Nearness d-algebras

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"This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday."

Abstract

In this paper, we have studied the notion of nearness d-algebras which is a generalization of nearness BCK-algebras (or $NBCK$-algebra for short). Therefore, it is defined that the notions of nearness d-subalgebra, nearness d-ideal in nearness d-algebras. Afterward, we investigated relations among them and gave some examples.

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1 Introduction

Set theory is very important tool especially for engineers and mathematicians. They use set theory as a base in their studies. Researchers defined new approaches when ordinary set theory is insufficient. Because, the real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [27]. In response to this situation Zadeh [28] introduced fuzzy set theory as an alternative to probability theory. Then, rough set theory, proposed by Pawlak in 1982, focusesed on the uncertainty caused by indiscernible elements with different values in decision attributes [20]. Worldwide, there has been a rapid growth in interest in rough set theory and its applications in recent years. In 1999, Molodtsov [7] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. Moreover, to overcome these difficulties, Molodtsov [7]

introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches.

In 2002, Peters introduced near set theory as a generalization of rough set theory. In this theory, Peters depends on the features of objects to define the nearness of objects and consequently, the classification of our universal set with respect to the available information of the objects. The concept of near set theory was motivated by image analysis and inspired by a study of the perception of the nearness of familiar physical objects. Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real valued function representing a feature of physical objects such as images or behaviors of individual biological organisms. In [23], an indiscernibility relation that depends on the features of the objects in order to define the nearness of the objects was given. In more recent studies, it has been accepted as a generalized approach theory to investigate the nearness of similar non-empty sets (see [21], [22], [26], [25], and [24]).

We may be used near set theory to turn elements in algebraic structures into concrete elements. Because, in the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points. But this is not useful for real life problems. All researchers who study algebraic structures consider abstract elements. But using them in our study some time is insufficent. We use perceptual objects (non-abstract points) in near set theory. Perceptual objects have some features such as colour, degree of maturation for an apple. In the algebraic structures constructed on nearness approximation spaces or weak nearness approximation spaces, the basic tool is consideration of upper approximations of the subsets of perceptual objects. There are two important differences between ordinary algebraic structures and nearness algebraic structures. The first one is working with non-abstract points while the second one is considering of upper approximations of the subsets of perceptual objects for the closeness of binary operations.

The number sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are very useful in the field of engineering. But, we emphasize that the elements of the sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ have one and only one property. Having just one feature is not valuable to study for us. Because, many things has multiple features in real life. So, we must be take attention perceptible elements which has more than one property. Since the elements of the sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ has one and only one property, upper and lower approximation's and itself of these sets are equal to each other for $r = n$ $(n \in \mathbb{Z}^+)$. The sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are algebraic structures and moreover nearness algebraic structures.

Many researchers defined algebraic structures in different sets such as fuzzy set, rough set and soft set. For example, they defined fuzzy over a group as a fuzzy group. But, nearness algebraic structures defined on a set unlike them. Therefore, we think that the nearness algebraic structure must be studied in which has property of $G \subset N_r(B)^*G$, where G is a nearness algebraic structure.

In 2012, İnan and Öztürk analyzed the concept of nearness groups and investigated their basic properties [2, 3]. Then, in [4], the nearness semigroups and nearness rings were established and their basic properties were investigated (and other algebraic approaches of near sets in [10], [11],[13] [14],[16], [18]).

In 2015, Öztürk, Çelik Siner, and Jun introduced BCK-algebras on nearness approximation spaces [12]. Furthermore, the some properties of nearness BCK-algebras was checked out, and several examples were given about nearness BCK-algebras. Afterwards, in [15], Öztürk and Jun have defined quotient $NBCK$-algebras by considering an operation on the set of all cosets defined via ideals and also analyzed some properties of the quotient $NBCK$-algebras.

In this article, it is defined that the notion of nearness d-algebras which is a generalization of nearness BCK-algebras and given several examples. Furthermore, we show that the notions of nearness d-subalgebra, nearness d-ideal in nearness d-algebras, and investigate relations among

them.

2 Preliminaries

An object description is determined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$, which is a subset of an object space $\mathcal{O}$. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \to \mathbb{R}$ (set of reals). In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \to \mathbb{R}^L, \Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$ a vector containing measurements (returned values) associated with each functional value $\varphi_i(x)$, where the description length $\mid \Phi \mid = L$ (see [21]).

