



On nilpotent and solvable pair of polygroups

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Abstract

In this paper, we introduce nilpotent and solvable pair of polygroups. By a pair of polygroups (P, N) , we mean a polygroup P with a normal sub-polygroup N of P . In addition, we obtain a necessary and sufficient condition between nilpotency (solvability) of pair of polygroups and pair of groups. In particular, we extend the theory of groups to a pair of polygroups. Finally, we study the relationship between nilpotent and solvable pair of polygroups.

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1 Introduction

Pair of groups has some applications in group theory, for instance, Ellis [8, 9] introduced the notion of Schur multiplier and capability of pair of groups and obtained some results. Also, Hassanzadeh [12] introduced the concept of nilpotency for pair of groups. By a pair of groups (G, N) we mean a group G with a normal subgroup N . Clearly, the nilpotency of a group G implies the nilpotency of (G, N) . Also, if (G, N) is nilpotent, then so is N . A pair of groups is nilpotent if and only if the lower (upper) central series terminates at finite steps.

The theory of algebraic hyperstructures is a well-established branch in algebraic theory. Application of hyperstructures have been studied in many branches of mathematics such as geometry, automata, probabilities, and so on [3, 4]. Fundamental relation is an important concept in hypergroups. It is a way to connect hypergroups and groups. Koskas [15] defined the fundamental

relation β^* on hypergroups mainly, it has been studied by Corsini [5, 6] and Fereni [11]. The fundamental relation β^* is the smallest equivalence relation on a hypergroup H such that the quotient H/β^* is a group. In 1934, Marty [16] introduced the concept of polygroups as a special hypergroup. Polygroup theory extends some well-known group results and introduce new topics in polygroups. For example, Nilpotent and solvable polygroups are vital notations of group theory (see [7]). Polygroups have been discussed by Corsini ([5, 6]), Ameri ([1, 2]), Comer [3], Davvaz [7], Jafarpour [14], and so on.

Now, in this paper, we study on polygroup analogue of nilpotent pair of groups. In addition, we introduce the notion of solvable pair of groups and ataine some properties of it. Then for a polygroup P and a normal sub-polygroup N we introduce the concept of nilpotent and solvable pair of polygroups. Also, we obtain a necessary and sufficient condition between nilpotent (solvable) pair of polygroups and nilpotent (solvable) pair of groups. A well-known property of nilpotent groups is that the intersection of a normal subgroup of a nilpotent group G by the center of G is non-trivial. Now, we prove the same property on pair of polygroups. Finally, we investigate the relation between nilpotent and solvable pair of polygroups.

2 Preliminary

We begin our discussion with some fundamental definitions and results.

A hyperoperation \circ is a mapping from $H \times H$ into the family of non-empty subsets of H . A hypergroupoid (H, \circ) is a non-empty set H with a hyperoperation \circ . If A and B are non-empty subsets of H , then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Also, we use $x \circ A$ instead of $\{x\} \circ A$ and $A \circ x$ for

$A \circ \{x\}$. Generally, the singleton a is identified with its member a . The structure (H, \circ) is called a semihypergroup if $a \circ (b \circ c) = (a \circ b) \circ c$ for any $a, b, c \in H$. A semihypergroup (H, \circ) is called a hypergroup if for any $x \in H$, $x \circ H = H \circ x = H$. The condition that for any $a, b \in H$ there exist x and y such that $a \in b \circ x$ and $a \in y \circ b$, is called the reproduction axiom.

Definition 2.1. [7] *An algebraic structure $(P, \cdot, e, {}^{-1})$, where $e \in P$ and ${}^{-1}$ is an unitary operation on P , is called polygroup if for any $x, y, z \in P$ the following conditions hold:*

- (i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- (ii) $e \cdot x = x \cdot e = x$,
- (iii) $x \in y \cdot z \Leftrightarrow y \in x \cdot z^{-1} \Leftrightarrow z \in y^{-1} \cdot x$.

Let $(P_1, \cdot, e_1, {}^{-1})$ and $(P_2, *, e_2, {}^{-1})$ be two polygroups. Then $(P_1 \times P_2, \circ)$, where \circ is defined as follows, is a polygroup (see [7]).

$$(x_1, y_1) \circ (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2, \text{ and } y \in y_1 * y_2\}.$$

Note. From now on for a polygroup $(P, \cdot, e, {}^{-1})$ and $x, y \in P$ we use xy instead of $x \cdot y$.

Definition 2.2. [7] *A non-empty subset K of a polygroup P is a sub-polygroup of P if for any $a, b \in K$ we have $ab \subseteq K$ and $a^{-1} \in K$. Also, a sub-polygroup N of P is normal if for any $a \in P$, $a^{-1}Na \subseteq N$.*

Also, for a sub-polygroup K of P and $x \in P$, denote the left (right) coset of K by xK (Kx) and suppose P/K is the set of all left (right) cosets of K in P . Note that for a normal sub-polygroup N of P , we have $Nx = xN$ and for all $x, y \in P$ we have $Nxy = Nz$ for all $z \in xy$. Also, $(P/N, \odot, N, {}^{-1})$ is a polygroup, where

$$(Nx) \odot (Ny) = \{Nz \mid z \in xy\} \text{ and } (Nx)^{-1} = Nx^{-1}$$

A polygroup is called commutative if for any $x, y \in P$, $xy = yx$. Let (H, \circ) and (H, \star) be two hypergroups. A map $f : (H, \circ) \rightarrow (H, \star)$ is called a homomorphism if for any $a, b \in H$, $f(a \circ b) \subseteq f(a) \star f(b)$. Also, f is a good homomorphism if for any $a, b \in H$, $f(a \circ b) = f(a) \star f(b)$. Let (H, \circ) be a hypergroup. The relation $\rho \subseteq H \times H$ is an equivalence relation and A, B are two non-empty subsets of H . Then

$$A\bar{\rho}B \Leftrightarrow a\rho b, \quad \forall a \in A, \forall b \in B.$$

The relation ρ is called a strongly regular on the left (on the right) if for any $x, y, a \in H$ we have

$$x\bar{\rho}y \Leftrightarrow a \circ x\bar{\rho}a \circ y \quad (x \circ a\bar{\rho}y \circ a).$$

In addition, a strongly regular relation is strongly regular on the right and on the left.

