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A graph associated to a polygroup with respect to an automorphism

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Abstract

In this paper, we introduce and study, $\zeta^{\alpha}(P)$, the α-center of a polygroup $(P,\cdot)$ with respect to an automorphism α. Then we associate to P a graph Γ^{α}_{P}, whose vertices are elements of $P\setminus\zeta^{\alpha}(P)$ and x connected to y by an edge in case $x\cdot y\cdot\omega\neq y\cdot x^{\alpha}\cdot\omega$ or $y\cdot x\cdot\omega\neq x\cdot y^{\alpha}\cdot\omega$, where ω is the heart of P. We obtain some basic properties of this graph. In particular, we prove that if $\zeta^{\alpha}(P)\neq P$, then $dim(\Gamma^{\alpha}_{P})=2$. Moreover, we define a weak α-commutative polygroup to state that if $\Gamma^{\alpha}_{H}\cong\Gamma^{\beta}_{K}$ and H is a weak α-commutative, then K is a weak β-commutative. Also, we show that if H and K are two polygroups such that $\Gamma^{\alpha}_{H}\cong\Gamma^{\beta}_{K}$, then for some automorphisms η and λ, $\Gamma^{\eta}_{H\times A}\cong\Gamma^{\lambda}_{K\times B}$, where A and B are two weak commutative polygroups.

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1 Introduction

Graph theory have been applied in areas such as computer science, image capturing, networking, etc. Some extensive papers are on assigning a graph to a ring, group, polygroup and etc. This help you to study some properties of theses structures by the associated graph (see [1],[4],[5]). Furtheremore, in [1], we see that a finite group with some conditions on its graph is a solvabel group.

Marty [13], defined hypergroups and analize their properties. You can find the applications of hyperstructures in many areas, such as geometry, automata, probabilities, and so on. One of the most important subclasses of hypergroups are Polygroups, introduced by Bonansinga and Corsini [6], that appear in many contents such as nilpotent polygroups, fundamental relation of polygroups. Polygroups discussed by many scholars (see [2, 7, 8, 9, 10, 12]). One of the intersting problems in hypergroup theory is the relation between hypergroups (polygroups) and hypergraphs. Corsini

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studied the relations between hypergroups and hypergraphs and Farshi et. all in [11] studied the hypergraphs and hypergroups based on special elements.
Now, in this paper first we define $\zeta^{\alpha}(P)$, the notions of α-center of a polygroup with respect to an automorphism α. Then we define an α-graph Γ_P^{α}, as a generalization of classical group, with the vertex set $P\setminus\zeta^{\alpha}(P)$ and adjoint two vertices x and y if $x\cdot y\cdot\omega\neq y\cdot x^{\alpha}\cdot\omega$ or $y\cdot x\cdot\omega\neq x\cdot y^{\alpha}\cdot\omega$, where ω is the heart of P. We introduce the notion of weak α-commutative polygroups. Basically, we obtain an isomorphism between $\Gamma_{P\times A}^{\alpha\times i}$ and $\Gamma_{H\times B}^{\beta\times i}$ in which $\Gamma_P^{\alpha}\cong\Gamma_H^{\beta}$, i is the identity automorphism and A, B are two weak commutative polygroups with the same order.

2 Preliminaries

We recall some basic definitions which are proposed by the pioneers of this subject.
Let G be a group and $\alpha\in Aut(G)$. For two elements $x, y\in G$, we say x and y commute under the automorphism α whenever $yx = xy^{\alpha}$.

Hyperstructure theory was first identified by Marty [13] in 1934 when he defined hypergroups and started to analyze their properties. A *hyperstructure (or hypergroupoid)* is a non-empty set H with a *hyperoperation* $\circ$ defined on H, that is, a mapping of $H\times H$ into the family of non-empty subsets of H. If $(x, y)\in H\times H$, then its image under $\circ$ is denoted by $x\circ y$. If A, B are two non-empty subsets of H, then $A\circ B$ is given by $A\circ B=\bigcup\{x\circ y | x\in A, y\in B\}$. We use $x\circ A$ instead of $\{x\}\circ A$ and $A\circ x$ for $A\circ\{x\}$. Generally, the singleton a is identified with its member a. The structure $(H,\circ)$ is called a *semihypergroup* if $a\circ(b\circ c)=(a\circ b)\circ c$ for any $a, b, c\in H$, and a semihypergroup $(H,\circ)$ is a *hypergroup* if

$$x\circ H = H\circ x = H, \quad \text{for any } x\in H,$$

which is called the *reproduction axiom*.

Definition 2.1. [10] *A polygroup is an algebraic structure $(P,\cdot,e,^{-1})$, where "$\cdot$" is a hyperoperation on P, "$^{-1}$" is an unary operation on P and $e\in P$ such that the following axioms hold for any $x, y, z\in P$,*
(i) $(x\cdot y)\cdot z = x\cdot(y\cdot z)$;
(ii) $e\cdot x = x\cdot e = x$;
(iii) $x\in y\cdot z$ *implies* $y\in x\cdot z^{-1}$ *and* $z\in y^{-1}\cdot x$.

It is easy to see that for any $x\in P$, $e\in(x\cdot x^{-1})\cap(x^{-1}\cdot x)$ and $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$, where $A^{-1}=\{a^{-1}\mid a\in A\}$.

A non-empty subset K of a polygroup P is called a *subpolygroup* of P if $a, b\in K$ implies $a\cdot b\subseteq K$ and $a\in K$ implies $a^{-1}\in K$. A subpolygroup N of a polygroup P is called *normal* if $a^{-1}\cdot N\cdot a\subseteq N$, for any $a\in P$. Also, for a subpolygroup K of P and $x\in P$, denote the left (right) coset of K by $x\cdot K$ $(K\cdot x)$ and suppose P/K is the set of all left (right) cosets of K in P. Note that for a normal subpolygroup N of P, we have $N\cdot x = x\cdot N$ and for all $x, y\in P$ we have $N\cdot x\cdot y = N\cdot z$ for all $z\in x\cdot y$. Also, $(P/N,\odot,N,^{-1})$ is a polygroup, where

$$(N\cdot x)\odot(N\cdot y)=\{N\cdot z\mid z\in x\cdot y\} \ \textit{and} \ (N\cdot x)^{-1}=N\cdot x^{-1}.$$

A polygroup in which $x\cdot y = y\cdot x$ for all $x, y\in P$ is called *commutative* polygroup.
Let P be a polygroup and $\rho\subseteq P\times P$ be an equivalence relation on P. For non-empty subsets A and B of P, we define $A\overline{\overline{\rho}}B\iff(\forall a\in A$ and $\forall b\in B$ we get $a\rho b)$. Then the relation ρ is called a

strongly regular on the left (on the right) if $x\rho y \Longrightarrow a \cdot x\overline{\overline{\rho}}a \cdot y(x \cdot a\overline{\overline{\rho}}y \cdot a)$, for any $x, y, a \in P$. In addition, ρ is called *strongly regular* if it is strongly regular on the right and on the left (see [10]).

