



Implicative soju ideals of BCK-algebras

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“This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday.”

Abstract

Molodtsov proposed soft set theory to address uncertainty in a parameter manner, and Atanassov introduced intuitionistic fuzzy set which is very useful in providing a flexible model to elaborate uncertainty and vagueness involved in decision making. Using intuitive fuzzy set and soft set, Jun et al. introduced a new notion, “soju structure”, as a kind of hybrid structure. The concept of implicative soju ideal in BCK-algebra will be introduced and several properties will be investigated in this article. We will discuss the relationship between soju subalgebra, soju ideal, positive implicative soju ideal, commutative soju ideal and implicative soju ideal. Also, characterizations of implicative soju ideal will be established.

Article Information

Corresponding Author: S.Z. Song; Received: March 2021; Accepted: Invited paper; Paper type: Original.

Keywords:

(Positive implicative, commutative) soju ideal, implicative soju ideal.



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1 Introduction

Soft set theory, as the general version of fuzzy set theory, was proposed by Molodtsov [20] to address uncertainty in a parameter manner. Intuitionistic fuzzy set, introduced by Atanassov [4, 5], is also a generalization of fuzzy set theory, and it is very useful in providing a flexible model to elaborate uncertainty and vagueness involved in decision making. Using soft set theory and intuitionistic fuzzy set theory to study several aspects including algebraic structures is being done by many scholars (see [1, 2, 3, 6, 7, 8, 10, 12, 11, 14, 15, 16, 18, 22, 23, 25]). Jun et al. used intuitionistic fuzzy set and soft set to introduce a new structure, so called a soju structure, which is a kind of hybrid structure, and then it is applied to BCK/BCI-algebras (see [17, 21, 24]).

In this paper, we introduce the implicative soju ideal in BCK-algebra and investigate several properties. We investigate the relationship between soju subalgebra, soju ideal, positive implicative soju ideal, commutative soju ideal and implicative soju ideal. We provide examples to show that any soju subalgebra (resp., soju ideal, positive implicative soju ideal and commutative soju ideal) is not an implicative soju ideal. We find and present conditions under which soju subalgebra (resp., soju ideal, positive implicative soju ideal and commutative soju ideal) can be implicative soju ideal. We discuss characterizations of implicative soju ideal.

2 Preliminaries

This section lists the already well-known concepts for BCK-algebras and soju structures related to this paper.

2.1 Basic concepts about BCK-algebras

A *BCK-algebra* is defined to be an algebra $(X; *, 0)$ that satisfies the following conditions:

$$(I_1) ((\tilde{x} * \tilde{y}) * (\tilde{x} * \tilde{z})) * (\tilde{z} * \tilde{y}) = 0,$$

$$(I_2) (\tilde{x} * (\tilde{x} * \tilde{y})) * \tilde{y} = 0,$$

$$(I_3) \tilde{x} * \tilde{x} = 0,$$

$$(I_4) \tilde{x} * \tilde{y} = 0, \tilde{y} * \tilde{x} = 0 \Rightarrow \tilde{x} = \tilde{y},$$

$$(K) 0 * \tilde{x} = 0$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.

We define an order relation “ \leq ” on a BCK-algebra X as follows:

$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} * \tilde{y} = 0). \quad (1)$$

Every BCK-algebra X satisfies:

$$(\forall \tilde{x} \in X)(\tilde{x} * 0 = \tilde{x}), \quad (2)$$

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\tilde{x} \leq \tilde{y} \Rightarrow \tilde{x} * \tilde{z} \leq \tilde{y} * \tilde{z}, \tilde{z} * \tilde{y} \leq \tilde{z} * \tilde{x}), \quad (3)$$

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{y}) * \tilde{z} = (\tilde{x} * \tilde{z}) * \tilde{y}). \quad (4)$$

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{z}) * (\tilde{y} * \tilde{z}) \leq \tilde{x} * \tilde{y}). \quad (5)$$

A BCK-algebra X is said to be

- *commutative* (see [19]) if $\tilde{x} * (\tilde{x} * \tilde{y}) = \tilde{y} * (\tilde{y} * \tilde{x})$ for all $\tilde{x}, \tilde{y} \in X$.
- *positive implicative* (see [19]) if $(\tilde{x} * \tilde{z}) * (\tilde{y} * \tilde{z}) = (\tilde{x} * \tilde{y}) * \tilde{z}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.
- *implicative* (see [19]) if $\tilde{x} = \tilde{x} * (\tilde{y} * \tilde{x})$ for all $\tilde{x}, \tilde{y} \in X$.