Theorem 2.3. [5] *If (H, \cdot) is a hypergroup and ρ is a strongly regular relation on H , then $(\frac{H}{\rho}, \otimes)$ is a group with the following operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \quad \text{for all } z \in xy.$$

In [15], Koskas defined the relation $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x); x \in H\}$ is the diagonal relation on H and

$$a\beta_n b \Leftrightarrow \exists (x_1, \dots, x_n) \in H^n, \{a, b\} \subseteq \prod_{i=1}^n x_i.$$

The transitive closure of β^* , is a strongly regular relation. Freni proved that if H is a hypergroup, then $\beta = \beta^*$ (see [10]).

The kernel of the canonical map $\pi : P \rightarrow \frac{P}{\beta^*}$, denote by ω_P or ω , is called the core of P . It is easy to prove that $\omega_P = \beta^*(e)$ and $\beta^*(x)^{-1} = \beta^*(x^{-1})$ for all $x \in P$ (see [7]).

Theorem 2.4. [7] *Assume P_1 and P_2 are two polygroups. Then $\omega_{P_1 \times P_2} = \omega_{P_1} \times \omega_{P_2}$.*

Theorem 2.5. [7] *Let A be a non-empty subset of P . Then the intersection of all sub-polygroups of P containing A , denoted by $\langle A \rangle$ is equal to $\bigcup \{x_1^{\epsilon_1} \dots x_k^{\epsilon_k} | x_i \in A, k \in \mathbb{N}, \epsilon_i \in \{1, -1\}\}$.*

Definition 2.6. [7] *The lower central series of P is the sequence $\dots \subseteq \gamma_1(P) \subseteq \gamma_0(P) = P$, where $\gamma_0(P) = P$ and for $k > 0$,*

$$\gamma_{k+1}(P) = \langle \{h \in P \mid xy \cap hyx \neq \emptyset \text{ such that } x \in \gamma_k(P) \text{ and } y \in P\} \rangle.$$

Also, P is called a nilpotent polygroup if for some $n \in \mathbb{N}$, $\gamma_n(P) \subseteq \omega$. The smallest such n is called a class of P .

In [2] it is proved that for any $x, y \in P$ we have

$$\{h \in P \mid xy \cap hyx \neq \emptyset\} = \{h \in P \mid h \in [x, y]\},$$

where $[x, y]$ is the commutator of two elements x, y and is defined by $[x, y] = \{t \mid t \in xyx^{-1}y^{-1}\}$.

Now, we recall the notion of the commutator of subgroups. Let H and K be two subgroups of a group G . Then the commutator of H and K , denoted by $[H, K]$ is a group generated by $\{[h, k] \mid h \in H, k \in K\}$, where $[h, k] = hkh^{-1}k^{-1}$.

Definition 2.7. [12] *Let (G, N) be a pair of groups. Put $\gamma_1(G, N) = N$ and for $i > 1$ $\gamma_{i+1}(G, N) = [\gamma_i(G, N), G]$. Then*

$$N = \gamma_1(G, N) \geq \gamma_2(G, N) \geq \dots$$

is called the lower central series of (G, N) . Also, a pair (G, N) is called nilpotent if $\gamma_n(G, N) = \{e\}$, where e is the identity element of G .

3 Nilpotent pair of polygroups

In this section, by the notion of lower central series for a pair of polygroups the concept of nilpotent pair of polygroups is introduced. Also, some basic results are given (see Theorems ??, ?? and ??).

Definition 3.1. Let N be a normal sub-polygroup of P . Then (P, N) is called a pair of polygroups. Also, if H and M are sub-polygroups of P and N , respectively, then (H, M) is called a sub-pair of (P, N) .

Let N_1 and N_2 be two sub-polygroups of P_1 and P_2 , respectively. By a homomorphism from (P_1, N_1) to (P_2, N_2) we mean a homomorphism from P_1 to P_2 which sends N_1 to N_2 .

Note. From now on, let (P, N) be a pair of polygroups, $k, n \in \mathbb{N}$ and for any $x \in P$ put $\bar{x} = \beta^*(x)$.

Definition 3.2. Assume $l_1(P, N) = N$ and for any $k > 1$,

$$l_k(P, N) = \langle \{h \in P \mid h \in [x, y] \text{ such that } x \in l_{k-1}(P, N) \text{ and } y \in P\} \rangle.$$

Then (P, N) is called a nilpotent pair of polygroups (NP) if for some k we have $l_k(P, N) \subseteq \omega_N$. The smallest such k is called the nilpotency class of (P, N) .

It is clear that if P is a nilpotent polygroup, then (P, N) is nilpotent.

Example 3.3. (i) Let $P_1 = \{e, a, b, c\}$. We define \cdot on P_1 as follows:

| | | | | |
|---------|-----|---------------|-------|--------|
| \cdot | e | a | b | c |
| e | e | a | b | c |
| a | a | a | P_1 | c |
| b | b | $\{e, a, b\}$ | b | b, c |
| c | c | $\{a, c\}$ | c | P_1 |

Then (P_1, \cdot) is a polygroup (see [7]). Consider sub-polygroup $N_1 = \{e, a\}$ of P_1 . It is easy to see that $l_1(P_1, N_1) = \omega_{N_1} = \{e, a\} = N_1$ and so a pair (P_1, N_1) is an NP.