Let P be a polygroup and β^* be the smallest equivalence relation on P such that the quotient P/β^*, the set of all equivalence classes, is a group. In this case, β^* is called the *fundamental equivalence relation* on P and P/β^* is called the *fundamental group*. The product $\otimes$ in P/β^* is as follows:

$$\beta^*(x) \otimes \beta^*(y) = \beta^*(z) \ for \ all \ z \in x \cdot y.$$

Let $\mathbf{U}_P$ be the set of finite products of elements of P and $u \subseteq U_P$. We define the relation β as follows:

$$x\beta y \text{ if and only if } \{x, y\} \subseteq u \quad (I).$$

We have $\beta^* = \beta$ for hypergroups. Since polygroups are certain subclasses of hypergroups, we have $\beta^* = \beta$. The kernal of the canonical map $\varphi : P \to P/\beta^*$ is called the *core* (or heart) of P and is denoted by ω_P (or ω). Here, we also denote by ω_P the unit of P/β^*. It is easy to prove that $\omega_P = \beta^*(e)$ and $\beta^*(x)^{-1} = \beta^*(x^{-1})$ for all $x \in P$(see [10]).

Let $(H, \cdot, e_1, ^{-1})$ and $(H', \star, e_2, ^{-1})$ be two polygroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \star f(b)$, for any $a, b \in H$. We say that f is a *good homomorphism* if $f(a \cdot b) = f(a) \star f(b)$ for any $a, b \in H$.

Definition 2.2. [10] *Let P be a polygroup and A be a non-empty subset of H. By $< A >$ we mean the intersection of all subpolygroups of P containing A.*

It is easy to verify that

$$< A > = \cup\{x_1^{\epsilon_1} \cdot ... \cdot x_k^{\epsilon_k} | x_i \in A, k \in \mathbb{N}, \epsilon_i \in \{1, -1\}\}.$$

Also, $< A, B >$ is used for $< A \cup B >$.

Let G be a group and $Z(G)$ be the center of it. A graph Γ_G, whose vertices are elements of $G \setminus Z(G)$ and x connected to y by an edge in case $xy \neq yx$, was first considered by Paul Erdos. The set of vertices of Γ_G is denoted by $V(G)$. A *path* ρ is a sequence $v_0e_1v_1...e_kv_k$ whose terms are alternately distinct vertices and distinct edges, such that the ends of e_i are v_{i-1} and v_i for any i, $1 \leq i \leq k$. In this case, ρ is called a path between v_0 and v_k and the number k is called the *length* of ρ. If v_0 and v_k are adjacent in Γ by an edge e_{k+1}, then $\rho \cup \{e_{k+1}\}$ is called a *cycle*. The *length* of a cycle define the number of its edges. The length of the shortest cycle in a graph Γ is called *girth* of Γ and denoted by $girth(\Gamma)$. If v and w are vertices of Γ, then $d(v, w)$ denotes the length of the shortest path between v and w. The largest distance between all pairs of the vertices of Γ is called the *diameter* of Γ, and is denoted by $diam(\Gamma)$. A graph is *connected* if there is a path between each pair of the vertices of Γ. A subset S of the vertices of a connected graph Γ is called a *cut set* if $\Gamma \setminus S$ is not a connected graph. For a graph Γ and a subset S of the vertex set $V(\Gamma)$, denoted by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is said to be a *dominating* set.

Notation. Let $(P, \cdot, e, ^{-1})$, $(P/\beta^*, \otimes, e, ^{-1})$ be a polygroup and fundamental group, respectively from now on. Consider $n \in \mathbb{N}$, $Aut(P)$ is the set of all automorphism of P and $\alpha \in Aut(P)$. Also, for any $x \in P$, set $\overline{x} = \beta^*(x)$.

3 α-Center of polygroups

In this section, first we define and study the α-center of a polygroup P, denoted by $\zeta^\alpha(P)$. Then, we redefine the cener of a polygroup, denoted by $\zeta(P)$. Finally, we obtain a relation between $\zeta^\alpha(P)$

and $\zeta(P)$. This help us to see that $\zeta^{\alpha}(P)=\zeta^{\alpha^{-1}}(P)$.
Note. For any $x\in P$ we use x^{α} and xy instead of $\alpha(x)$ and $x\cdot y$, respectively.

Definition 3.1. *The set $\zeta^{\alpha}(P)$ is called α-center of P defined as follows:*

$$\zeta^{\alpha}(P)=\{x\in P|xy\omega=yx^{\alpha}\omega \ \text{ for any } y\in P\}.$$

Example 3.2. *Let $P=\{e,a,b\}$. We define the operation $\cdot$ and automorphism α on P as follows:*

.	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$$\alpha(x)=\begin{cases} b, & x=a\\ a, & x=b\\ e, & x=e\end{cases}$$

Clearly, $\omega=\{e\}$ and so for any $y\in P$ we have $ey\omega=ye^{\alpha}\omega$. Then $e\in\zeta^{\alpha}(P)$. Since $ab\omega\neq ba^{\alpha}\omega$ and $ba\omega\neq ab^{\alpha}\omega$ we conclude $a,b\notin\zeta^{\alpha}(P)$. Therefore $\zeta^{\alpha}(P)=\{e\}$.

