Proof. Let $x \in G$. Then

(ND-1) there exists $e \in G$ such that $e * e = e$ and by hypothesis, there exists $x \in G$ such that $x * x \neq e$.

(ND-2) there exists $e \in G$ such that $e * e = e$ and for any $x \neq e$, we have $e * x = x \neq e$.

(ND-3) there exist $x = e$ and $y = e \in G$, such that $x * y = y * x = e$, and so $x = y$ and there exist $x \neq e, y \neq e \in G$, such that $x * y = y * x = e$, and so $x \neq x^{-1}$. □

Corollary 3.9. *The Klein four-group $(G, *, e)$ is not a neutro-d-algebra.*

Example 3.10. *Consider neutro-d-algebra $(X, *, 0)$ in Example 3.2. There exists some $x \in X$ such that $x * x \neq 0$, but $(X, *, 0)$ is not a group.*

Proof. Since for all $x \in G, x^{-1} = x$, the condition (ND-3) does not hold. □

Let $X \neq \emptyset$ be a finite set. We denote $\mathcal{N}_D(X)$ and $\mathcal{N}_{ND}(X)$ by the set of all neutro-d-algebras and neutro-d-algebras of Type 3 that are constructed on X, respectively.

Theorem 3.11. *Let $(X, *, 0)$ be a neutro d-algebra. Then*

(*i*) *If $|X| = 1$, then $(X, *, 0)$ is a trivial d-algebra.*

(*ii*) *If $|X| = 2$, then $|\mathcal{N}_D(X)| = 2$ and $|\mathcal{N}_{ND}(X)| > \infty$.*

(*iii*) *If $|X| = 3$, then there exists $\emptyset \neq Y \subseteq X$, such that $(Y, *', 0)$ is a nontrivial or trivial d-algebra.*

Proof. We consider only cases (*ii*) and (*iii*), because the others are immediate.

(*ii*) Let $X = \{0, x\}$. Then we have 2 trivial neutro-d-algebras $(X, *_1), (X, *_2)$ and an infinite number of trivial neutro-d-algebras of Type 3 $(X, *, 0)$ in Tables 1 and , where $w \notin X$.

Table 1: Neutro-d-algebras of orders 2

$*_1$	0	x
0	0	x
x	0	x

,

$*_2$	0	x
0	x	0
x	x	0

,

$*$	0	x
0	x	0
x	w	0

(*iii*) Let $X = \{0, x, y\}$. Now consider $Y = \{0, x\}$ and define a neutro-d-algebra $(X, *', 0)$ as the following table. Clearly $(Y, *', 0)$ is a non-trivial d-algebra. □

Table 2: Neutro-d-algebras of orders 3

$*'$	0	x	y
0	0	0	y
x	x	0	0
y	0	y	x

.

Theorem 3.12. *Every d-algebra can be extended to a neutro-d-algebra.*

Proof. Let $(X, *, 0)$ be a d-algebra, $\alpha \notin X$, and U be the universe of discourse that strictly includes $X \cup \{\alpha\}$. For all $x, y \in X \cup \{\alpha\}$, define $*_\alpha$ on $X \cup \{\alpha\}$ by $x *_\alpha y = x * y$ where, $x, y \in X$ and if $\alpha \in \{x, y\}$, define $x *_\alpha y$ as indeterminate or $x *_\alpha y \notin X \cup \{\alpha\}$. Then $(X \cup \{\alpha\}, *_\alpha, 0)$ is a neutro-d-algebra. □

Example 3.13. *Let $X = \{0, 1, 2, 3, 4, 5\}$ and consider Table 3.*

Table 3: Neutro-d-algebras and neutro-d-algebra of Type 3

$*_1$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	2	0	a	2	0	3
2	1	1	0	0	2	0
3	3	0	1	2	0	5
4	0	4	0	1	4	0
5	4	0	1	0	2	3

,

$*_2$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	0	0	0	5
2	2	2	0	0	5	0
3	3	3	3	0	0	2
4	0	1	0	4	1	2
5	5	0	4	0	0	x

and

$*_3$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	t	0	s	0
2	2	2	0	y	0	3
3	3	1	3	0	z	0
4	4	4	4	4	0	1
5	0	2	0	2	0	w

.