(ii) Let $P_2 = N_2 = \{0, 1\}$ be the cyclic group of order two. Then $l_1(P_2, N_2) = N_2$ and $l_2(P_2, N_2) = \omega_{N_2} = \{0\}$ and so pair (P_2, N_2) is an NP.

Theorem 3.4. Let (P, N) be a pair of polygroups and H be a normal sub-polygroup of P . If (P, N) is an NP, then $(P/H, N/H)$ is an NP.

Proof. First, we prove that for any $n \geq 0$, $l_n(P/H, N/H) = l_n(P, N)H/H$. We proceed by induction on n and show that

$$l_n\left(\frac{P}{H}, \frac{N}{H}\right) \subseteq \frac{l_n(P, N)H}{H} \text{ and } l_n\left(\frac{(P, N)H}{H}\right) \subseteq l_n\left(\frac{P}{H}, \frac{N}{H}\right).$$

For $n = 1$, the inclusions are obvious. Now, suppose that $l_n\left(\frac{P}{H}, \frac{N}{H}\right) \subseteq \frac{l_n(P, N)H}{H}$ holds and $yH \in l_{n+1}\left(\frac{P}{H}, \frac{N}{H}\right)$. Therefore, by Definition ??, $yH \in [aH, bH]$, where $aH \in l_n\left(\frac{P}{H}, \frac{N}{H}\right)$ and $bH \in \frac{P}{H}$. Then by hypotheses of induction, $aH = a'H$ for some $a' \in l_n(P, N)$. Therefore, $yH \in [aH, bH] = [a'H, bH]$ and so by Definition ??, $yH = y'H$ such that $y' \in [a', b] \subseteq l_{n+1}(P, N)$. Hence, $yH \in \frac{l_{n+1}(P, N)H}{H}$.

Conversely, if $yH \in \frac{l_{n+1}(P, N)H}{H}$, then there exists $t \in l_{n+1}(P, N)$ such that $yH = tH$. Therefore, for some $a \in l_n(P, N), b \in P$ we have $t \in [a, b]$. Thus, $yH = tH \in [aH, bH]$, where $aH \in l_n\left(\frac{P}{H}, \frac{N}{H}\right)$ and so $yH \in l_{n+1}\left(\frac{P}{H}, \frac{N}{H}\right)$. Hence, the proof is complete. \square

In what follows we see that the converse of Theorem ?? is not true in general.

Example 3.5. Let $P = S_3$ be the symmetric group and $N = H = A_3$ be the alternative group of degree 3. Then $(P/H, N/H)$ is an NP but (S_3, A_3) is not nilpotent because by Definition ??, $\gamma_1(S_3, A_3) = A_3$ and so $\gamma_2(S_3, A_3) = [A_3, S_3] \subseteq A_3$. If $[A_3, S_3] = \{e\}$, then S_3 is an abelian group, a contradiction. Thus, $[A_3, S_3] \neq \{e\}$. And so $\gamma_2(S_3, A_3) \neq \{e\}$. By a similar method for any $n \in \mathbb{N}$, $\gamma_n(S_3, A_3) \neq \{e\}$. Hence, (S_3, A_3) is not nilpotent.

Corollary 3.6. If (P, N) is an NP and $I \subseteq N$ is a subnormal polygroup of P . Then $(P/I, N/I)$ is an NP.

Theorem 3.7. Assume (P_1, N_1) and (P_2, N_2) are two pairs of polygroups. Then, for all $k \geq 0$ we have:

$$l_k(P_1 \times P_2, N_1 \times N_2) = l_k(P_1, N_1) \times l_k(P_2, N_2).$$

Proof. By induction on k we prove that $l_k(P_1 \times P_2, N_1 \times N_2) = l_k(P_1, N_1) \times l_k(P_2, N_2)$. The first step of induction holds. Now suppose that $(a, b) \in l_k(P_1 \times P_2, N_1 \times N_2)$, and the result holds for $k - 1$. By Definition ??, for some $(x_1, y_1) \in l_{k-1}(P_1 \times P_2, N_1 \times N_2)$ and $(z_1, z_2) \in P_1 \times P_2$ we have $(a, b) \in [(x_1, y_1), (z_1, z_2)]$. Since $[(x_1, y_1), (z_1, z_2)] = ([x_1, z_1], [y_1, z_2])$ we get $a \in [x_1, z_1]$ and $b \in [y_1, z_2]$. By the hypotheses of induction, we conclude $a \in l_{k-1}(P_1, N_1)$ and $b \in l_{k-1}(P_2, N_2)$. Hence, $(a, b) \in l_k(P_1, N_1) \times l_k(P_2, N_2)$. A similar calculation gives the converse. \square

Theorem 3.8. Consider (P_1, N_1) and (P_2, N_2) are two pairs of polygroups. Then $(P_1 \times P_2, N_1 \times N_2)$ is an NP if and only if (P_1, N_1) and (P_2, N_2) are NP.

Proof. (\Leftarrow) Since (P_1, N_1) and (P_2, N_2) are NP, then there exist $k_1, k_2 \in \mathbb{N}$ such that $l_{k_1}(P_1, N_1) \subseteq \omega_{N_1}$ and $l_{k_2}(P_2, N_2) \subseteq \omega_{N_2}$. Put $k = \text{lcm}\{k_1, k_2\}$. Thus, by Theorems ?? and ??, we get

$$l_k(P_1 \times P_2, N_1 \times N_2) = l_k(P_1, N_1) \times l_k(P_2, N_2) \subseteq l_{k_1}(P_1, N_1) \times l_{k_2}(P_2, N_2) \subseteq \omega_{N_1} \times \omega_{N_2}.$$

Therefore, $(P_1 \times P_2, N_1 \times N_2)$ is an NP.