Theorem 3.3. *For any $x,y\in P$, $\overline{x}\otimes\overline{y}=\overline{y}\otimes\overline{x^{\alpha}}$ if and only if $xy\omega=yx^{\alpha}\omega$.*

Proof. ($\Rightarrow$) Let $\overline{x}\otimes\overline{y}=\overline{y}\otimes\overline{x^{\alpha}}$. Then for any $t\in xy$ and $t'\in yx^{\alpha}$ we have $\overline{t}=\overline{t'}$ and so $\overline{t^{-1}}\otimes\overline{t'}=\overline{e}$. Then for any $r\in t^{-1}t'$ we have $\overline{r}=\overline{e}$ and so $t^{-1}t'\subseteq\omega$, i.e $(t^{-1}t')\omega=\omega$. Then, $t'\omega=t\omega$ and so $xy\omega=yx^{\alpha}\omega$.
($\Leftarrow$) Assume $xy\omega=yx^{\alpha}\omega$. Then for any $t\in xy$ there exists $t'\in yx^{\alpha}$ such that , $t'\omega=t\omega$. Then $\overline{t'}=\overline{t}$ and so $\overline{x}\otimes\overline{y}=\overline{y}\otimes\overline{x^{\alpha}}$. □

We recall that for a group G, $Z^{\alpha}(G)=\{y\in G;[x,y]_{\alpha}=e\}$, where $[x,y]_{\alpha}=x^{-1}y^{-1}xy^{\alpha}$ (see *[3]*).

Theorem 3.4. *$\zeta^{\alpha}(P)$ is a normal subpolygroup of P.*

Proof. First we show that for any $x_1,x_2\in\zeta^{\alpha}(P)$, $x_1x_2\subseteq\zeta^{\alpha}(P)$. Suppose $r\in x_1x_2$. By Theorem 3.3, for any $y\in P$, $\overline{x_1}\otimes\overline{y}=\overline{y}\otimes\overline{x_1^{\alpha}}$ and $\overline{x_2}\otimes\overline{y}=\overline{y}\otimes\overline{x_2^{\alpha}}$, which implies that

$$\begin{aligned}\overline{r}\otimes\overline{y} &= (\overline{x_1}\otimes\overline{x_2})\otimes\overline{y}\\ &= \overline{x_1}\otimes(\overline{x_2}\otimes\overline{y})\\ &= \overline{x_1}\otimes(\overline{y}\otimes\overline{x_2^{\alpha}})\\ &= (\overline{x_1}\otimes\overline{y})\otimes\overline{x_2^{\alpha}}\\ &= \overline{y}\otimes(\overline{x_1^{\alpha}}\otimes\overline{x_2^{\alpha}})\\ &= \overline{y}\otimes\overline{r^{\alpha}}.\end{aligned}$$

Then by Theorem 3.3, we have $ry\omega=yr^{\alpha}\omega$. Therefore, $r\in\zeta^{\alpha}(P)$ and $x_1x_2\subseteq\zeta^{\alpha}(P)$. Now, we show that $x_1^{-1}\in\zeta^{\alpha}(P)$. Since $x_1\in\zeta^{\alpha}(P)$, for all $y\in P$ we have

$$\begin{aligned}x_1y\omega=yx_1^{\alpha}\omega &\Longrightarrow \overline{x_1}\otimes\overline{y}=\overline{y}\otimes\overline{x_1^{\alpha}}\\ &\Longrightarrow \overline{x_1^{-1}}\otimes(\overline{x_1}\otimes\overline{y})\otimes\overline{x_1^{-\alpha}}=\overline{x_1^{-1}}\otimes(\overline{y}\otimes\overline{x_1^{\alpha}})\otimes\overline{x_1^{-\alpha}}\\ &\Longrightarrow \overline{y}\otimes\overline{x_1^{-\alpha}}=\overline{x_1^{-1}}\otimes\overline{y}\\ &\Longrightarrow yx_1^{-\alpha}\omega=x_1^{-1}y\omega.\end{aligned}$$

and so $x_1^{-1} \in \zeta^\alpha(P)$. Therefore, $\zeta(P)$ is a subpolygroup of P. We show $\zeta^\alpha(P)$ is a normal subpolygroup. Suppose $z \in \zeta^\alpha(P)$, then for any $x, y \in P$ and $r \in x^{-1}zx$ we have

$$\begin{aligned}
\overline{r} \otimes \overline{y} &= \overline{x^{-1}} \otimes \overline{z} \otimes (\overline{x} \otimes \overline{y}), \\
&= \overline{x^{-1}} \otimes (\overline{x} \otimes \overline{y}) \otimes \overline{z^\alpha}, \\
&= \overline{y} \otimes \overline{z^\alpha}, \\
&= \overline{z} \otimes \overline{y}, \\
&= \overline{z} \otimes \overline{y} \otimes (\overline{x^{-\alpha}} \otimes \overline{x^\alpha}), \\
&= (\overline{z} \otimes (\overline{y} \otimes \overline{x^{-\alpha}})) \otimes \overline{x^\alpha}), \\
&= (\overline{y} \otimes \overline{x^{-\alpha}}) \otimes \overline{z^\alpha} \otimes \overline{x^\alpha}, \\
&= \overline{y} \otimes \overline{r^\alpha}.
\end{aligned}$$

Then by Theorem 3.3, $r \in \zeta^\alpha(P)$ and so $x^{-1}\zeta^\alpha(P)x \subseteq \zeta^\alpha(P)$. Thus $\zeta^\alpha(P)$ is a normal subpolygroup of P. □

In the following we obtain a necessary and sufficient condition between elements of P and the fundamental group P/β^*.

Corollary 3.5. *Let P be a polygroup. Then $x \in \zeta^\alpha(P)$ if and only if $\beta^*(x) \in Z^\alpha(P/\beta^*)$.*

Proof. Suppose $x \in P$. Then

$$\begin{aligned}
x \in \zeta^\alpha(P) &\Leftrightarrow xy\omega = yx^\alpha\omega, \text{ for any } \ y \in P \\
&\Leftrightarrow \overline{x} \otimes \overline{y} = \overline{y} \otimes \overline{x^\alpha} \text{ (by Theorem 3.3)} \\
&\Leftrightarrow [\overline{y}, \overline{x}]_\alpha = \overline{e} \\
&\Leftrightarrow \overline{x} \in Z^\alpha(P/\beta^*).
\end{aligned}$$

□

Theorem 3.6. *Consider $\alpha \in Aut(P)$. Then $\overline{\alpha} \in Aut(P/\beta^*)$, where $\overline{\alpha} : P/\beta^* \to P/\beta^*$ is defined by $\overline{\alpha}(\overline{x}) = \overline{x^\alpha}$.*