Then

(i) If $a = 2$, then $(X, *_1, 0)$ is a neutro-d-algebra and $(X \setminus \{3, 4, 5\}, *_1, 0)$ is a d-algebra.

(ii) $(X, *_2, 0)$ is a neutro-d-algebra and $(X \setminus \{4, 5\}, *_2, 0)$ is a d-algebra.

(iii) If $s = t = y = z = 0, w = 3$, then $(X, *_3, 0)$ is a neutro-d-algebra and for $s = t = 1, y = 2, z = 3$, $(X \setminus \{5\}, *_3, 0)$ is a d-algebra. If $s = t = y = z = 0, w = \sqrt{2}$, then $(X, *_3, 0)$ is a neutro-d-algebra of Type 3 where $s, t \in \{0, 1\}, x \in \{4, 5\}, y \in \{2, 0\}, z \in \{3, 0\}$ and $w \in \{3, \sqrt{2}\}$.

Definition 3.14. *Let $(X, *, 0)$ be a neutro d-algebra. If there exist $x, y, z \in X$ such that $x * y = 0$ and $y * z = 0$ imply $x * z = 0$ and there exist $x, y, z \in X$ such that $x * y = 0$ and $y * z = 0$ imply $x * z \neq 0$ or indeterminate, then $(X, *, 0)$ is called a neutro transitive d-algebra. For instance, neutro d-algebra $(X, *_1, 0)$ in Example 3.13, is a neutro transitive d-algebra, which $a = 2$.*

Theorem 3.15. *Let $(X, *, 0)$ be a neutro d-algebra and $x, y \in X$. Then $(X, *, 0)$ is transitive if and only if $(X, \leq_X)$ is a neutro-poset, which $x \leq_X y$ if and only if $x * y = 0$.*

Proof. Since $(X, *, 0)$ is a neutro-d-algebra, by $(ND$-1), there exists $x \in X$ such that $x * x = 0$ and there exists $x \in X$ such that $x * x \neq 0$ or indeterminate. It concludes that $x \leq_X x$. If $x \leq_X y$ and $y \leq_X x$, then $x * y = 0$ and $y * x = 0$. Now, by $(ND$-3), we have $x = y$ or $x \neq y$. If $x \leq_X y$ and $y \leq_X z$, then $x * y = 0$ and $y * z = 0$. Since $(X, *, 0)$ is a neutro transitive d-algebra, we get $x * z = 0$ or $x * z \neq 0$. Thus $(X, \leq_X)$ is a neutro-poset.

Convesely, let $(X, \leq_X)$ be a neutro-poset. Then there exist $x, y, z \in X$ such that $x * y = 0$ and $y * z = 0$ imply $x * z = 0$ and there exist $x', y', z' \in X$ such that $x' * y' = 0$ and $y' * z' = 0$ imply $x' * z' \neq 0$. □

Theorem 3.16. *An algebra $(X, *, 0)$ is a neutro-d-algebra if and only if it satisfies the following conditions:*

$(ND$-1$')$ there exist $x, y \in X$ such that $x \leq_X x$ and there exist $x, y \in X$ such that $x \not\leq_X x$,

$(ND$-2$')$ there exist $x, y \in X$ such that $0 \leq_X x$ and there exist $x, y \in X$ such that $0 \not\leq_X x$,

$(ND$-3$')$ there exist $x, y \in X$ if $x \leq_X y$ and $y \leq_X x$, then $x = y$ and there exist $x, y \in X$ if $x \leq_X y$ and $y \leq_X x$, then $x \neq y$.