A subset K of a BCK-algebra X is called an *ideal* of X if it satisfies:

$$0 \in K, \quad (6)$$

$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} * \tilde{y} \in K, \tilde{y} \in K \Rightarrow \tilde{x} \in K). \quad (7)$$

A subset K of a BCK-algebra X is called

- a *commutative ideal* of X (see [19]) if it satisfies (6) and

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{y}) * \tilde{z} \in K, \tilde{z} \in K \Rightarrow \tilde{x} * (\tilde{y} * (\tilde{y} * \tilde{x})) \in K). \quad (8)$$

- a *positive implicative ideal* of X (see [19]) if it satisfies (6) and

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * \tilde{y}) * \tilde{z} \in K, \tilde{y} * \tilde{z} \in K \Rightarrow \tilde{x} * \tilde{z} \in K). \quad (9)$$

- an *implicative ideal* of X (see [19]) if it satisfies (6) and

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)((\tilde{x} * (\tilde{y} * \tilde{x})) * \tilde{z} \in K, \tilde{z} \in K \Rightarrow \tilde{x} \in K). \quad (10)$$

For more information on BCK-algebra, please refer to the book [19].

2.2 Basic concepts about soju structures

In what follows, let W be an initial universe set unless otherwise specified.

Definition 2.1. [17] *Let X be a set of parameters. For any subset K of X , let $\lambda := (\varphi_\lambda, \eta_\lambda)$ be an intuitionistic fuzzy set in K and (\tilde{F}, K) be a soft set over W . Then the pair $(K, \langle \lambda; \tilde{F} \rangle)$ is called a soju structure related to K (briefly, K -soju structure) over $([0, 1], W)$. An X -soju structure over $([0, 1], W)$ is simply called a soju structure over $([0, 1], W)$.*

Given a soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$, we consider the following sets:

$$\equiv_{\tilde{F}}^\lambda := \{\{x, y\} \subseteq X \mid \varphi_\lambda(x) = \varphi_\lambda(y), \eta_\lambda(x) = \eta_\lambda(y), \tilde{F}(x) = \tilde{F}(y)\}, \quad (11)$$

$$\Omega_{\tilde{F}}^\lambda := \left\{ \frac{x}{(y,z)} \in \frac{X}{X \times X} \mid \begin{array}{l} \varphi_\lambda(x) \geq \min\{\varphi_\lambda(y), \varphi_\lambda(z)\} \\ \eta_\lambda(x) \leq \max\{\eta_\lambda(y), \eta_\lambda(z)\} \\ \tilde{F}(x) \supseteq \tilde{F}(y) \cap \tilde{F}(z) \end{array} \right\}. \quad (12)$$

It is clear that

$$(\forall x, y, z \in X) \left(\frac{x}{(y,z)} \in \Omega_{\tilde{F}}^\lambda \Leftrightarrow \frac{x}{(z,y)} \in \Omega_{\tilde{F}}^\lambda \right), \quad (13)$$

$$(\forall x, y, a, b \in X) \left(\frac{x}{(y,y)} \in \Omega_{\tilde{F}}^\lambda, \frac{y}{(a,b)} \in \Omega_{\tilde{F}}^\lambda \Rightarrow \frac{x}{(a,b)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (14)$$

$$(\forall x, y, z, a \in X) \left(\frac{x}{(y,z)} \in \Omega_{\tilde{F}}^\lambda, \frac{y}{(a,a)} \in \Omega_{\tilde{F}}^\lambda \Rightarrow \frac{x}{(a,z)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (15)$$

$$(\forall x, y, z, a \in X) \left(\frac{x}{(y,z)} \in \Omega_{\tilde{F}}^\lambda, \frac{a}{(x,z)} \in \Omega_{\tilde{F}}^\lambda \Rightarrow \frac{a}{(y,z)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (16)$$

$$(\forall x, y, z, a, b \in X) \left(\frac{x}{(y,z)} \in \Omega_{\tilde{F}}^\lambda, \frac{y}{(a,b)}, \frac{z}{(a,b)} \in \Omega_{\tilde{F}}^\lambda \Rightarrow \frac{x}{(a,b)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (17)$$