The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe an object of interest. Sample objects $X \subseteq \mathcal{O}$ are near each if and only if the objects have similar descriptions. Recall that each φ defines a description of an object. Then, $\triangle_{\varphi_i}$ denote $\triangle_{\varphi_i} = \mid \varphi_i(x') - \varphi_i(x) \mid$, where $x', x \in \mathcal{O}$. The difference φ leads to a description of the indiscernibility relation " $\sim_B$ " introduced by Peters in [21].

Definition 2.1. [21] *Let* $x, x' \in \mathcal{O}, B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$*, where description length* $i \leq \mid \Phi \mid$.

Comparing object descriptions is the basic idea in the near set approach to object recognition. Sets of object X, X' are called near each other if those sets contain the objects with at least partial matching descriptions.

Definition 2.2. [21] *Let* $X, X' \subseteq \mathcal{O}, B \subseteq \mathcal{F}$. *Then* X *is called near* X' *if there exist* $x \in X, x' \in X', \varphi_i \in B$ *such that* $x \sim_{\varphi_i} x'$.

A weak nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$, where the approximation space is defined with respect to a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, $\sim_{B_r}$ indiscernibility relation B_r defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, and collection of partitions (families of neighbourhoods) $N_r(B)$. This relation $\sim_{B_r}$ defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$. These classes form a new set called the quotient set $\mathcal{O}/\sim_{B_r}$, where $\mathcal{O}/\sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\}$. In effect, each choice of probe functions B_r defines a partition $\xi_{\mathcal{O}, B_r}$ on a set of objects $\mathcal{O}$, namely, $\xi_{\mathcal{O}, B_r} = \mathcal{O}/\sim_{B_r}$. We consider $X \subseteq \mathcal{O}$, then upper approximation of X defined by $N_r(B)^- X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$, and lower approximation of X defined by $N_r(B)_- X = \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}$, (see [10], [21]).

Theorem 2.3. [15] *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ *be a weak nearness approximation space and* $X, Y \subset \mathcal{O}$, *then the following statements hold;*
  i) $N_r(B)_- X \subseteq X \subseteq N_r(B)^- X$,
  ii) $N_r(B)^- (X \cup Y) = (N_r(B)^- X) \cup (N_r(B)^- Y)$,
  iii) $N_r(B)_- (X \cup Y) \supseteq (N_r(B)_- X) \cup (N_r(B)_- Y)$,
  iv) $N_r(B)^- (X \cap Y) \subseteq (N_r(B)^- X) \cap (N_r(B)^- Y)$,
  v) $N_r(B)_- (X \cap Y) = (N_r(B)_- X) \cap (N_r(B)_- Y)$,
  vi) $X \subseteq Y$ *implies* $N_r(B)^- X \subseteq N_r(B)^- Y$,
  vii) $X \subseteq Y$ *implies* $N_r(B)_- X \subseteq N_r(B)_- Y$.

Definition 2.4. [15] *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ *be a weak nearness approximation space;* $\emptyset \neq \mathcal{X} \subseteq \mathcal{O}$, *and* 0 *be a constant on* $\mathcal{O}$. *A subset* $\mathcal{X}$ *of the set* $\mathcal{O}$ *is called BCK-algebra on nearness approximation space* $\mathcal{O}$ *or NBCK-algebra for short if the following properties are satisfied for all* $a, b, c \in \mathcal{X}$

$NBCI - i)$ $((a \circledast b) \circledast (a \circledast c)) \circledast (c \circledast b) = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCI - ii)$ $(a \circledast (a \circledast b)) \circledast b = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCI - iii)$ $a \circledast a = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCK - iv)$ $0 \circledast a = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCI - v)$ *If* $a \circledast b = 0$ *and* $b \circledast a = 0$, *then* $a = b$ *property holds in* $\mathcal{X}$,

where $\circledast : \mathcal{X} \times \mathcal{X} \to N_r(B)^- \mathcal{X}$ *is a well-defined operation.*

"$\preceq$" *relation defined on* $\mathcal{X}$, *for all* $a, b \in \mathcal{X}$, $a \preceq b :\Leftrightarrow a \circledast b = 0$.
NBCK-algebra again can be defined with this relation:
$NBCI - i')$ $(a \circledast b) * (a \circledast c) \preceq c \circledast b$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCI - ii')$ $a * (a \circledast b) \preceq b$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCI - iii')$ $a \preceq a$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCK - iv')$ $0 \preceq a$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCI - v')$ *If* $a \preceq b$ *and* $b \preceq a$, *then* $a = b$ *property holds in* $\mathcal{X}$,

$NBCI - vi')$ $a \preceq b \Leftrightarrow a \circledast b = 0$ *property holds in* $N_r(B)^- \mathcal{X}$.