(\Rightarrow) Since $(P_1 \times P_2, N_1 \times N_2)$ is an NP, we conclude that there exists $k \in \mathbb{N}$ such that $l_k(P_1 \times P_2, N_1 \times N_2) \subseteq \omega_{N_1 \times N_2}$. Then by Theorems ?? and ??, we have $l_k(P_1, N_1) \subseteq \omega_{N_1}$ and $l_k(P_2, N_2) \subseteq \omega_{N_2}$. Hence, (P_1, N_1) and (P_2, N_2) are NP. \square

Example 3.9. Let (P_1, N_1) and (P_2, N_2) pairs as Example ??. Then we define \cdot on $P_1 \times P_2 \cong P$ as follows:

| | | | | | | | | |
|---------|-----|-----|---------------|---------------|------------------|------------------|------------------|------------------|
| \cdot | e | a | b | c | d | f | g | h |
| e | e | a | b | c | d | f | g | h |
| a | a | e | c | b | f | d | h | g |
| b | b | c | b | c | $\{e, b, d, g\}$ | $\{a, c, f, h\}$ | g | h |
| c | c | b | c | b | $\{a, c, f, h\}$ | $\{e, b, d, g\}$ | h | g |
| d | d | f | $\{e, b, d\}$ | a, c, f | d | f | $\{d, g\}$ | $\{f, h\}$ |
| f | f | d | $\{a, c, f\}$ | $\{e, b, d\}$ | f | d | $\{f, h\}$ | $\{d, g\}$ |
| g | g | h | $\{b, g\}$ | $\{c, h\}$ | g | h | $\{e, b, d, g\}$ | $\{a, c, f, h\}$ |
| h | h | g | $\{c, h\}$ | $\{b, g\}$ | h | g | $\{a, c, f, h\}$ | $\{e, b, d, g\}$ |

Now, (P, \cdot) is a polygroup [7]. Also, Theorem ?? implies $(P_1 \times P_2, N_1 \times N_2)$ is an NP.

Theorem 3.10. Consider (P_1, N_1) and (P_2, N_2) are two pairs of polygroups and $\phi : (P_1, N_1) \longrightarrow (P_2, N_2)$ is a good homomorphism. If ϕ is one to one and $(K_1, K_1 \cap N_1)$ is a nilpotent sub-polygroup of (P_1, N_1) , then $\phi(K_1, K_1 \cap N_1)$ is a nilpotent sub-polygroup of (P_2, N_2) .

Proof. By induction on n we show that $l_n(\phi(K_1, K_1 \cap N_1)) = \phi(l_n(K_1, K_1 \cap N_1))$. For $n = 1$, it is clear. Now, assume $l_n(\phi(K_1, K_1 \cap N_1)) = \phi(l_n(K_1, K_1 \cap N_1))$ holds and $z \in l_{n+1}(\phi(K_1, K_1 \cap N_1))$. Then there exist $x \in l_n(\phi(K_1, K_1 \cap N_1))$ and $y \in \phi(K_1)$ such that $z \in [x, y]$. By hypotheses of induction $x \in \phi(l_n(K_1, K_1 \cap N_1))$ and so for some $a \in l_n(K_1, K_1 \cap N_1)$ we have $x = \phi(a)$. Also, $y \in \phi(K_1)$ implies that $y = \phi(b)$ for some $b \in K_1$. Therefore, $z \in [\phi(a), \phi(b)] = \phi[a, b]$ and so $z \in \phi(l_{n+1}(K_1, K_1 \cap N_1))$. Therefore,

$$l_{n+1}(\phi(K_1, K_1 \cap N_1)) \subseteq \phi(l_{n+1}(K_1, K_1 \cap N_1)).$$

Conversely, let $z \in \phi(l_{n+1}(K_1, K_1 \cap N_1))$. Then for some $c \in l_{n+1}(K_1, K_1 \cap N_1)$ we have $z = \phi(c)$. Since $c \in l_{n+1}(K_1, K_1 \cap N_1)$ we conclude that there exist $a \in l_n(K_1, K_1 \cap N_1)$ and $b \in K_1$ such that $c \in [a, b]$. Then $z = \phi(c) \in \phi[a, b] = [\phi(a), \phi(b)]$ and so by hypotheses of induction and $a \in l_n(K_1, K_1 \cap N_1)$ we have $\phi(a) \in l_n(\phi(K_1, K_1 \cap N_1))$. Therefore $z \in l_{n+1}(\phi(K_1, K_1 \cap N_1))$. Consequently, $l_n(\phi(K_1, K_1 \cap N_1)) = \phi(l_n(K_1, K_1 \cap N_1))$. \square

In what follows, consider β_1^* is the restriction of β^* on N . Then by Definitions ??, ?? and Theorem ??, we get a relationship between pair of groups and polygroups.