Proof. First we prove that for any $x \in P$, $\overline{x} = \overline{e}$ if and only if $\overline{x^\alpha} = \overline{e}$. If $\overline{x} = \overline{e}$, then there exist $z_1, ..., z_n \in P$ such that $\{x, e\} \in \prod_{i=1}^{n} z_i$. Then $x \in \prod_{i=1}^{n} z_i$ and $e \in \prod_{i=1}^{n} z_i$ and so $x^\alpha \in \prod_{i=1}^{n} z_i^\alpha$ and $e^\alpha \in \prod_{i=1}^{n} z_i^\alpha$. Therefore, $\{x^\alpha, e = e^\alpha\} \in \prod_{i=1}^{n} z_i^\alpha$ implies $\overline{x^\alpha} = \overline{e}$. By the similar way we have the converse. Thus, $\overline{\alpha}$ is well defined and one to one. Now, for any $\overline{y} \in P/\beta^*$, consider $x = y^{\alpha^{-1}}$. Then $\overline{\alpha}(\overline{y^{\alpha^{-1}}}) = (\overline{y^{\alpha^{-1}}})^\alpha = \overline{y}$ and so $\overline{\alpha}$ is onto. For $\overline{x_1}, \overline{x_2} \in P/\beta^*$, we have

$$\overline{\alpha}(\overline{x_1} \otimes \overline{x_2}) = \{\overline{\alpha}(\overline{t}) : t \in x_1x_2\} = \{\overline{t^\alpha} : t \in x_1x_2\} = \{\overline{z} : z \in x_1^\alpha x_2^\alpha\} = \overline{x_1^\alpha} \otimes \overline{x_2^\alpha} = \overline{\alpha}(\overline{x_1}) \otimes \overline{\alpha}(\overline{x_2}).$$

Thus, $\overline{\alpha} \in Aut(P/\beta^*)$. □

Definition 3.7. *Let $\alpha \in Aut(P)$. Then:*
(i) P is called an α-commutative if $xy = yx^\alpha$ for any $x, y \in P$.
(ii) P is called a weak α-commutative if $xy\omega = yx^\alpha\omega$ for any $x, y \in P$.
(iii) Consider $(G, \cdot)$ is a group and $\beta \in Aut(G)$. Then G is called a β-Abelian group if $x \cdot y = y \cdot x^\beta$ for any $x, y \in G$.

Clearly, $\zeta^\alpha(P) = P$ if and only if P is weak α-commutative.

Theorem 3.8. *If P is an α-commutative polygroup, then P/β^* is an $\overline{\alpha}$-Abelian group. Moreover, P is a weak α-commutative if and only if P/β^* is an $\overline{\alpha}$-Abelian group.*

Proof. By Theorem 3.6, $\overline{\alpha} \in Aut(P/\beta^*)$. Since P is an α-commutative polygroup, we have $xy = yx^\alpha$ and so $xy\omega = yx^\alpha\omega$ for any $x, y \in P$. Then by Theorems 3.3 and 3.6, $\overline{x} \otimes \overline{y} = \overline{y} \otimes \overline{x^\alpha} = \overline{y} \otimes \overline{x}^{\overline{\alpha}}$ and so P/β^* is an $\overline{\alpha}$-Abelian group. Moreover, since P is a weak α-commutative polygroup, we have $xy = yx^\alpha$ and so by Theorem 3.3, $\overline{x} \otimes \overline{y} = \overline{y} \otimes \overline{x}^{\overline{\alpha}}$. Therefore, P/β^* is an $\overline{\alpha}$-Abelian group. □

Definition 3.9. *For $x \in P$ the α-centeralizer x in P is defined by*

$$C^\alpha(x) = \{y \in P \mid yx\omega = xy^\alpha\omega\}.$$

Clearly, $\zeta^\alpha(P) = \bigcap\limits_{x \in P} C^\alpha(x)$.

Theorem 3.10. *For any $x \in P$, $C^\alpha(x)$ is a subpolygroup of P.*

Proof. Since $e \in C^\alpha(x)$ we have $C^\alpha(x) \neq \oslash$. Now, we show that for any $y, z \in C^\alpha(x)$, $zy \subseteq C^\alpha(x)$ and $z^{-1} \in C^\alpha(x)$. For this let $r \in zy$. Since $zx\omega = xz^\alpha\omega$ and $yx\omega = xy^\alpha\omega$, by Theorem 3.3, we have $\overline{z} \otimes \overline{x} = \overline{x} \otimes \overline{z^\alpha}$ and $\overline{y} \otimes \overline{x} = \overline{x} \otimes \overline{y^\alpha}$. Also, $\overline{r} = \overline{z} \otimes \overline{y}$. Thus,

$$\overline{r} \otimes \overline{x} = \overline{z} \otimes \overline{y} \otimes \overline{x} = \overline{z} \otimes \overline{x} \otimes \overline{y^\alpha} = \overline{x} \otimes \overline{z^\alpha} \otimes \overline{y^\alpha} = \overline{x} \otimes \overline{r^\alpha},$$

and so $zy \subseteq C^\alpha(x)$. By the same manipulation of Theorem 3.4, $z^{-1} \in C^\alpha(x)$. Consequently, $C^\alpha(x)$ is a subpolygroup of P. □

In [10], the center of a polygroup, denoted by $Z(P)$, is defined as $\langle\{x \in P \mid xy\omega = yx\omega \text{ for any } y \in P\}\rangle$. Now, we redefine the center of a polygroup as follows.

Definition 3.11. *The set $\{x \in P \mid xy\omega = yx\omega \text{ for any } y \in P\}$, denoted by $\zeta(P)$, is called the center of a polygroup.*

Example 3.12. *Suppose P is a polygroup as in Example 3.2. Since for any $y \in P$, $ey\omega = ye\omega$, $ay\omega = ya\omega$ and $by\omega = yb\omega$ we conclude $\zeta(P) = P$.*

Similar to Theorem 3.4, we can prove that $\zeta(P)$ is a normal subpolygroup of P.