Definition 2.5. [15] *Let* $\mathcal{X}$ *be an NBCK-algebra, and* S *be a non-empty subset of* $\mathcal{X}$. *Then,* S *is called a subalgebra of* $\mathcal{X}$ *if* $0 \in N_r(B)^- S$ *and* $a \circledast b \in N_r(B)^- S$ *for all* $a, b \in S$.

For other notions and definitions not mentioned in this paper, the readers are referred to [5] [6], [8], [9], [12], [16], [15] and [22].

3 Nearness d-algebras

Definition 3.1. *Let* $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ *be a weak nearness approximation space;* $\emptyset \neq \mathcal{X} \subseteq \mathcal{O}$, *and* 0 *be a constant on* $\mathcal{O}$. *A subset* $\mathcal{X}$ *of the set* $\mathcal{O}$ *is called d-algebra on weak nearness approximation space* $\mathcal{O}$ *or nearness d-algebra for short if the following properties are satisfied for all* $a, b, c \in \mathcal{X}$

$Nd - i)$ $a \circledast a = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$Nd - ii)$ $0 \circledast a = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$Nd - iii)$ *If* $a \circledast b = 0$ *and* $b \circledast a = 0$, *then* $a = b$ *property holds in* $\mathcal{X}$, *where* $\circledast : \mathcal{X} \times \mathcal{X} \to N_r(B)^- \mathcal{X}$ *is a well-defined operation.*

A nearness BCK-algebra is a nearness d-algebra $(\mathcal{X}, *, 0)$ satisfying the following additional axioms:

$NBCK - i)$ $((a \circledast b) \circledast (a \circledast c)) \circledast (c \circledast b) = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

$NBCK - ii)$ $(a \circledast (a \circledast b)) \circledast b = 0$ *property holds in* $N_r(B)^- \mathcal{X}$,

for all $a, b, c \in \mathcal{X}$. Therefore, nearness d-algebra is a generalization of nearness BCK-algebras.

Example 3.2. *Let* $\mathcal{O} = \{0, a, b, c, d, e\}$ *be a set of perceptual objects,* $r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ *be a set of probe functions, and* $\mathcal{X} = \{a, b, c\} \subset \mathcal{O}$. *Values of the probe functions*

$$\varphi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\beta_2, \beta_3, \beta_4\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$$

are given in Table 1.

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Table 1

	0	a	b	c	d	e
φ_1	β_1	β_1	β_2	β_3	β_1	β_4
φ_2	β_2	β_3	β_2	β_3	β_3	β_4
φ_3	β_2	β_3	β_2	β_4	β_1	β_5

</div>

Let us now determine the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ of elements of $\mathcal{O}$:

$$[0]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(0) = \beta_1\} = \{0, a, d\} = [a]_{\varphi_1} = [d]_{\varphi_1},$$
$$[b]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(b) = \beta_2\} = \{b\},$$
$$[c]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(c) = \beta_3\} = \{c\},$$
$$[e]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(e) = \beta_4\} = \{e\}.$$

Then, we get $\xi_{\varphi_1} = \left\{ [0]_{\varphi_1}, [b]_{\varphi_1}, [c]_{\varphi_1}, [e]_{\varphi_1} \right\}$.

$$[0]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(0) = \beta_2\} = \{0, b\} = [b]_{\varphi_2},$$
$$[a]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(a) = \beta_3\} = \{a, c, d\}, = [c]_{\varphi_2} = [d]_{\varphi_2},$$
$$[e]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(d) = \beta_4\} = \{e\}.$$

Thus, we have $\xi_{\varphi_2} = \left\{ [0]_{\varphi_2}, [a]_{\varphi_2}, [e]_{\varphi_2} \right\}$.