Definition 2.2. [17] *Let X be a BCK-algebra. A soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ is called a soju subalgebra of X if the following condition is valid.*

$$(\forall x, y \in X) \left(\frac{x*y}{(x,y)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (18)$$

Definition 2.3. [17] *Let X be a BCK-algebra. A soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ is called a soju ideal of X if the following conditions are valid.*

$$(\forall x \in X) \left(\frac{0}{(x,x)} \in \Omega_{\tilde{F}}^\lambda \right), \quad (19)$$

$$(\forall x, y \in X) \left(\frac{x}{(x*y, y)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (20)$$

Lemma 2.4. [17] *Every soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of a BCK-algebra X satisfies:*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \frac{x}{(y,y)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (21)$$

Lemma 2.5. [17] *Every soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of a BCK-algebra X satisfies:*

$$(\forall x, y, z \in X) \left(x * y \leq z \Rightarrow \frac{x}{(y,z)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (22)$$

Definition 2.6. [24] *Let X be a BCK-algebra. A soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ is called a positive implicative soju ideal of X if it satisfies (19) and*

$$(\forall x, y, z \in X) \left(\frac{x*z}{((x*y)*z, y*z)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (23)$$

Definition 2.7. [21] Let X be a BCK-algebra. A soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ is called a commutative soju ideal of X if it satisfies (19) and

$$(\forall x, y, z \in X) \left(\frac{x*(y*(y*x))}{((x*y)*z, z)} \in \Omega_{\tilde{F}}^{\lambda} \right). \quad (24)$$

Recall that every positive implicative and commutative soju ideal is a soju ideal (see [21, 24]).

Lemma 2.8. [21] Given a soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of a BCK-algebra X , the following are equivalent.

- (i) $(X, \langle \lambda; \tilde{F} \rangle)$ is a commutative soju ideal of X .
- (ii) $(X, \langle \lambda; \tilde{F} \rangle)$ satisfies:

$$(\forall x, y \in X) \left(\frac{x*(y*(y*x))}{(x*y, x*y)} \in \Omega_{\tilde{F}}^{\lambda} \right). \quad (25)$$

3 Implicative soju ideals of BCK-algebras

In this section, we define an implicative soju ideal in a BCK-algebra, and investigate several properties. The symbol X in this section represent a BCK-algebra unless otherwise specified.

Definition 3.1. A soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ is called an implicative soju ideal of X if it satisfies (19) and

$$(\forall x, y, z \in X) \left(\frac{x}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^{\lambda} \right). \quad (26)$$

Example 3.2. (1) Let $X = \{0, 1, 2, 3\}$ be a set with the Cayley table which is given in Table 1.

Table 1: Cayley table for the binary operation “ $*$ ”

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then X is a BCK-algebra (see [19]). Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju structure over $([0, 1], W)$ which is defined by Table 2,

Table 2: Tabular representation of $(X, \langle \lambda; \tilde{F} \rangle)$

X	$\varphi_{\lambda}(x)$	$\eta_{\lambda}(x)$	$\tilde{F}(x)$
0	0.77	0.21	β_1
1	0.58	0.39	β_3
2	0.33	0.54	β_2
3	0.45	0.32	β_4

where $\beta_1 \supseteq \beta_2 \supseteq \beta_3 \supseteq \beta_4 \neq \emptyset$ in 2^W . It is routine to verify that $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X .

(2) Given an implicative ideal K of X , define a soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ as follows:

$$\lambda := (\varphi_{\lambda}, \eta_{\lambda}) : X \rightarrow [0, 1] \times [0, 1], \quad x \mapsto \begin{cases} (t, s) & \text{if } x \in K, \\ (0, 1) & \text{otherwise,} \end{cases}$$

$$\tilde{F} : X \rightarrow 2^W, \quad x \mapsto \begin{cases} \alpha & \text{if } x \in K, \\ \beta & \text{otherwise,} \end{cases}$$

where $(t, s) \in (0, 1] \times [0, 1)$ and $\alpha, \beta \in 2^W$ with $t + s \leq 1$ and $\alpha \supseteq \beta$. Then $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X .

We set up the relationship between soju ideal and implicative soju ideal.