Theorem 3.11. Let $(G, H) = (P/\beta^*, N/\beta_1^*)$. Then for any $k \geq 1$ we have:

$$\gamma_k(G, H) = \langle \{\bar{x} | x \in l_k(P, N)\} \rangle (I).$$

Proof. We prove by induction on k . Since $\gamma_1(G, H) = H = \langle \{\bar{x} | x \in N\} \rangle$, the case $k = 1$ is clear. Now, suppose that (I) holds and $\bar{a} \in \langle \{\bar{x} | x \in l_{k+1}(P, N)\} \rangle$. Then, $a \in l_{k+1}(P, N)$ and so for some $x \in l_k(P, N)$ and $y \in P$ we have $a \in [x, y]$. Hypotheses of induction imply that $\bar{x} \in \gamma_k(G, H)$. Then by Theorem ??, $\bar{a} = [\bar{x}, \bar{y}] \in \gamma_{k+1}(G, H)$. Thus, $\langle \{\beta^*(x) | x \in l_k(P, N)\} \rangle \subseteq \gamma_k(G, H)$.

Conversely, let $\bar{a} \in \gamma_{k+1}(G, H) = [\gamma_k(G, H), G]$. Then $\bar{a} = [\bar{x}, \bar{y}]$ for some $\bar{x} \in \gamma_k(G, H)$, $\bar{y} \in G$ and so $\bar{a} = \bar{x} \otimes \bar{y} \otimes \bar{x}^{-1} \otimes \bar{y}^{-1}$ i.e $\bar{x} \otimes \bar{y} = \bar{a} \otimes \bar{y} \otimes \bar{x}$. By hypotheses of induction $x \in l_k(P, N)$. Thus, by Theorem ??, there exist $c \in xy$ and $d \in ayx$ such that $\bar{c} = \bar{d}$. Let $t \in yx$. By reproduction axioms of polygroups for $c \in xy$ and $t \in yx$ there exists $u \in P$ such that $c \in ut$. Then $c \in ut \subseteq uyx$. Definition ??(ii), implies that

$$u \in c(yx)^{-1} = cx^{-1}y^{-1} \subseteq xyx^{-1}y^{-1} = [x, y].$$

Then $u \in [x, y] = l_{k+1}(P, N)$. Therefore, by Theorem ??,

$$\begin{aligned} \bar{u} = [\bar{x}, \bar{y}] &\Leftrightarrow \bar{u} = \bar{x} \otimes \bar{y} \otimes \bar{x}^{-1} \otimes \bar{y}^{-1} \\ &\Leftrightarrow \bar{u} \otimes \bar{y} \otimes \bar{x} = \bar{x} \otimes \bar{y} = \bar{c} = \bar{d} = \bar{a} \otimes \bar{y} \otimes \bar{x}. \end{aligned}$$

Therefore, $\bar{a} = \bar{u} \in \langle \{\bar{t} | t \in l_{k+1}(P, N)\} \rangle$. \square

Example 3.12. Consider P_1 and N_1 are as Example ??(i). Then $G = P_1/\beta^* = \{\bar{e} = \bar{a} = \bar{b} = \bar{c}\}$ and $H = N_1/\beta_1^* = \{\bar{e} = \bar{a}\}$. In addition, $\gamma_1(G, H) = \{\bar{e}\}$ and $l_1(P_1, N_1) = \omega_{N_1} = \{e, a\}$. Therefore, $\langle \{\bar{x} | x \in l_1(P_1, N_1)\} \rangle = \{\bar{e}\}$. Thus, $\gamma_1(G, H) = \langle \{\bar{x} | x \in l_1(P, N)\} \rangle$

Now, we are ready to state one of the main results of this section. In fact, we get a connection between NP and nilpotent pair of groups.

Theorem 3.13. *The pair (P, N) is an NP if and only if $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})$ is a nilpotent pair of groups.*

Proof. (\Leftarrow) Since (P, N) is an NP, for some $k \in \mathbb{N}$ we have $l_k(P, N) \subseteq \omega_N$. By Theorem 3.10, we have

$$\gamma_k\left(\frac{P}{\beta^*}, \frac{N}{\beta_1^*}\right) = \langle \{\bar{t}; t \in l_k(P, N) \subseteq \omega_N\} \rangle = \{\bar{t}; t \in \omega_N\} = \{\bar{e}\}.$$

It follows that $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})$ is a nilpotent pair of groups.

(\Rightarrow) Let $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})$ be a nilpotent pair of groups. Then for some k , $\gamma_k(\frac{P}{\beta^*}, \frac{N}{\beta_1^*}) = \{\bar{e}\}$. We show that $l_k(P, N) \subseteq \omega_N$. Let $t \in l_k(P, N)$. By Theorem ??, we get $\bar{t} \in \gamma_k(\frac{P}{\beta^*}, \frac{N}{\beta_1^*}) = \{\bar{e}\}$. Hence, $l_k(P, N) \subseteq \omega_N$. Therefore, (P, N) is an NP. \square

Definition 3.14. *Assume $Z_0(P, N) = \omega_N$ and for $i \in \mathbb{N}$,*

$$Z_i(P, N) = \langle \{n \in N \mid nyZ_{i-1}(P, N) = ynZ_{i-1}(P, N) \text{ for any } y \in P\} \rangle.$$

Then series $Z_0(P, N) \subseteq Z_1(P, N) \subseteq \dots$ is called the upper central series of (P, N) .

Example 3.15. *Let $P = \{e, a, b, c, d, f, g\}$ and $N = \{e, a, b\}$. Define \cdot on P as follows:*

| | | | | | | | |
|---------|-----|-----|------------|------------|------------|------------|------------|
| \cdot | e | a | b | c | d | f | g |
| e | e | a | b | c | d | f | g |
| a | a | e | b | c | d | f | g |
| b | b | b | $\{e, a\}$ | g | f | d | c |
| c | c | c | f | $\{e, a\}$ | g | b | d |
| d | d | d | g | f | $\{e, a\}$ | c | b |
| f | f | f | c | d | b | g | $\{e, a\}$ |
| g | g | g | d | b | c | $\{e, a\}$ | f |

Now, (P, \cdot) is a polygroup [7]. Also, $Z_0(P, N) = \omega_N = \{e, a\}$ and $Z_1(P, N) = \langle \{n \in N \mid nyZ_0(P, N) = ynZ_0(P, N) \text{ for any } y \in P\} \rangle = \{e, a\}$.