$$[0]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(0) = \beta_2\} = \{0, b\} = [b]_{\varphi_3},$$
$$[a]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(a) = \beta_3\} = \{a\},$$
$$[c]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(c) = \beta_4\} = \{c\},$$
$$[d]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(d) = \beta_1\} = \{d\},$$
$$[e]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(e) = \beta_5\} = \{e\}.$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [0]_{\varphi_3}, [a]_{\varphi_3}, [c]_{\varphi_3} [d]_{\varphi_3}, [e]_{\varphi_3} \right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$N_1(B)^- \mathcal{X} = \bigcup_{[x]_{\varphi_i} \cap \mathcal{X} \neq \emptyset} [x]_{\varphi_i} = \{0, a, b, c, d\}.$$

Considering the following table of operation :

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Table 2

$\circledast$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	d	d
c	c	a	a	0	0
d	d	d	d	d	0

</div>

In that case; $\mathcal{X} := \{a, b, c\}$ is a d-algebra by Definition 3.1. But, $\mathcal{X}$ is not a nearness BCK-algebra, since $(b \circledast (b \circledast c)) \circledast c = (b \circledast d) \circledast c = d \circledast c = d \neq 0$.

Definition 3.3. *Let $\mathcal{X}$ be a nearness d-algebra, and S be a non-empty subset of $\mathcal{X}$. Then, S is called a nearness d-subalgebra of $\mathcal{X}$ if $a \circledast b \in N_r(B)^- S$ for all $a, b \in S$. Also, S is called an upper-near d-subalgebra of $\mathcal{X}$ if $0 \in N_r(B)^- S$ and $a \circledast b \in N_r(B)^- S$ for all $a, b \in N_r(B)^- S$.*

Example 3.4. *From Example 3.2, we have $N_1(B)^- \mathcal{X} = \{0, a, b, c, d\} \subset \mathcal{O}$. Then, we can write*

$$N_1(B)^- (N_1(B)^- \mathcal{X}) = \bigcup_{[x]_{\varphi_i} \cap N_1(B)^- \mathcal{X} \neq \emptyset} [x]_{\varphi_i} = \{0, a, b, c, d\}.$$

In that case; $N_1(B)^- \mathcal{X} = \{0, a, b, c, d\}$ is a nearness d-algebra by Table 2. Let $S := \{b, c\} \subset \mathcal{X}$. Thus, we get

$$N_1(B)^- S = \bigcup_{[x]_{\varphi_i} \cap S \neq \emptyset} [x]_{\varphi_i} = \{0, b, c, d\}.$$

Then, $S := \{b, c\}$ isn't a nearness d-subalgebra of $\mathcal{X}$, for $c \circledast b = a \notin N_1(B)^- S$ by Table 2. If $S := \{a, c\} \subset \mathcal{X}$ then,

$$N_1(B)^- S = \bigcup_{[x]_{\varphi_i} \cap S \neq \emptyset} [x]_{\varphi_i} = \{0, a, c, d\}.$$

From Table 2, $S := \{b, c\}$ is a nearness d-subalgebra of $\mathcal{X}$.

Definition 3.5. *Let $\mathcal{X}$ be a d-algebra on $\mathcal{O}$, $B_r \subseteq \mathcal{F}$ where $r \leq |B|$ and $B \subseteq \mathcal{F}$, $\sim_{B_r}$ be an indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on $\mathcal{X}$, if $x \sim_{B_r} y$, where $x, y \in \mathcal{X}$ implies $(x \circledast b) \sim_{B_r} (y \circledast b)$, and $(b \circledast x) \sim_{B_r} (b \circledast y)$ for all $a, b \in \mathcal{X}$.*

Theorem 3.6. *Let $\mathcal{X}$ be a nearness d-algebra, $\sim_{B_r}$ be a congruence indiscernibility relation on $\mathcal{X}$, and S be a non-empty subset of $\mathcal{X}$. If $a \circledast b \in S$ for all $a, b \in S$, then S is an upper-near d-subalgebra of $\mathcal{X}$.*