Theorem 3.3. *Every implicative soju ideal is a soju ideal.*

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be an implicative soju ideal of X . If we take $y = x$ and $z = y$ in (26) and use (I_3) and (2), then we obtain (20). Therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal of X . \square

Corollary 3.4. *Every implicative soju ideal is a soju subalgebra.*

Proof. Straightforward. \square

The example below shows that there exist a soju ideal and a soju subalgebra which is not an implicative soju ideal.

Example 3.5. *Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation “*” which is given in Table 3. Then*

Table 3: Cayley table for the binary operation “*”

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

$(X, *, 0)$ is a BCK-algebra (see [19]). Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju structure over $([0, 1], W)$, where $W = \mathbb{Z}$, which is defined by Table 4. It is routine to check that $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal, and hence a soju subalgebra

Table 4: Tabular representation of $(X, \langle \lambda; \tilde{F} \rangle)$

X	$\varphi_\lambda(x)$	$\eta_\lambda(x)$	$\tilde{F}(x)$
0	0.65	0.29	\mathbb{N}
1	0.57	0.42	$2\mathbb{N}$
2	0.43	0.32	$2\mathbb{N}$
3	0.34	0.54	$4\mathbb{N}$
4	0.34	0.54	$4\mathbb{N}$

of X . But it is not an implicative soju ideal of X since $\frac{1}{((1*(3*1))*2, 2)} \notin \Omega_F^\lambda$.

It is natural to ask under what conditions any soju ideal or any soju subalgebra can be an implicative soju ideal. Below we will provide those conditions.

Lemma 3.6. *Every soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies:*

$$(\forall x, y \in X) \left(\frac{x}{(y, 0)} \in \Omega_F^\lambda \Leftrightarrow \frac{x}{(y, y)} \in \Omega_F^\lambda \right). \tag{27}$$

Proof. Straightforward. \square

Theorem 3.7. *Given a soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X , the following are equivalent.*

- (i) $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X .

(ii) $(X, \langle \lambda; \tilde{F} \rangle)$ satisfies:

$$(\forall x, y \in X) \left(\frac{x}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^{\lambda} \right). \quad (28)$$

Proof. Assume that $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X . Taking $z = 0$ in (26) and using (2) induces $\frac{x}{(x*(y*x), 0)} = \frac{x}{((x*(y*x))*0, 0)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y \in X$. It follows from Lemma 3.6 that $\frac{x}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y \in X$.

Conversely, suppose $(X, \langle \lambda; \tilde{F} \rangle)$ satisfies (28). Since $(X, \langle \lambda; \tilde{F} \rangle)$ of X is a soju ideal of X , we have $\frac{x*(y*x)}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y, z \in X$. It follows from (14) and (28) that $\frac{x}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y, z \in X$. Therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X . \square

Lemma 3.8. *Every soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies:*

$$(\forall x, y \in X) \left(\frac{x}{(y, y)} \in \Omega_{\tilde{F}}^{\lambda}, \frac{y}{(x, x)} \in \Omega_{\tilde{F}}^{\lambda} \Rightarrow \{x, y\} \in \equiv_{\tilde{F}}^{\lambda} \right). \quad (29)$$

$$(\forall x, y \in X) \left(\frac{x}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^{\lambda} \Rightarrow \{x, x*(y*x)\} \in \equiv_{\tilde{F}}^{\lambda} \right). \quad (30)$$

$$(\forall x, y \in X) \left(\{x, y\} \in \equiv_{\tilde{F}}^{\lambda}, \frac{y}{(a, b)} \in \Omega_{\tilde{F}}^{\lambda} \Rightarrow \frac{x}{(a, b)} \in \Omega_{\tilde{F}}^{\lambda} \right). \quad (31)$$

Proof. It is clear that (29) and (31) are established. Let $x, y \in X$ be such that $\frac{x}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^{\lambda}$. Since $x*(y*x) \leq x$, it follows from Lemma 2.4 that $\frac{x*(y*x)}{(x, x)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y \in X$. Hence $\{x, x*(y*x)\} \in \equiv_{\tilde{F}}^{\lambda}$ by (29). \square

Theorem 3.9. *Given a soju subalgebra $(X, \langle \lambda; \tilde{F} \rangle)$ of X , the following are equivalent.*

(i) $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X .