Proof. $(ND\text{-}1') \Leftrightarrow (ND\text{-}1)$ Let $(X,*,0)$ be a neutro-d-algebra. If there exists $x \in X$ such that $x * x = 0$ and there exists $x \in X$ such that $x * x \neq 0$ or indeterminate, then by definition, there exists $x \in X$ such that $x \leq_X x$) and there exists $x \in X$ such that $x \not\leq_X x$. The converse is similar. By a similar way, we can see that $(ND\text{-}2') \Leftrightarrow (ND\text{-}2)$ and $(ND\text{-}3') \Leftrightarrow (ND\text{-}3)$. □

4 Extension of neutro d-algebras

In this section, we construct neutro d-algebras by Cartesian product and some operations.

Let $(X,*,0)$ be a neutro d-algebra. For all $x,y \in X$, define $x *' y = \begin{cases} x & \text{if } x * y \neq 0 \\ 0 & \text{otherwise} \end{cases}$.

Theorem 4.1. *Let $(X,*,0)$ be a neutro d-algebra. Then $(X,*',0)$ is a neutro d-algebra.*

Proof. By definition, the proof is clear. □

Definition 4.2. *Let $(X_1,*_1,0_1)$ and $(X_2,*_2,0_2)$ be d-algebras, where $X_1 \cap X_2 = \emptyset$. For some $x,y \in X = X_1 \cup X_2$, define an operation $*$ as follows:*

$$x * y = \begin{cases} x *_1 y & \text{if } x,y \in X_1 \setminus X_2 \\ x *_2 y & \text{if } x,y \in X_2 \setminus X_1 \\ 0_1 & \text{if } x \in X_1, y \in X_2 \\ 0_2 & \text{if } x \in X_2, y \in X_1 \end{cases},$$

where $0_1 * 0_2 = 0_2$ and $0_2 * 0_1 = 0_1$.

Example 4.3. *Consider $(X_1 = \{0_1, a_1\}, *_1, 0_1)$ and $(X_2 = \{0_2, a_2\}, *_2, 0_2)$ are neutro-d-algebras. Then we have neutro-d-algebras $(X_1 \cup X_2, *, 0_1)$ and $(X_1 \cup X_2, *, 0_2)$ in Table 4.*

Table 4: Neutro-d-algebras of orders 2 and 4

$*_1$	0_1	a_1
0_1	0_1	a_1
a_1	0_1	a_1

,

$*_2$	0_2	a_2
0_2	a_2	0_2
a_2	a_2	0_2

, and

$*$	0_1	0_2	a_1	a_2
0_1	0_1	0_1	a_1	0_1
0_2	0_2	a_2	0_2	0_2
a_1	0_1	0_1	a_1	0_1
a_2	0_2	a_2	0_2	0_2

.

Theorem 4.4. *Let $(X_1,*_1,0_1)$ and $(X_2,*_2,0_2)$ be d-algebras, where $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Then*

(i) $(X,*,0_1)$ is a neutro-d-algebra;

(ii) $(X,*,0_2)$ is a neutro-d-algebra.

Proof. (i) We only prove $(ND\text{-}3)$. Let $x * y = 0_1$. It follows that $x \in X_1$ and $y \in X_2$ or $x,y \in X_1$. If $x,y \in X_1$, then since $(X_1,*_1,0_1)$ is a d-algebra, $y * x = 0_1$ implies $x = y$. But for $x \in X_1$ and $y \in X_2$, we have $y * x \neq 0_1$ so $(ND\text{-}3)$ is valid in any cases. Other items are clear.

(ii) It is similar to item (i). □

Example 4.5. *Let $X_1 = \{a, b\}$ and $X_2 = \{w, x, y, z\}$. Then $(X_1, *, a)$ and $(X_2, *, w)$ are d-algebras. So by Theorem 4.4, $(X_1 \cup X_1, *, a)$ and $(X_1 \cup X_1, *, w)$ are neutro-d-algebras in Table 5.*

Table 5: d-algebras and neutro-d-algebra

$*$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	b	a	a	a	a	a
w	a	w	w	w	w	w
x	w	w	x	w	w	w
y	w	w	y	x	w	w
z	w	w	z	x	x	w

Definition 4.6. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two neutro-d-algebras, where $X_1 \cap X_2 = \emptyset$. Define $*$ on $X_1 \cup X_2$, by* $x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1 \setminus X_2 \\ x *_2 y & \text{if } x, y \in X_2 \setminus X_1 \\ y & \text{otherwise} \end{cases}$.