Proof. Let $a, b \in N_r(B)^- S$. Then, we have $[a]_{B_r} \cap S \neq \emptyset$ and $[b]_{B_r} \cap S \neq \emptyset$. Hence, there exists elements $x, y \in S$ such that $x \in [a]_{B_r}$ and $y \in [b]_{B_r}$. In this case, $x \circledast y \in S$ by hypothesis, $a \sim_{B_r} x$ and $b \sim_{B_r} y$, thus $(a \circledast b) \sim_{B_r} (x \circledast b)$ and $(x \circledast b) \sim_{B_r} (x \circledast y)$ and so $(a \circledast b) \sim_{B_r} (x \circledast y)$. Hence $x \circledast y \in [a \circledast b]_{B_r}$, and so $[a \circledast b]_{B_r} \cap S \neq \emptyset$. Therefore, we get $a \circledast b \in N_r(B)^- S$. Also, $a \circledast a = 0 \in N_r(B)^- S$ by Definition 3.3. $\square$

Definition 3.7. *Let $(\mathcal{X}, \circledast, 0)$ be a nearness d-algebra, and I be a non-empty subset of $\mathcal{X}$. Then,*

   i) A set I is called a nearness d-ideal of $\mathcal{X}$ if it satisfies:

   NI1) $a \circledast b \in N_r(B)^- I$ and $b \in I$ implies $a \in I$ for all $a, b \in \mathcal{X}$,

   NI2) $a \in I$ and $b \in \mathcal{X}$ imply $a \circledast b \in N_r(B)^- I$.

   ii) A set I is called an upper-near d-ideal of $\mathcal{X}$ if it satisfies:

   UNI1) $0 \in N_r(B)^- I$,

   UNI2) $a \circledast b \in N_r(B)^- I$ and $b \in I$ implies $a \in I$ for all $a, b \in \mathcal{X}$,

   UNI3) $a \in N_r(B)^- I$ and $b \in N_r(B)^- \mathcal{X}$ implies $a \circledast b \in N_r(B)^- I$.

Example 3.8. *Let* $\mathcal{O} = \{0, a, b, c, d, e, f\}$ *be a set of perceptual objects,* $r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ *be a set of probe functions, and* $\mathcal{X} = \{a, b, c, d, e\} \subset \mathcal{O}, I = \{a, b, c\} \subset \mathcal{X}$. *Values of the probe functions*

$$\varphi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

are given in Table 3.

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Table 3

	0	a	b	c	d	e	f
φ_1	β_1	β_1	β_2	β_2	β_3	β_5	β_4
φ_2	β_1	β_4	β_4	β_2	β_3	β_2	β_5
φ_3	β_2	β_1	β_4	β_1	β_2	β_4	β_3

</div>

We find the near equivalence classes according to the indiscernibility relation of $\sim_{B_r}$ *of elements of* $\mathcal{O}$:

$$[0]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(0) = \beta_1\} = \{0, a\} = [a]_{\varphi_1},$$
$$[b]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(b) = \beta_2\} = \{b, c\} = [c]_{\varphi_1},$$
$$[d]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(d) = \beta_3\} = \{d\},$$
$$[e]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(e) = \beta_5\} = \{e\},$$

$$[f]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(f) = \beta_4\} = \{f\}.$$

Then, we get that $\xi_{\varphi_1} = \left\{ [0]_{\varphi_1}, [b]_{\varphi_1}, [d]_{\varphi_1}, [e]_{\varphi_1}, [f]_{\varphi_1} \right\}$.

$$[0]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(0) = \beta_1\} = \{0\},$$
$$[a]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(a) = \beta_4\} = \{a, b\} = [b]_{\varphi_2},$$
$$[c]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(c) = \beta_2\} = \{c, e\} = [e]_{\varphi_2},$$
$$[d]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(d) = \beta_3\} = \{d\},$$
$$[f]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(d) = \beta_5\} = \{f\}.$$

Thus, we have that $\xi_{\varphi_2} = \left\{ [0]_{\varphi_2}, [a]_{\varphi_2}, [c]_{\varphi_2}, [d]_{\varphi_2}, [f]_{\varphi_2} \right\}$.

$$[0]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(0) = \beta_2\} = \{0, d\},$$
$$[a]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(a) = \beta_1\} = \{a, c\} = [c]_{\varphi_3},$$
$$[b]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(b) = \beta_4\} = \{b, e\} = [e]_{\varphi_3},$$
$$[f]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(f) = \beta_3\} = \{f\}.$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [0]_{\varphi_3}, [a]_{\varphi_3}, [b]_{\varphi_3}, [f]_{\varphi_3} \right\}$. *Therefore, for* $r = 1$, *a set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. *Then, we can write*

$$N_1(B)^- \mathcal{X} = \bigcup_{[x]_{\varphi_i} \cap \mathcal{X} \neq \emptyset} [x]_{\varphi_i} = \{0, a, b, c, d, e, f\}.$$