Theorem 3.16. *Let for any $x \in P$, $xx^{-1} \subseteq Z_i(P, N)$. Then $Z_i(P, N)$ is a normal sub-polygroup of N and $[Z_{i+1}(P, N), P] \subseteq Z_i(P, N)$.*

Proof. First, we prove $Z_i(P, N)$ is a normal sub-polygroup of N . For this let $z \in Z_i$ and $n \in N$. Then by Definition ??, we have $zn^{-1}Z_{i-1}(P, N) = n^{-1}zZ_{i-1}(P, N)$. Therefore, by $nn^{-1} \subseteq Z_i(P, N)$ we get

$$nzn^{-1}Z_{i-1}(P, N) = nn^{-1}zZ_{i-1}(P, N) \subseteq nn^{-1}zZ_i(P, N) = Z_i(P, N).$$

So, $nzn^{-1} = nzn^{-1}e \in nzn^{-1}Z_{i-1}(P, N) \subseteq Z_i(P, N)$. Hence, $nzn^{-1} \subseteq Z_i(P, N)$ and $Z_i(P, N)$ is normal.

Now, we show $[Z_{i+1}(P, N), P] \subseteq Z_i(P, N)$. Let $x \in Z_{i+1}(P, N)$ and $y \in P$. Since $Z_i(P, N)$ is a normal sub-polygroup, by Definition ??, and $xx^{-1} \subseteq Z_i(P, N)$ we conclude that

$$\begin{aligned} x^{-1}y^{-1}Z_i(P, N) = y^{-1}x^{-1}Z_i(P, N) &\Leftrightarrow xyx^{-1}y^{-1}Z_i(P, N) = xy y^{-1}x^{-1}Z_i(P, N) \\ &\Leftrightarrow [x, y]Z_i(P, N) = Z_i(P, N) \\ &\Leftrightarrow [x, y] \subseteq Z_i(P, N). \end{aligned}$$

\square

Example 3.17. Assume P and N are as Example ???. Then for any $x \in P$ we have $xx^{-1} \subseteq Z_1(P, N)$ and $Z_1(P, N) = \{e, a\}$ is a normal sub-polygroup of N .

Theorem 3.18. Let for any $x \in P$ and $i \in \mathbb{N}$, $xx^{-1} \subseteq Z_i(P, N)$. Then (P, N) is an NP if and only if $Z_{r-1}(P, N) = N$, for some $r > 0$.

Proof. (\Leftarrow) It is enough to show that there exists $i \in \mathbb{N}$ such that $l_i(P, N) \subseteq \omega_N$. We argue by induction on i that for any $i \in \mathbb{N}$, $l_i(P, N) \subseteq Z_{r-i}(P, N)$. For $i = 1$, we have $l_1(P, N) = N = Z_{r-1}(P, N) = N$. Now, assume $l_i(P, N) \subseteq Z_{r-i}(P, N)$ and $a \in l_{i+1}(P, N)$. Then for some $x \in l_i(P, N)$ and $y \in P$ we have $a \in [x, y]$. By hypotheses of induction and Theorem ??, $a \in Z_{r-i-1}(P, N)$. Then $l_i(P, N) \subseteq Z_{r-i}(P, N)$. If $i = r$, then $l_r(P, N) \subseteq Z_0(P, N) = \omega_N$ and so (P, N) is an NP.

(\Rightarrow) By induction we show that $l_{r-i}(P, N) \subseteq Z_i(P, N)$, for any $0 \leq i \leq n$. For $i = 0$, we have $l_r(P, N) \subseteq Z_0(P, N) = \omega_N$. Let $l_{r-i}(P, N) \subseteq Z_i(P, N)$ and $a \in l_{r-i-1}(P, N)$ and $b \in P$. Then by Definition ??, $[a, b] \subseteq l_{r-i}(P, N)$. The hypothesis of induction implies $[a, b] \subseteq Z_i(P, N)$. Therefore, $abZ_i(P, N) = baZ_i(P, N)$ and so by Definition ??, $a \in Z_{i+1}(P, N)$ i.e for every $i \in \mathbb{N}$, $l_{r-i}(P, N) \subseteq Z_i(P, N)$. If $i = r - 1$, then $Z_{r-1}(P, N) = N$. \square

Example 3.19. Let P_1 and N_1 be as for Example ???. Then pair (P_1, N_1) is an NP and $Z_0(P_1, N_1) = \omega_{N_1} = N_1$.

Now, we generalize a well-known property of nilpotent groups in pair of polygroups.

Theorem 3.20. Assume for any $x \in P$, $xx^{-1} \subseteq Z_i(P, N)$ and M is a non-trivial normal sub-polygroup of P such that $M \cap N \neq \omega_N$. Then (P, N) is an NP implies that $M \cap Z(P, N) \neq \omega_N$.