(ii) $(X, \langle \lambda; \tilde{F} \rangle)$ satisfies:

$$(\forall x, y, z, a \in X) \left((x*(y*x))*z \leq a \Rightarrow \frac{x}{(z, a)} \in \Omega_{\tilde{F}}^{\lambda} \right). \quad (32)$$

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be an implicative soju ideal of X . Then it is a soju ideal of X (see Theorem 3.3). If we combine (28) with (30), then $\{x, x*(y*x)\} \in \equiv_{\tilde{F}}^{\lambda}$. Let $x, y, z, a \in X$ be such that $(x*(y*x))*z \leq a$. Then $\frac{x*(y*x)}{(z, a)} \in \Omega_{\tilde{F}}^{\lambda}$ by Lemma 2.5. Hence $\frac{x}{(z, a)} \in \Omega_{\tilde{F}}^{\lambda}$ by (31).

Conversely, suppose that $(X, \langle \lambda; \tilde{F} \rangle)$ satisfies (32). Since $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju subalgebra of X , we have $\frac{0}{(x, x)} = \frac{x*x}{(x, x)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x \in X$. Since $(x*(y*x))*((x*(y*x))*z) \leq z$, we get $\frac{x}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y, z \in X$ by (32). Consequently, $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X . \square

Theorem 3.10. *In an implicative BCK-algebra, every soju ideal is an implicative soju ideal.*

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju ideal of an implicative BCK-algebra X . Since X is an implicative BCK-algebra, we get $x = x*(y*x)$ for all $x, y \in X$. Hence

$$(x*((x*(y*x))*z))*z = (x*z)*((x*(y*x))*z) = ((x*(y*x))*z)*((x*(y*x))*z) = 0,$$

that is, $x*((x*(y*x))*z) \leq z$ for all $x, y, z \in X$. It follows from Lemma 2.5 that $\frac{x}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^{\lambda}$ for all $x, y, z \in X$. Therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X . \square

Corollary 3.11. *If every BCK-algebra X satisfies either of the following:*

(i) $(\forall x, y \in X) ((x*(x*y))* (x*y) = y*(y*x)),$

(ii) $(\forall x, y \in X) ((x*(x*y))* (y*x) = y*(y*x)),$

(iii) $(\forall x, y \in X) ((x*(x*y))* (x*y) = (y*(y*x))* (y*x)),$

then every soju ideal is an implicative soju ideal.

The following example shows that even if X is an implicative BCK-algebra, any soju subalgebra of X may not be an implicative soju ideal of X .

Example 3.12. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation “ $*$ ” which is given in Table 5.

Table 5: Cayley table for the binary operation “ $*$ ”

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is an implicative BCK-algebra (see [19]). Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju structure over $([0, 1], W)$, where $W = \mathbb{Z}$, which is defined by Table 6. It is routine to verify that $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju subalgebra of X .

Table 6: Tabular representation of $(X, \langle \lambda; \tilde{F} \rangle)$

X	$\varphi_\lambda(x)$	$\eta_\lambda(x)$	$\tilde{F}(x)$
0	0.63	0.19	$2\mathbb{Z}$
1	0.56	0.28	$8\mathbb{N}$
2	0.47	0.32	$8\mathbb{N}$
3	0.47	0.51	$4\mathbb{Z}$
4	0.33	0.51	$4\mathbb{N}$

But it is not an implicative soju ideal of X since $\frac{3}{((3*(2*3))*1, 1)} \notin \Omega_{\tilde{F}}^\lambda$.

We build the relationship between positive implicative soju ideal and implicative soju ideal.

Theorem 3.13. Every implicative soju ideal is a positive implicative soju ideal.

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be an implicative soju ideal of X . Then $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal of X (see Theorem 3.3). Let $x, y, z \in X$. Since $((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$ and

$$(x * z) * (x * (x * z)) = (x * (x * (x * z))) * z = (x * z) * z,$$

it follows from Lemma 2.5 that

$$\frac{(x * z) * (x * (x * z))}{(y * z, (x * y) * z)} = \frac{(x * z) * z}{(y * z, (x * y) * z)} \in \Omega_{\tilde{F}}^\lambda. \quad (33)$$

Note that $\frac{x * z}{((x * z) * (x * (x * z))), (x * z) * (x * (x * z))} \in \Omega_{\tilde{F}}^\lambda$ by Theorem 3.7, and hence $\{x * z, (x * z) * (x * (x * z))\} \in \equiv_{\tilde{F}}^\lambda$ by (30). Thus $\frac{x * z}{(y * z, (x * y) * z)} \in \Omega_{\tilde{F}}^\lambda$ by (31), and therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is a positive implicative soju ideal of X . \square

The following example shows that the converse of Theorem 3.13 is not true in general.