Theorem 4.7. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two neutro-d-algebras. Then*

(*i*) *$(X_1 \cup X_2, *, 0_1)$ is a neutro-d-algebra;*

(*ii*) *$(X_1 \cup X_2, *, 0_2)$ is a neutro-d-algebra.*

Proof. It is obvious. □

Definition 4.8. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two neutro-d-algebras. Define $*$ on $X_1 \times X_2$, by $(x, y) * (x', y') = (x *_1 x', y *_2 y')$, where $(x, y), (x', y') \in X_1 \times X_2$.*

Theorem 4.9. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two neutro-d-algebras. Then $(X_1 \times X_2, *, (0_1, 0_2))$ is a neutro-d-algebra.*

Proof. We prove only the item (ND-3). Since $(X_1, *_1, 0_1)$ is a neutro-d-algebra, there exist $x_1, x_1' \in X_1$ and $y_1, y_1' \in X_1$ such that if $x_1 *_1 x_1' = x_1' *_1 x_1 = 0_1$ and $y_1 *_1 y_1' = y_1' *_1 y_1 = 0_1$, then $x_1 = x_1'$ and $y_1 \neq y_1'$. By the similar way, $(X_2, *_2, 0_2)$ is a neutro-d-algebra, so there exist $x_2, x_2' \in X_2$ and $y_2, y_2' \in X_2$ such that if $x_2 *_2 x_2' = x_2' *_2 x_2 = 0_2$ and $y_2 *_2 y_2' = y_2' *_2 y_2 = 0_2$, then $x_2 = x_2'$ and $y_2 \neq y_2'$. It follows that there exist $(x_1, x_2), (x_1', x_2'), (y_1, y_2), (y_1', y_2') \in X_1 \times X_2$ such that if $(x_1, x_2) * (x_1', x_2') = (x_1', x_2') * (x_1, x_2) = (0_1, 0_2)$ and $(y_1, y_2) * (y_1', y_2') = (y_1', y_2') * (y_1, y_2) = (0_1, 0_2)$, then $(x_1, x_2) = (x_1', x_2')$ and $(y_1, y_2) \neq (y_1', y_2')$. □

Definition 4.10. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two neutro-d-algebras. A map $f : (X_1, *_1, 0_1) \to (X_2, *_2, 0_2)$ is called a neutro d-homomorphism, if there exist $x, y \in X$ such that $f(x *_1 y) = f(x) *_2 f(y)$ and there exist $x, y \in X$ such that $f(x *_1 y) \neq f(x) *_2 f(y)$ and $f(0_1) = 0_2$.*

Theorem 4.11. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two neutro-d-algebras. Then*

(*i*) *the projection mappings π_{X_1} and π_{X_2} are neutro d-homomorphisms;*

(*ii*) *the inclusion mappings i_{X_1} and i_{X_2} are neutro d-homomorphisms.*

Proof. The proof is straightforward. □

Definition 4.12. *Let $(X, *, 0)$ be a neutro-d-algebra and $Y \subseteq X$. Then Y is called a neutro-d-subalgebra if $(Y, *, 0)$ is a neutro-d-algebra.*

Let $(X, *, 0)$ be a neutro-d-algebra and $Y \subseteq X$. If Y is a neutro-d-subalgebra of X, then $(X, *, 0)$ is only trivial neutro-d-subalgebra of X.