Proof. By Theorem ??, there exists a positive integer c such that $N = Z_c(P, N)$. It follows $M \cap Z_c(P, N) \neq \omega_N$. Let i , be a less integer such that $M \cap Z_i(P, N) \neq \omega_N$. Now, by Theorem ??, we have

$$[M \cap Z_i(P, N), P] < M \cap Z_{i-1}(P, N) = \omega_N.$$

By Definition ??, $M \cap Z_i(P, N) < M \cap Z_1(P, N)$. Hence,

$$M \cap Z_i(P, N) < M \cap Z_1(P, N) < M \cap Z_i(P, N).$$

Therefore, $M \cap Z_1(P, N) = M \cap Z_i(P, N) \neq \omega_N$. \square

Theorem 3.21. [7] Let (G, \cdot) be a group. Then $(P_G, \circ, e, {}^{-1})$ is a polygroup, where $P_G = G \cup \{a\}$, $a \notin G$ and \circ is defined as follows:

- (1) $a \circ a = e$,
- (2) $e \circ x = x \circ e = x$, $\forall x \in G$,
- (3) $a \circ x = x \circ a = x$, $\forall x \in G - \{e, a\}$,
- (4) $x \circ y = x.y$, $\forall (x, y) \in G^2; y \neq x^{-1}$,
- (5) $x \circ x^{-1} = x^{-1} \circ x = \{e, a\}$, $\forall x \in G - \{e, a\}$.

In addition, P_G is a nilpotent polygroup if and only if G is a nilpotent group.

Example 3.22. Assume G is the quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. Since G is nilpotent by Theorem ??, we conclude that $(P_G, \circ, e, {}^{-1})$ is a nilpotent polygroup and so (P_G, N) is nilpotent too. Put $M = \{1, -1, j, -j, a\}$ and $N = \{1, -1, i, -i, a\}$. Clearly, $\omega_N = \{1, a\}$, $M \cap N \neq \omega_N$ and $x \circ x^{-1} = \{1, a\} \subseteq Z(P_G, N) = \{1, -1, a\}$, for any $x \in P_G$. In addition, $-1 \in M \cap Z(P_G, N)$ and $-1 \notin \omega_N$ imply that $M \cap Z(P, N) \neq \omega_N$.

Corollary 3.23. If (P, N) is an NP and $N \neq \omega_N$, then $Z(P, N) \neq \omega_N$.

4 Solvable pair of polygroups

The goal of this section is to define the notion of solvable pair of polygroups. Also, we present some results on this topic. Basically, we get a relationship between solvable pair of polygroups and solvable pair of groups. Finally, we prove that every nilpotent pair of polygroups is solvable. First we define a solvable pair of groups. For a pair of groups (G, N) , let $(G, N)^{(0)} = N$, $(G, N)^{(1)} = [N, G]$ and for $k > 1$, $(G, N)^{(k+1)} = [(G, N)^{(k)}, (G, N)^{(k)}]$. The following normal series is called *derived series* of (G, N) (see [12]),

$$N \geq (G, N)^{(1)} \geq (G, N)^{(2)} \geq \dots$$

Definition 4.1. *The pair of groups (G, N) is called solvable if for some $k \in \mathbb{N}$ we have $(G, N)^{(k)} = \{e\}$.*

Note that for an abelian group G and a normal subgroup N of G the pair (G, N) is solvable. Since $(G, N)^{(1)} = [N, G] = \{e\}$.

Here we define a solvable pair of polygroups. In addition, we study the connection between solvable pair of polygroups and solvable pair of groups.

Definition 4.2. *Assume $i_1(P, N) = N$ and for $k > 1$,*

$$i_k(P, N) = \langle \{h \in P \mid h \in [x, y] \text{ such that } x, y \in i_{k-1}(P, N)\} \rangle.$$

Then (P, N) is called solvable pair of polygroups (SP) if $i_k(P, N) \subseteq \omega_N$.

By the same manipulation of the previous section we can obtain the following results, so the proofs are omitted.

Theorem 4.3. *Let (P, N) be a pair of polygroups, H be a normal sub-polygroup of P and H be a sub-polygroup of N . Then for any $n \geq 0$, $i_n(P/H, N/H) = i_n(P, N)H/H$.*

Example 4.4. *Assume P is as Example ??, $N = \{e, a, b\}$ and $H = \{e, a\}$. Then $P/H = \{eH = aH, bH, cH, dH, fH, gH\}$, $NH = \{eH = aH, bH\}$. In addition $i_1(P, N) = N = \{e, a, b\}$, $i_2(P, N) = \{e, a\}$ and $i_1(P/H, N/H) = N/H = \{eH = aH, bH\}$, $i_2(P/H, N/H) = \{eH\}$. Therefore, $i_2(P/H, N/H) = \{H\} = i_2(P, N)H/H$.*

Corollary 4.5. *If (P, N) is an SP and $H \subseteq N$, then $(P/H, N/H)$ is an SP.*

Theorem 4.6. *Let (P_1, N_1) , and (P_2, N_2) be two pairs of polygroups. Then, for any $k \geq 0$ we have:*

$$i_k(P_1 \times P_2, N_1 \times N_2) = i_k(P_1, N_1) \times i_k(P_2, N_2).$$

Corollary 4.7. *Let (P_1, N_1) and (P_2, N_2) be two pairs of polygroups. Then $(P_1 \times P_2, N_1 \times N_2)$ is an SP if and only if (P_1, N_1) and (P_2, N_2) are SP.*

Example 4.8. *Let (P_1, N_1) , (P_2, N_2) and (P, N) be as Example ??, Clearly, $i_1(P_1, N_1) = \omega_{N_1}$ and $i_2(P_2, N_2) = \omega_{N_2}$. By Corollary ??, (P, N) is an SP.*

Theorem 4.9. *Let (P_1, N_1) and (P_2, N_2) be two pairs of polygroups and $\phi : (P_1, N_1) \rightarrow (P_2, N_2)$ be a good homomorphism. If ϕ is one to one and $(K_1, K_1 \cap N_1)$ is a solvable sub-polygroup of (P_1, N_1) , then $\phi(K_1, K_1 \cap N_1)$ is a solvable sub-polygroup of (P_2, N_2) .*