Example 3.14. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation “ $*$ ” which is given in Table 7. Then $(X, *, 0)$ is a BCK-algebra (see [19]). Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju structure over $([0, 1], W)$, where $W = \mathbb{Z}$, which is defined by Table 8.

It is routine to verify that $(X, \langle \lambda; \tilde{F} \rangle)$ is a positive implicative soju ideal of X . But it is not an implicative soju ideal of X since $\frac{2}{((2*(4*2))*3, 3)} \notin \Omega_{\tilde{F}}^\lambda$.

Table 7: Cayley table for the binary operation “ $*$ ”

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Table 8: Tabular representation of $(X, \langle \lambda; \tilde{F} \rangle)$

X	$\varphi_\lambda(x)$	$\eta_\lambda(x)$	$\tilde{F}(x)$
0	0.66	0.09	$2\mathbb{Z}$
1	0.54	0.35	$4\mathbb{Z}$
2	0.42	0.42	$4\mathbb{N}$
3	0.49	0.23	$16\mathbb{N}$
4	0.34	0.52	$8\mathbb{N}$

Proposition 3.15. *Every implicative soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies:*

$$(\forall x, y \in X) \left(\frac{x*(x*y)}{(y*(y*x), y*(y*x))} \in \Omega_{\tilde{F}}^\lambda \right). \quad (34)$$

Proof. Let $x, y \in X$. Since

$$(x*(x*y))*(y*(x*(x*y))) \leq (x*(x*y))*(y*x) = (x*(y*x))*(x*y) \leq y*(y*x),$$

it follows from (2) and Lemma 2.4 that

$$\frac{((x*(x*y))*(y*(x*(x*y))))*0}{(y*(y*x), y*(y*x))} = \frac{(x*(x*y))*(y*(x*(x*y)))}{(y*(y*x), y*(y*x))} \in \Omega_{\tilde{F}}^\lambda. \quad (35)$$

Using (26) leads to $\frac{x*(x*y)}{(((x*(x*y))*(y*(x*(x*y))))*0, 0)} \in \Omega_{\tilde{F}}^\lambda$, and it is equivalent to

$$\frac{x*(x*y)}{(((x*(x*y))*(y*(x*(x*y))))*0, ((x*(x*y))*(y*(x*(x*y))))*0)} \in \Omega_{\tilde{F}}^\lambda \quad (36)$$

by Lemma 3.6. Therefore (34) is induced by (14), (35) and (36). \square

We consider conditions for a positive implicative soju ideal to be an implicative soju ideal.

Lemma 3.16. [24] *Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju ideal of X . Then it is a positive implicative soju ideal of X if and only if it satisfies:*

$$(\forall x, y \in X) \left(\frac{x*y}{((x*y)*y, (x*y)*y)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (37)$$

Theorem 3.17. *If every positive implicative soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (34), then it is an implicative soju ideal of X .*

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a positive implicative soju ideal of X that satisfies the condition (34). Then $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal of X . Hence

$$\frac{x*(y*x)}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^\lambda \quad (38)$$

for all $x, y, z \in X$. Since $(y*(y*x))*(y*x) \leq x*(y*x)$, we get

$$\frac{(y*(y*x))*(y*x)}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda \quad (39)$$

by Lemma 2.4. Using Lemma 3.16 gives

$$\frac{y*(y*x)}{((y*(y*x))*(y*x), (y*(y*x))*(y*x))} \in \Omega_{\tilde{F}}^\lambda, \quad (40)$$

and so

$$\frac{x*(x*y)}{((y*(y*x))*(y*x), (y*(y*x))*(y*x))} \in \Omega_{\tilde{F}}^\lambda \quad (41)$$

by (14) and (34). Using (14), (39) and (40) induces

$$\frac{y*(y*x)}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda. \quad (42)$$