Theorem 3.13. *Let $fix(\alpha) = \{x \in P : x^\alpha x^{-1} \subseteq \omega\}$. Then $\zeta^\alpha(P) = \zeta(P) \cap fix(\alpha)$.*

Proof. Let $x \in \zeta^\alpha(P)$. Then $yx^\alpha\omega = xy\omega$ for any $y \in P$. Suppose $y = e$, then $x^\alpha\omega = x\omega$ and so $yx\omega = xy\omega$. Therefore, $x \in \zeta(P) \cap fix(\alpha)$. Hence, $\zeta^\alpha(P) \subseteq \zeta(P) \cap fix(\alpha)$. Now, assume $x \in \zeta(P) \cap fix(\alpha)$. Then for any $y \in P$, $yx\omega = xy\omega$ and $x^\alpha = x$. Therefore, $yx^\alpha\omega = xy\omega$ for any $y \in P$ i.e $x \in \zeta^\alpha(P)$. Therefore, $\zeta^\alpha(P) = \zeta(P) \cap fix(\alpha)$. □

Corollary 3.14. *If α fixes every element of $\zeta(P)$, then $\zeta^\alpha(P) = \zeta(P)$.*

Proof. Let $x \in \zeta(P)$. Then $xy\omega = yx\omega$. Since α fixes every element of $\zeta(P)$, we have $x^\alpha = x$. Then $xy\omega = yx^\alpha\omega$ and so $x \in \zeta^\alpha(P)$. Thus, $\zeta(P) \subseteq \zeta^\alpha(P)$. Now, by Theorem 3.13, $\zeta^\alpha(P) = \zeta(P) \cap fix(\alpha) \subseteq \zeta(P)$. Therefore, $\zeta^\alpha(P) = \zeta(P)$.

□

Proposition 3.15. $\zeta^\alpha(P) = \zeta^{\alpha^{-1}}(P)$.

Proof. Since $fix(\alpha) = fix(\alpha^{-1})$. Then by Theorem 3.13,

$$\zeta^{\alpha}(P) = \zeta(P) \cap fix(\alpha) = \zeta(P) \cap fix(\alpha^{-1}) = \zeta^{\alpha^{-1}}(P).$$

Thus, $\zeta^{\alpha}(P) = \zeta^{\alpha^{-1}}(P)$.

□

4 Graph of polygroups

In this section, we associate a graph Γ_P^{α} to a polygroup P, whose vertices are elements of $P \setminus \zeta^{\alpha}(P)$ and x connected to y by edge in case $xy\omega \neq yx^{\alpha}\omega$ or $yx\omega \neq xy^{\alpha}\omega$. The set of vertices of Γ_P^{α} is denoted by $V(G)$. Basically, we study polygroups throughout its isomorphic α-graphs (see Proposition 4.8, Theorem 4.12 and corollaries 4.22, 4.24).

Example 4.1. *Let $P = \{e, a, b, c\}$. We define the operation $\cdot$ and automorphism α on P as follows:*

.	e	a	b	c
e	e	a	b	c
a	a	P	$\{a,b,c\}$	$\{a,b,c\}$
b	b	$\{a,b,c\}$	P	$\{a,b,c\}$
c	c	$\{a,b,c\}$	$\{a,b,c\}$	P

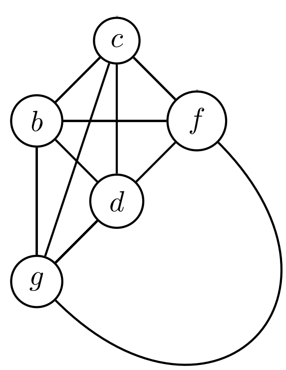
$$\alpha(x) = \begin{cases} b, & x = a \\ a, & x = b \\ x, & otherwise \end{cases}$$

Since $\overline{e} = \overline{a} = \overline{b} = \overline{c} = P$ and so $\overline{x} \otimes \overline{y} = \overline{y} \otimes \overline{x^{\alpha}}$ for any $x, y \in P$. Hence, $xy\omega = yx^{\alpha}\omega$. Thus $\zeta^{\alpha}(P) = P$. Therefore, the α-graph is empty.

Example 4.2. *Let $P = \{e, a, b, c, d, f, g\}$. We define the operation$\cdot$ and automorphism α on P as follows:*

.	e	a	c	b	f	d	g
e	e	a	c	b	f	d	g
a	a	e	c	b	f	d	g
c	c	c	$\{e,a\}$	f	b	g	d
b	b	b	g	$\{e,a\}$	d	f	c
f	f	f	d	c	g	b	$\{e,a\}$
d	d	d	f	g	c	$\{e,a\}$	b
g	g	g	b	d	$\{e,a\}$	c	f

$$\alpha(x) = \begin{cases} d, & x = b \\ b, & x = d \\ g, & x = f \\ f, & x = g \\ x, & otherwise \end{cases}$$



Then $(P, \cdot, e, ^{-1})$ is a non-commutative polygroup (see [10]). Since $\overline{a} = \overline{e} = \{e, a\}$ and $\overline{x} = x$ for any $x \neq \{a, e\}$, we conclude that $\zeta^{\alpha}(P) = \{e, a\}$. Then for each $y, x \in P \setminus \zeta^{\alpha}(P)$ we have $\overline{x} \otimes \overline{y} \neq \overline{y} \otimes \overline{x^{\alpha}}$ and so by Theorem 3.3, $xy\omega \neq yx^{\alpha}\omega$ which implies x and y are adjacent by an edge. Therefore, we have a connected graph.

Let $(P_1, \cdot, e_1, ^{-1})$ and $(P_2, *, e_2, ^{-1})$ be two polygroups. Then $(P_1 \times P_2, \circ)$, where $\circ$ is defined as follows, is a polygroup (see [10]).

$$(x_1, y_1) \circ (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2, \text{ and } y \in y_1 * y_2\}.$$

Corollary 4.3. [10] *If N_1, N_2 are normal subpolygroups of P_1 and P_2 respectively, then $N_1 \times N_2$ is a normal subpolygroup of $P_1 \times P_2$ and $(P_1 \times P_2)/(N_1 \times N_2) \cong P_1/N_1 \times P_2/N_2$.*

Corollary 4.4. [10] *Suppose ω_P, ω_K and $\omega_{P\times K}$ are the hearts of polygroups P, K and $P \times K$, respectively. Then $\omega_{P\times K} = \omega_P \times \omega_K$.*

Lemma 4.5. *Let $(H, \cdot, e, ^{-1})$ and $(K, *, e, ^{-1})$ be two polygroups. Then for each $h_1, h_2 \in H$ and $k_1, k_2 \in K$ we have*

$$(h_1, k_1) \circ (h_2, k_2) \circ \omega_{H\times K} = (h_2, k_2) \circ (h_1^{\alpha}, k_1^{\beta}) \circ \omega_{H\times K},$$

if and only if $h_1 \cdot h_2 \cdot \omega_H = h_2 \cdot h_1^{\alpha} \cdot \omega_H$ and $k_1 * k_2 * \omega_K = k_2 * k_1^{\beta} * \omega_K$. In addition, $\zeta^{\alpha\times\beta}(H \times K) = \zeta^{\alpha}(H) \times \zeta^{\beta}(K)$.