Theorem 4.10. Consider $(G, H) = (P/\beta^*, N/\beta^*)$. Then for any $k \geq 1$ we have:

$$(G, H)^{(k)} = \langle \{\bar{x} | x \in i_k(P, N)\} \rangle.$$

Proof. We prove by induction on k . For $k = 1$, we have $(G, H)^{(1)} = H = \langle \{\bar{x} | x \in N\} \rangle$. Now, suppose that $\bar{a} \in \langle \{\bar{x} | x \in i_{k+1}(P, N)\} \rangle$. Then, $a \in i_{k+1}(P, N)$ and so by Definition ??, there exist $x, y \in i_k(P, N)$ such that $a \in [x, y]$. By hypotheses of induction we have $\bar{x}, \bar{y} \in (G, H)^{(k)}$. It follows that $\bar{a} = [\bar{x}, \bar{y}] \in (G, H)^{(k+1)}$. Therefore, $(G, H)^{(k)} \subseteq \langle \{\bar{x} | x \in i_k(P, N)\} \rangle$.

Conversely, let $\bar{a} \in (G, H)^{(k+1)} = [(G, H)^k, (G, H)^k]$. Therefore, for some $\bar{x}, \bar{y} \in (G, H)^{(k)}$, $\bar{a} = [\bar{x}, \bar{y}]$. Using hypotheses of induction we have $x, y \in i_k(P, N)$. Since $\bar{x} \otimes \bar{y} = \bar{a} \otimes \bar{y} \otimes \bar{x}$, by Theorem ??, we conclude that there exist $c \in xy$ and $d \in ayx$ such that $\bar{c} = \bar{d}$. By the same proof of Theorem ??, there exists $u \in P$ such that $u \in [x, y] \subseteq i_{k+1}(P, N)$. Hence,

$$\bar{u} \otimes \bar{y} \otimes \bar{x} = \bar{x} \otimes \bar{y} = \bar{c} = \bar{d} = \bar{a} \otimes \bar{y} \otimes \bar{x}.$$

So we have $\bar{a} = \bar{u} \in \langle \{\bar{t}; t \in i_{k+1}(P, N)\} \rangle$. Therefore, $(G, H)^{(k)} \supseteq \langle \{\beta^*(x) | x \in i_k(P, N)\} \rangle$. \square

Theorem 4.11. Pair (P, N) is an SP if and only if $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})$ is a solvable pair of groups.

Proof. (\Rightarrow) Since (P, N) is a solvable pair of groups $i_k(P, N) \subseteq \omega_N$, by Theorem ??,

$$\left(\frac{P}{\beta^*}, \frac{N}{\beta_1^*}\right)^{(k)} = \langle \{\bar{t}; t \in i_k(P, N) \subseteq \omega_N\} \rangle.$$

So $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})$ is a solvable pair of groups.

(\Leftarrow) Consider $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})$ is solvable. Then there exists k such $(\frac{P}{\beta^*}, \frac{N}{\beta_1^*})^{(k)} = \bar{e}$. Let $t \in i_k(P, N)$. By Theorem ??, $\bar{t} \in (\frac{P}{\beta^*}, \frac{N}{\beta_1^*})^{(k)}$. Therefore, $i_k(P, N) \subseteq \omega_N$. This completes the proof. \square

Now, we study the connection between nilpotent and solvable pair of polygroups.

Theorem 4.12. Every nilpotent pair of polygroups is solvable.

Proof. Let (P, N) be an NP. It is easy to see that for any $k \in \mathbb{N}$, $i_k(P, N) \subseteq l_k(P, N)$. Now, since (P, N) is an NP we get that for some n , $l_n(P, N) \subseteq \omega_N$ and so $i_n(P, N) \subseteq \omega_N$. Therefore, (P, N) is an SP. \square

Example 4.13. Assume P_1 and N_1 are as in Example ??. Pair (P_1, N_1) is an NP and by Theorem ??, we have (P_1, N_1) is an SP. In addition, $(\frac{P_1}{\beta^*}, \frac{N_1}{\beta_1^*}) = \frac{P_1}{\beta^*} = \{\bar{e}\}$ and so $(\frac{P_1}{\beta^*}, \frac{N_1}{\beta_1^*})$ is a solvable pair of groups.

A polygroup is called *proper* if it is not a group.

In what follows we show that the converse of Theorem ??, is not true in general.

Example 4.14. Let S_3 be the symmetric group and A_3 be the alternative group of degree 3. Then by definition of derived series of pair (G, N) we have $(S_3, A_3)^{(0)} = A_3$ and so $(S_3, A_3)^{(1)} = [A_3, A_3] = \{e\}$. It follows that (S_3, A_3) is solvable. But by Example ??, (S_3, A_3) is not nilpotent.

Theorem 4.15. Every proper pair of polygroups of order less than 61 is solvable.

Proof. Let (P, N) be a proper pair of polygroups of order less than 61. Then, $(P/\beta^*, N/\beta^*)$ is a pair of groups of order less than 60 that is solvable (see [13]). Therefore, Theorem ??, follows that (P, N) is an SP. \square

5 Conclusion

In this paper, for a polygroup P and a normal sub-polygroup N of P the notions of nilpotent and solvable pairs of polygroups (P, N) were defined. Moreover, the notion of nilpotent pair of polygroups (P, N) as a generalization of nilpotent polygroup was introduced. Some examples have been used to clarify the concept of nilpotent (solvable) pair of polygroup. In addition, a connection between nilpotent (solvable) pair of groups and nilpotent (solvable) pairs of polygroup was obtained. Especially, the relationship between nilpotent and solvable pair of polygroups were investigated. This work can be used on Engel pair of polygroups, too.

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