Also, if we use (14), (34) and (42), then we get

$$\frac{x*(x*y)}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda. \quad (43)$$

Since $(x * y) * z \leq x * y \leq x * (y * x)$, it follows from Lemma 2.4 that

$$\frac{(x*y)*z}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda. \quad (44)$$

Since $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal of X , we get $\frac{x*y}{((x*y)*z, z)} \in \Omega_{\tilde{F}}^\lambda$. Combine this with (44) and use (15) to induce

$$\frac{x*y}{(x*(y*x), z)} \in \Omega_{\tilde{F}}^\lambda. \quad (45)$$

Also, combine (38) with (45) and use (16) to induce

$$\frac{x*y}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^\lambda. \quad (46)$$

Since $\frac{x}{(x*(x*y), x*y)} \in \Omega_{\tilde{F}}^\lambda$, it follows from (15) and (41) that

$$\frac{x}{((y*(y*x))*(y*x), x*y)} \in \Omega_{\tilde{F}}^\lambda. \quad (47)$$

By using (15), (39) and (47), we obtain $\frac{x}{(x*(y*x), x*y)} \in \Omega_{\tilde{F}}^\lambda$ which implies from (17), (38) and (46) that $\frac{x}{((x*(y*x))*z, z)} \in \Omega_{\tilde{F}}^\lambda$. Therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X . \square

We look into the relationship between commutative soju ideal and implicative soju ideal.

Lemma 3.18. [21] *Given a soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X , the following are equivalent.*

- (i) $(X, \langle \lambda; \tilde{F} \rangle)$ is a commutative soju ideal of X .
- (ii) $(X, \langle \lambda; \tilde{F} \rangle)$ satisfies:

$$(\forall x, y \in X) \left(\frac{x*(y*(y*x))}{(x*y, x*y)} \in \Omega_{\tilde{F}}^\lambda \right). \quad (48)$$

Theorem 3.19. *Every implicative soju ideal is a commutative soju ideal.*

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be an implicative soju ideal of X . Then $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal of X (see Theorem 3.3). For every $x, y \in X$, we have

$$\begin{aligned} & (x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq (x * (y * (y * x))) * (y * x) \\ & = (x * (y * x)) * (y * (y * x)) \leq x * y \end{aligned}$$

which implies from Lemma 2.4 that $\frac{(x*(y*(y*x)))*(y*(x*(y*(y*x))))}{(x*y, x*y)} \in \Omega_{\tilde{F}}^\lambda$. It follows from (14) and Theorem 3.7 that $\frac{x*(y*(y*x))}{(x*y, x*y)} \in \Omega_{\tilde{F}}^\lambda$. Therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is a commutative soju ideal of X by Lemma 3.18. \square

Table 9: Cayley table for the binary operation “ $*$ ”

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 10: Tabular representation of $(X, \langle \lambda; \tilde{F} \rangle)$

X	$\varphi_\lambda(x)$	$\eta_\lambda(x)$	$\tilde{F}(x)$
0	0.77	0.21	β_1
1	0.52	0.41	β_3
2	0.52	0.41	β_3
3	0.43	0.34	β_2

The next example shows that the converse of Theorem 3.19 is not true in general.

Example 3.20. Let $X = \{0, 1, 2, 3\}$ be a set with the Cayley table which is given in Table 9. Then X is a BCK-algebra (see [19]). Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju structure over $([0, 1], W)$ which is defined by Table 10

where $\beta_1 \supseteq \beta_2 \supseteq \beta_3 \supseteq \beta_4 \neq \emptyset$ in 2^W . It is routine to verify that $(X, \langle \lambda; \tilde{F} \rangle)$ is a commutative soju ideal, and hence a soju ideal, of X . Since $\frac{1}{(1*(2*1), 1*(2*1))} = \frac{1}{(0, 0)} \notin \Omega_{\tilde{F}}^\lambda$, we know that $(X, \langle \lambda; \tilde{F} \rangle)$ is not an implicative soju ideal of X by Theorem 3.7.

We provide conditions for a commutative soju ideal to be an implicative soju ideal.