Proof. ($\Rightarrow$) If $(h_1, k_1) \circ (h_2, k_2) \circ \omega_{H\times K} = (h_2, k_2) \circ (h_1^{\alpha}, k_1^{\beta}) \circ \omega_{H\times K}$, then

$$\{(r_1, s_1) \circ \omega_{H\times K} \mid r_1 \in h_1 \cdot h_2, s_1 \in k_1 * k_2\} = \{(r_2, s_2) \circ \omega_{H\times K} \mid r_2 \in h_2 \cdot h_1^{\alpha}, s_2 \in k_2 * k_1^{\beta}\}.$$

Thus for all $(r_1, s_1) \circ \omega_{H\times K}$ where $r_1 \in h_1^{\alpha} \cdot h_2$ and $s_1 \in k_1^{\beta} * k_2$, there exists $(r_2, s_2) \circ \omega_{H\times K}$ such that $r_2 \in h_2 \cdot h_1^{\alpha}$ and $s_2 \in k_2 * k_1^{\beta}$ and

$$(r_1, s_1) \circ \omega_{H\times K} = (r_2, s_2) \circ \omega_{H\times K}. \quad (II)$$

By Corollaries 4.3 and 4.4, we have $\omega_{H\times K} \cong \omega_H \times \omega_K$ and $\frac{H\times K}{\omega_{H\times K}} \cong \frac{H}{\omega_H} \times \frac{K}{\omega_K}$ and so there exists an automorphism $\varphi : \frac{H}{\omega_H} \times \frac{K}{\omega_K} \to \frac{H\times K}{\omega_{H\times K}}$ such that $\varphi(h \cdot \omega_H, k * \cdot\omega_K) = (h, k) \circ \omega_{H\times K}$ for any $h \in H$ and $k \in K$. Since $(r_1, s_1) \circ \omega_{H\times K}$, and $(r_2, s_2) \circ \omega_{H\times K} \in \frac{H\times K}{\omega_{H\times K}} \cong^{\varphi} \frac{H}{\omega_H} \times \frac{K}{\omega_K}$, we get $(r_1, s_1) \circ \omega_{H\times K} = \varphi(r_1 \cdot \omega_H, s_1 * \omega_K)$ and $(r_2, s_2) \circ \omega_{H\times K} = \varphi(r_2 \cdot \omega_H, s_2 * \omega_K)$ and so, by (II) we have $\varphi(r_1 \cdot \omega_H, s_1 * \omega_K) = \varphi(r_2 \cdot \omega_H, s_2 * \omega_K)$. Since φ is isomorphism we obtain $r_1 \cdot \omega_H = r_2 \cdot \omega_H$ and $s_1 * \omega_K = s_2 * \omega_K$. Thus,

$$\beta^*_H(r_1) = \beta^*_H(r_2) \text{ and } \beta^*_K(s_1) = \beta^*_K(s_2). \quad (III)$$

On the other hand, $r_1 \in h_1 \cdot h_2$ and $r_2 \in h_2 \cdot h_1^{\alpha}$, then $\beta^*_H(r_1) = \beta^*_H(h_1) \otimes \beta^*_H(h_2)$ and $\beta^*_H(r_2) = \beta^*_H(h_2) \otimes \beta^*_H(h_1^{\alpha})$. Therefore, by Theorem 3.3, and (III) we have $h_1 \cdot h_2 \cdot \omega = h_2 \cdot h_1^{\alpha} \cdot \omega$. By the similar way, $k_1 * k_2 * \omega_K = k_2 * k_1^{\beta} * \omega_K$.
($\Leftarrow$) The proof of converse is similar. $\square$

Corollary 4.6. *Let $\alpha \in Aut(H)$ and $\beta \in Aut(K)$. Then H and K are α and β-commutative polygroups, respectively if and only if $H \times K$ is $\alpha \times \beta$-commutative.*

Proof. Polygroups H and K are α and β-commutative if and only if $\zeta^{\alpha}(H) = H$ and $\zeta^{\beta}(K) = K$ if and only if $\zeta^{\alpha\times\beta}(H \times K) = \zeta^{\alpha}(H) \times \zeta^{\beta}(K) = H \times K$ if and only if $H \times K$ is $\alpha \times \beta$-commutative. $\square$

Definition 4.7. *Let $(P,\cdot,e,^{-1}),(H,*,e,^{-1})$ be two polygroups and $\alpha\in Aut(P)$ and $\beta\in Aut(H)$. The graphs Γ^{α}_{P} and Γ^{β}_{H} are said to be isomorphic with respect to α and β ($\Gamma^{\alpha}_{P}\cong\Gamma^{\beta}_{H}$) if there is a bijection map $\varphi: P\backslash\zeta^{\alpha}(P)\longrightarrow H\backslash\zeta^{\beta}(H)$ preserving edges, means that for each $x,y\in P\backslash\zeta^{\alpha}(P)$, $xy\omega_P\neq yx^{\alpha}\omega_P$ if and only if $\varphi(x)*\varphi(y)*\omega_H\neq\varphi(y)*(\varphi(x))^{\beta}*\omega_H$.*

Proposition 4.8. *Let H and K be two subpolygroups of P, $\alpha,\beta\in Aut(P)$ and $\Gamma^{\alpha}_{H}\cong\Gamma^{\beta}_{K}$. If H is not a weak α-commutative, then K is not a weak β-commutative.*

Proof. Let k_1,k_2 be two arbitary elements of K. Since $\Gamma^{\alpha}_{H}\cong\Gamma^{\beta}_{K}$ we have a bijection $\theta: H\setminus\zeta^{\alpha}(H)\to K\setminus\zeta^{\beta}(K)$. Since H is not a weak α-commutative, then there exist $h_1,h_2\in H$ such that $h_1h_2\omega\neq h_2h_1^{\alpha}\omega$ and so by Definition 4.7, $\theta(h_1)\theta(h_2)\omega\neq\theta(h_2)\theta(h_1)^{\beta}\omega$. Take $k_1=\theta(h_1)$ and $k_2=\theta(h_2)$. Then $k_1k_2\omega\neq k_2k_1^{\beta}\omega$. Therefore, K is not a weak α-commutative. □

The numbers of elements in a polygroup P is called the *order* of P. Now, we obtain an isomorphism between $\Gamma^{\alpha\times i}_{P\times A}$ and $\Gamma^{\beta\times i}_{H\times B}$ whenever $\Gamma^{\alpha}_{P}\cong\Gamma^{\beta}_{H}$, i is the identity automorphism and A and B are two weak commutative polygroups with the same order.