Furthermore,

$$N_1\left(B\right)^- I = \bigcup_{[x]_{\varphi_i} \cap I \neq \emptyset} [x]_{\varphi_i} = \{0, a, b, c, e\}.$$

Considering the following table of operation:

Table 4

⊛	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	a	0	0	0	0	0	0
b	b	b	0	0	e	e	b
c	c	a	a	0	0	0	e
d	d	d	d	d	0	d	e
e	e	d	d	d	d	0	e
f	f	f	f	f	f	f	0

In that case; $(\mathcal{X}, \circledast, 0)$ *is a d-algebra on* $\mathcal{O}$ *by Definition 3.1, in other words,* $\mathcal{X}$ *is a nearness d-algebra. Then,* I *is not a nearness d-ideal of* $\mathcal{X}$*, since* $a, e \in \mathcal{X}$*,* $e \circledast a = d \in N_r\left(B\right)^- I$ *and* $a \in I$ *but* $e \notin I$*. Hence,* I *is an upper-near d-ideal of* $\mathcal{X}$*.*

Example 3.9. *Let* $\mathcal{O} = \{0, a, b, c, d, e, f\}$ *be a set of perceptual objects,* $r = 1$*,* $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ *be a set of probe functions, and* $\mathcal{X} = \{0, a, b, c, d, e\} \subset \mathcal{O}, I = \{0, a, d, e\} \subset \mathcal{X}$*. Values of the probe functions*

$$\varphi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$$

are given in Table 5.

Table 5

	0	a	b	c	d	e	f
φ_1	β_1	β_1	β_2	β_2	β_3	β_5	β_4
φ_2	β_1	β_4	β_2	β_2	β_3	β_3	β_2
φ_3	β_2	β_1	β_4	β_5	β_1	β_3	β_4

Now, we determine the near equivalence classes according to the indiscernibility relation of $\sim_{B_r}$ *of elements of* $\mathcal{O}$:

$$[0]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(0) = \beta_1\} = \{0, a\} = [a]_{\varphi_1},$$
$$[b]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(b) = \beta_2\} = \{b, c\} = [c]_{\varphi_1},$$
$$[d]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(d) = \beta_3\} = \{d\},$$
$$[e]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(e) = \beta_5\} = \{e\},$$
$$[f]_{\varphi_1} = \{y \in \mathcal{O} \mid \varphi_1(y) = \varphi_1(f) = \beta_4\} = \{f\}.$$

Then, we get that $\xi_{\varphi_1} = \left\{[0]_{\varphi_1}, [b]_{\varphi_1}, [d]_{\varphi_1}, [e]_{\varphi_1}, [f]_{\varphi_1}\right\}.$

$$[0]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(0) = \beta_1\} = \{0\},$$
$$[a]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(a) = \beta_4\} = \{a\},$$
$$[b]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(c) = \beta_2\} = \{b, c, f\} = [c]_{\varphi_2} = [f]_{\varphi_2},$$
$$[d]_{\varphi_2} = \{y \in \mathcal{O} \mid \varphi_2(y) = \varphi_2(d) = \beta_3\} = \{d, e\} = [e]_{\varphi_2}.$$

Thus, we have that $\xi_{\varphi_2} = \left\{ [0]_{\varphi_2}, [a]_{\varphi_2}, [b]_{\varphi_2}, [d]_{\varphi_2} \right\}.$

$$[0]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(0) = \beta_2\} = \{0\},$$
$$[a]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(a) = \beta_1\} = \{a, d\} = [d]_{\varphi_3},$$
$$[b]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(b) = \beta_4\} = \{b, f\} = [f]_{\varphi_3},$$
$$[c]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(c) = \beta_5\} = \{c\},$$
$$[e]_{\varphi_3} = \{y \in \mathcal{O} \mid \varphi_3(y) = \varphi_3(f) = \beta_3\} = \{e\}.$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [0]_{\varphi_3}, [a]_{\varphi_3}, [b]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3} \right\}.$ *Therefore, for* $r = 1$, *a set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}.$ *Then, we can write*

$$N_1(B)^- \mathcal{X} = \bigcup_{[x]_{\varphi_i} \cap \mathcal{X} \neq \emptyset} [x]_{\varphi_i} = \{0, a, b, c, d, e, f\}.$$