Lemma 3.21. If a soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (37), then

$$(\forall x, y \in X) ((x * y, (x * y) * y) \in \equiv_{\tilde{F}}^\lambda). \quad (49)$$

Proof. Assume that a soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (37) and let $x, y \in X$. Since $(x * y) * y \leq x * y$, it follows from Lemma 2.4 that $\frac{(x * y) * y}{(x * y, x * y)} \in \Omega_{\tilde{F}}^\lambda$. If we combine this with (37), then $(x * y, (x * y) * y) \in \equiv_{\tilde{F}}^\lambda$. \square

Corollary 3.22. Every positive implicative soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (49).

Lemma 3.23. If a soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (25), then

$$(\forall x, y \in X) ((x * y, x * (y * (y * x))) \in \equiv_{\tilde{F}}^\lambda). \quad (50)$$

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a soju ideal of X satisfies the condition (25). Since $x * y \leq x * (y * (y * x))$, we get $\frac{x * y}{(x * (y * (y * x)), x * (y * (y * x)))} \in \Omega_{\tilde{F}}^\lambda$ by Lemma 2.4. If we combine this with (25), then we obtain $(x * y, x * (y * (y * x))) \in \equiv_{\tilde{F}}^\lambda$ for all $x, y \in X$. \square

Corollary 3.24. Every commutative soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (50).

Theorem 3.25. If a commutative soju ideal $(X, \langle \lambda; \tilde{F} \rangle)$ of X satisfies the condition (37), then it is an implicative soju ideal of X .

Proof. Let $(X, \langle \lambda; \tilde{F} \rangle)$ be a commutative soju ideal of X that satisfies the condition (37). Then $(X, \langle \lambda; \tilde{F} \rangle)$ is a soju ideal of X . Let $x, y \in X$. Since $(y * (y * x)) * (y * x) \leq x * (y * x)$, it follows from Lemma 2.4 that

$$\frac{(y * (y * x)) * (y * x)}{(x * (y * x), x * (y * x))} \in \Omega_{\tilde{F}}^\lambda. \quad (51)$$

By the condition (37), we know that $\frac{y*(y*x)}{((y*(y*x))* (y*x), (y*(y*x))* (y*x))} \in \Omega_{\tilde{F}}^\lambda$ and so

$$(y * (y * x), (y * (y * x)) * (y * x)) \in \equiv_{\tilde{F}}^\lambda \quad (52)$$

by Lemma 3.21. Using (31), (51) and (52) leads to

$$\frac{y*(y*x)}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda. \quad (53)$$

On the other hand, since $x * y \leq x * (y * x)$, we get $\frac{x*y}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda$, and so

$$\frac{x*(y*(y*x))}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda \quad (54)$$

by (31) and Lemma 3.23. Since $\frac{x}{(x*(y*(y*x)), y*(y*x))} \in \Omega_{\tilde{F}}^\lambda$ by (20), we have $\frac{x}{(x*(y*x), x*(y*x))} \in \Omega_{\tilde{F}}^\lambda$ by (17), (53) and (54). Therefore $(X, \langle \lambda; \tilde{F} \rangle)$ is an implicative soju ideal of X by Theorem 3.7. \square

The following corollary is the result of combining Lemma 3.16 and Theorems 3.13, 3.19 and 3.25.

Corollary 3.26. *A soju structure $(X, \langle \lambda; \tilde{F} \rangle)$ over $([0, 1], W)$ is an implicative soju ideal of X if and only if it is both a commutative soju ideal and a positive implicative soju ideal of X .*

4 Conclusion

We introduced the concept of implicative soju ideal in BCK-algebra and further explored various properties. In consideration of A and B, which are described next:

- (1) $\left\{ \begin{array}{l} \text{A: Soju subalgebra} \\ \text{B: Implicative soju ideal} \end{array} \right.$
- (2) $\left\{ \begin{array}{l} \text{A: Soju ideal} \\ \text{B: Implicative soju ideal} \end{array} \right.$
- (3) $\left\{ \begin{array}{l} \text{A: Positive implicative soju ideal} \\ \text{B: Implicative soju ideal} \end{array} \right.$
- (4) $\left\{ \begin{array}{l} \text{A: Commutative soju ideal} \\ \text{B: Implicative soju ideal,} \end{array} \right.$

we looked into the relationship between A and B. We proved that B becomes A, and gave examples of A not being B. We found and presented conditions under which A can be B. Our future work will utilize ideas and results in this paper to study the substructures of algebraic systems related to BCK/BCI algebra.

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B02006812).

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