Corollary 4.13. *Let $(X, *, 0)$ be a neutro-d-algebra and $|X| = n$. Then there exist $m \leq n$ and $x_1, x_2, \ldots, x_m \in X$ such that $(\{0, x_1, x_2, \ldots, x_m\}, *, 0)$ is a neutro-d-subalgebra of X.*

Theorem 4.14. *Let $f : (X_1, *_1, 0_1) \to (X_2, *_2, 0_2)$ be a neutro d-homomorphism. Then*

(i) *if $0_1 *_1 0_1 = 0_1$, then $0_2 *_2 0_2 = 0_2$;*

(ii) *if there exists $x, y \in X_1$ such that $x *_1 y = 0_1$, then $f(x) *_2 f(y) = 0_2$;*

(iii) *if $0_2 *_2 0_2 = 0_2$ and $x \in Ker(f) = \{x \in X_1 \mid f(x) = 0_2\}$, then $x *_1 x \in Ker(f)$;*

(iv) *if $0_2 *_2 0_2 = 0_2$ and f is a one to one map, then $Ker(f)$ is not a neutro-d-subalgebra of X_1.*

Proof. (i) Let $0_1 *_1 0_1 = 0_1$. Since $f : (X_1, *_1, 0_1) \to (X_2, *_2, 0_2)$ is a neutro d-homomorphism, we get

$$0_2 = f(0_1) = f(0_1 *_1 0_1) = f(0_1) *_2 f(0_1) = 0_2 *_2 0_2.$$

It follows that $0_2 *_2 0_2 = 0_2$.

(ii) Let $x, y \in X_1$. Then $x *_1 y = 0_1$ implies $0_2 = f(0_1) = f(x *_1 y) = f(x) *_2 f(y)$.

(iii) Since $x \in Ker(f)$, we get that $0_2 = 0_2 *_2 0_2 = f(x) *_2 f(x) = f(x *_1 x)$. It follows that $x *_1 x \in Ker(f)$.

(iv) Let $x \in Ker(f)$. Then by item (i), $f(0_1) = 0_2 = f(x *_1 x)$. Since f is a one to one map, we get that for all $x \in Ker(f), x *_1 x = 0_1$ and so $Ker(f)$ is not a neutro-d-subalgebra of X_1. □

Theorem 4.15. *Let $f : (X_1, *_1, 0_1) \to (X_2, *_2, 0_2)$ be a neutro d-homomorphism. Then*

(i) *if Y is a neutro d-subalgebra of X_1, then $f(Y)$ is a neutro d-subalgebra of X_2;*

(ii) *if Z is a neutro d-subalgebra of X_2 and f is bijective, then $f^{-1}(Z)$ is a neutro d-subalgebra of X_1.*

Proof. (i) Since $f(0_1) = 0_2 \in f(Y)$, get $f(Y) \neq \emptyset$. If $a, b \in f(Y)$, then there exist $x, y \in X_1$ such that $a *_2 b = f(x) *_2 f(y) = f(x *_1 y) \in f(Y)$ and so $f(Y)$ a neutro d-subalgebra of X_2. Since there exists $x \in Y$ such that $x *_1 x = 0_1$ and there exists $y \in Y$ such that $y *_1 y \neq 0_1$, we get $f(x) *_2 f(x) = f(x *_1 x) = f(0_1) = 0_2$ and $f(y) *_2 f(y) = f(y *_1 y) \neq f(0_1) = 0_2$, so ($ND$-1) is valid. The proof of other items is similar.

(ii) It is similar to part (i). □

5 Conclusions

To conclude, the current paper has introduced the nove concept of neutro d-subalgebra and investigated some its properties. We showed that neutro BCK-subalgebras are neutro d-subalgebras and under some conditions, neutro d-subalgebras can be neutro BCK-subalgebras. Although the underlying set of any neutro d-subalgebra is finite set, it is presented that the class of neutro d-subalgebras has infinite cardinal, so it makes a fundamental different between of d-subalgebras and neutro d-subalgebras. We wish this research is important for next studies in neutro logical hyperalgebras. In our future studies, we hope to obtain more results regarding neutro (hyper)d-subalgebras and their applications.

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