Furthermore,

$$N_1(B)^- I = \bigcup_{[x]_{\varphi_i} \cap I \neq \emptyset} [x]_{\varphi_i} = \{0, a, d, e\}.$$

Considering the following table of operation:

Table 6

⊛	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	a	0	0	0	0	0	0
b	b	b	0	0	b	b	0
c	c	c	c	0	c	c	0
d	d	d	d	d	0	d	0
e	e	e	e	e	e	0	0
f	f	f	f	f	f	f	0

Then, $(\mathcal{X}, \circledast, 0)$ is a d-algebra on $\mathcal{O}$ by Definition 3.1, in other words, $\mathcal{X}$ is a nearness d-algebra. Moreover, I is a nearness d-ideal of $\mathcal{X}$.

Let $\mathcal{X}$ be a nearness d-algebra. Let $X \circledast Y = \{x \circledast y \mid x \in X, \text{ and } y \in Y\}$, where X and Y are subsets of $\mathcal{X}$.

Lemma 3.10. *Let* $\mathcal{X}$ *be a nearness d-algebra. If* $\sim_{B_r}$ *is a congruence indiscernibility relation on* $\mathcal{X}$, *then* $[x]_{B_r} \circledast [y]_{B_r} \subseteq [x \circledast y]_{B_r}$ *for all* $x, y \in \mathcal{X}.$

Definition 3.11. *Let $\mathcal{X} \subseteq \mathcal{O}$ and $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak near approximation spaces, $B_r \subseteq \mathcal{F}$ where $r \leq |B|$ and $B \subseteq \mathcal{F}$, $\sim_{B_r}$ be an indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a complete congruence indiscernibility relation on nearness d-algebra $\mathcal{X}$, if $[a]_{B_r} \circledast [b]_{B_r} = [a \circledast b]_{B_r}$ for all $a, b \in \mathcal{X}$.*

Theorem 3.12. *Let $\mathcal{X}$ be a nearness d-algebra, and I be a non-empty subset of $\mathcal{X}$. If $\sim_{B_r}$ is a complete congruence indiscernibility relation on $\mathcal{X}$, and I is nearness d-ideal of $\mathcal{X}$, then I is an upper-near d-ideal of $\mathcal{X}$.*

Proof. Let $a, b \in \mathcal{X}$ such that $a \circledast b, b \in N_r(B)^- I$. Then, $[a \circledast b]_{B_r} \cap I \neq \emptyset$ and since $\sim_{B_r}$ is a complete congruence indiscernibility relation, we have $[a]_{B_r} \circledast [b]_{B_r} \cap I \neq \emptyset$ and $[b]_{B_r} \cap I \neq \emptyset$. Therefore, there exist $c, d \in I$ such that $c \in [a]_{B_r} * [b]_{B_r} \cap I \neq \emptyset$ and $d \in [b]_{B_r} \cap I \neq \emptyset$. Thus, we have $c = x \circledast y; x \in [a]_{B_r}, y \in [b]_{B_r}$, and so we get $a \sim_{B_r} x$ and $b \sim_{B_r} y$, and also $b \sim_{B_r} d$. Hence, we obtain $y \sim_{B_r} d$ and so $[y]_{B_r} = [d]_{B_r}$. Since $d \in I$, we have $y \in I$. Combining $c = x \circledast y \in I$ and using $y \in I$, we have $x \in I$ by hypothesis. In this case, $[a]_{B_r} \cap I \neq \emptyset$ implies $a \in N_r(B)^- I$. Similarly, $(UNI3)$ is obtained. On the other hand, we have $a \circledast a = 0 \in N_r(B)^- I$ by hypothesis. This comletes the proof. $\qquad\square$

4 Conclusions

As a recent study of nearness BCK-algebras, this paper shows some notion of nearness d-algebras which is a generalization of nearness BCK-algebras. Also, it is defined that the notions of nearness d-subalgebra, nearness d-ideal in nearness d-algebras. Afterward, we investigate relations among them and give some examples. We believe that these properties will be more useful theoretical development for d-algebras theory. One can consider others types of algebra like nearness BCC-algebra, nearness BCH-algebra, nearness PU-algebra, nearness subtraction algebra and etc. based on the ideas and results of this paper.

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