Theorem 4.9. *Suppose $(P,\cdot,e,^{-1})$ and $(H,*,e,^{-1})$ are two polygroups, $(A,\star,e,^{-1})$ and $(B,\bullet,e,^{-1})$ are two weak commutative polygroups with the same order, $\alpha\in Aut(P),\beta\in Aut(H),\gamma\in Aut(A),\eta\in Aut(B)$ and $\zeta^{\alpha}(P)\neq P,\zeta^{\beta}(H)\neq H$. If $\Gamma^{\alpha}_{P}\cong\Gamma^{\beta}_{H}$, then $\Gamma^{\alpha\times\gamma}_{P\times A}\cong\Gamma^{\beta\times\eta}_{H\times B}$,*

Proof. Let $\phi:\Gamma^{\alpha}_{P}\longrightarrow\Gamma^{\beta}_{H}$ be a graph isomorphism. Then there is a one to one map $\phi: P\backslash\zeta^{\alpha}(P)\to H\backslash\zeta^{\beta}(H)$ such that $g_1\cdot g_2\cdot\omega_P=g_2\cdot g_1^{\alpha}\cdot\omega_P$ where $g_1,g_2\in P\backslash\zeta^{\alpha}(P)$ if and only if $\phi(g_1)*\phi(g_2)*\omega_H=\phi(g_2)*(\phi(g_1)^{\beta})*\omega_H$. Since A and B have the same order, we have a bijection map $\psi: A\to B$. By Lemma 4.5, we prove $\varphi:(g,a)\longrightarrow(\phi(g),\psi(a))$ is a graph isomorphism between $\Gamma^{\alpha\times\gamma}_{P\times A}$ and $\Gamma^{\beta\times\eta}_{H\times B}$. Moreover, we show that for $g_1,g_2\in P$ and $a_1,a_2\in A$, $(g_1,a_1)\circ(g_2,a_2)\circ\omega_{P\times A}=(g_2,a_2)\circ(g_1^{\alpha},a_1^{\gamma})\circ\omega_{P\times A}$ if and only if $\varphi(g_1,a_1)\circ\varphi(g_2,a_2)\circ\omega_{H\times B}=\varphi(g_2,a_2)\circ(\varphi(g_1,a_1)^{\beta\times\eta})\circ\omega_{H\times B}$. For this, we have

$$\begin{aligned}
&(g_1,a_1)\circ(g_2,a_2)\circ\omega_{P\times A}=(g_2,a_2)\circ(g_1^{\alpha},a_1^{\gamma})\circ\omega_{P\times A}\\
\Leftrightarrow\quad & g_1\cdot g_2\cdot\omega_P=g_2\cdot g_1^{\alpha}\cdot\omega_P \text{ and } \ a_1\star a_2\star\omega_A=a_2\star a_1^{\gamma}\star\omega_A \ \text{ (by Lemma4.5)}\\
\Leftrightarrow\quad & \phi(g_1)*\phi(g_2)*\omega_H=\phi(g_2)*(\phi(g_1))^{\beta}*\omega_H \text{ and}\\
& \psi(a_1)\bullet\psi(a_2)\bullet\omega_B=\psi(a_2)\bullet(\psi(a_1))^{\eta}\bullet\omega_B, \text{(since } B \text{ is a weak } \eta-\text{commutative)}\\
\Leftrightarrow\quad & (\phi(g_1),\psi(a_1))\circ(\phi(g_2),\psi(a_2))\circ\omega_{H\times B}=(\phi(g_2),\psi(a_2))\circ((\phi(g_1))^{\beta},(\psi(a_1))^{\eta})\circ\omega_{H\times B}\\
\Leftrightarrow\quad & \varphi(g_1,a_1)\circ\varphi(g_2,a_2)\circ\omega_{H\times B}=\varphi(g_2,a_2)\circ(\varphi(g_1,a_1))^{\beta\times\eta}\circ\omega_{H\times B}.
\end{aligned}$$

□

In the following, with some additional conditions we obtain a connected graph.

Theorem 4.10. *Let P be a polygroup such that $\zeta^{\alpha}(P)\neq P$. Then $diam(\Gamma^{\alpha}_{P})=2$.*

Proof. Let x,y be two distinct vertices of Γ^{α}_{P} and for any $z\in X$, $\beta^{*}(z)=\overline{z}$. First, we show that the followings hold.
(i) If x,y are adjacent, then $d(x,y)=1$.
(ii) If x,y are not adjacent, then $d(x,y)=2$.
(i) The proof is obviose.
(ii) Since x and y are not adjucent, $xy\omega=yx^{\alpha}\omega$. From $\zeta^{\alpha}(P)\neq P$, we have there exist $x',y'\in V(P)$ such that $xx'\omega\neq x'x^{\alpha}\omega$ or $x'x\omega\neq xx'^{\alpha}\omega$ and $yy'\omega\neq y'y^{\alpha}\omega$ or $y'y\omega\neq yy'\alpha\omega$. Now, if

$y, x^{'}$ or $x, y^{'}$ are adjacent, then $d(x,y) = 2$. Otherwise, $y, x^{'}$ and $x, y^{'}$ are not adjusent and so $yx^{'\alpha}\omega = x^{'}y \cdot \omega$ and $yx^{'}\omega = x^{'}y^{\alpha}\omega$, $xy^{'\alpha}\omega = y^{'}x\omega$ and $xy^{'}\omega = y^{'}x^{\alpha}\omega$. Therefore, by Theorem 3.3, $\overline{y} \otimes \overline{x^{'\alpha}} = \overline{x^{'}} \otimes \overline{y}$, $\overline{y} \otimes \overline{x^{'}} = \overline{x^{'}} \otimes \overline{y^{\alpha}}$ and $\overline{x} \otimes \overline{y^{'}} = \overline{y^{'}} \otimes \overline{x^{\alpha}}$, $\overline{x} \otimes \overline{y^{'\alpha}} = \overline{y^{'}} \otimes \overline{x}$. Now, we show that there is $r \in x^{'}y^{'}$ such that $\overline{r} \otimes \overline{x} \neq \overline{x} \otimes \overline{r^{\alpha}}$ and $\overline{r} \otimes \overline{y} \neq \overline{y} \otimes \overline{r^{\alpha}}$. For this, let $\overline{r} \otimes \overline{x} = \overline{x} \otimes \overline{r^{\alpha}}$. Then $\overline{x^{'}} \otimes \overline{y^{'}} \otimes \overline{x} = \overline{x} \otimes \overline{x^{'\alpha}} \otimes \overline{y^{'\alpha}}$ which by $\overline{x} \otimes \overline{y^{'\alpha}} = \overline{y^{'}} \otimes \overline{x}$ implies $\overline{x^{'}} \otimes \overline{x} \otimes \overline{y^{'\alpha}} = \overline{x} \otimes \overline{x^{'\alpha}} \otimes \overline{y^{'\alpha}}$ and so $\overline{x^{'}} \otimes \overline{x} = \overline{x} \otimes \overline{x^{'\alpha}}$ which is a contradiction (since $x, x^{'}$ are adjusent). Thus $\overline{r} \otimes \overline{x} \neq \overline{x} \otimes \overline{r^{\alpha}}$. Similarly, $\overline{r} \otimes \overline{y} \neq \overline{y} \otimes \overline{r^{\alpha}}$. Therefore, by Theorem 3.3, r is adjacent to both x and y, thus $d(x,y) = 2$. $\square$

Example 4.11. *Let P be a polygroup as in Example 4.2. Then $\zeta^{\alpha}(P) \neq P$. Therefore, by the proof of Theorem 4.10, if x, y are adjacent, then $d(x,y) = 1$, otherwise $d(x,y) = 2$. Therefore, Γ_P^{α} is connected.*

Theorem 4.12. *Let $\zeta^{\alpha}(P) \neq P$. If $x \in fix(\alpha)$ and $\{x\}$ is a dominating set for Γ_P^{α}, then $\zeta^{\alpha}(P) = \omega$, $x^2 \subseteq \omega$ and $C^{\alpha}(x) = \{e, x\}$.*

Proof. First, we show $\zeta^{\alpha}(P) = \omega$. Let $\{x\}$ be a dominating set for Γ_P^{α} and $z \in \zeta^{\alpha}(P)$ such that $z \notin \omega$. Thus $zy\omega = yz^{\alpha}\omega$ for any $y \in P$. By Theorem 3.3, $\overline{z} \otimes \overline{y} = \overline{y} \otimes \overline{z^{\alpha}}$. Suppose $r \in x \cdot z$. Then $\overline{r} \otimes \overline{x^{\alpha}} = \overline{x} \otimes \overline{z} \otimes \overline{x^{\alpha}} = \overline{x} \otimes \overline{x^{\alpha}} \otimes \overline{z^{\alpha}} = \overline{x} \otimes \overline{r^{\alpha}}$. By Theorem 3.3, and $x \in fix(\alpha)$, we have $xr^{\alpha}\omega = rx\omega$. By the similar way $rx^{\alpha}\omega = xr\omega$. Thus r is not adjacent to x and so $\{x\}$ is not a dominating set, that is a contradiction. Thus $\zeta^{\alpha}(P) = \omega$.

Now, we show $x^2 \subseteq \omega$. Since $x \in fix(\alpha)$, we have $\overline{x^{-1}} \otimes \overline{x^{\alpha}} = \overline{e} = \overline{x} \otimes \overline{x^{-1}}$ and so by Theorem 3.3, x and x^{-1} are not adjacent. But x is dominating set thus $x = x^{-1}$. Therefore, if $r \in x^2$, then $\overline{r} = \overline{x} \otimes \overline{x} = \overline{x} \circ \overline{x^{-1}} = \overline{e}$, which implies that $r \in \omega$. Therefore, $x^2 \subseteq \omega$.

Finally, since $\zeta^{\alpha}(P) = \omega$ and x is adjacent to all vertices of Γ_P^{α}, we get $C^{\alpha}(x) = \{e, x\}$. $\square$

Theorem 4.13. *Let $\zeta^{\alpha}(P) \neq P$ and S be a subset of $V(\Gamma_P^{\alpha})$. Then S is a dominating set if and only if $R^{\alpha}(S) \cap C^{\alpha}(S) \subseteq \zeta^{\alpha}(P) \cup S$, where $R^{\alpha}(S) = \{y \in P; yx^{\alpha}\omega = xy\omega, \text{ for any } y \in S\}$.*

Proof. ($\Rightarrow$) Suppose that S is a dominating set and $a \in R^{\alpha}(S) \cap C^{\alpha}(S)$. If $a \notin \zeta^{\alpha}(P) \cup S$, then by definition of dominating set, there exists an element $x \in S$ such that a and x are adjucent and so we have two cases.

Case (i) $ax\omega \neq xa^{\alpha}\omega$. Then $a \notin C^{\alpha}(S)$, that is a contradiction. It follows that $R^{\alpha}(S) \cap C^{\alpha}(S) \subseteq \zeta^{\alpha}(P) \cup S$.

Case (ii) $xa\omega \neq ax^{\alpha}\omega$. Then $a \notin R^{\alpha}(S)$ and since $R^{\alpha}(S) \cap C^{\alpha}(S) \subseteq R^{\alpha}(S)$ we conclude that $a \notin R^{\alpha}(S) \cap C^{\alpha}(S)$, that is a contradiction. Therefore, $R^{\alpha}(S) \cap C^{\alpha}(S) \subseteq \zeta^{\alpha}(P) \cup S$.

($\Leftarrow$) Now assume that $R^{\alpha}(S) \cap C^{\alpha}(S) \subseteq \zeta^{\alpha}(P) \cup S$. If an arbitary element $a \notin \zeta^{\alpha}(P) \cup S$, then by assumption, $a \notin C^{\alpha}(S) \cap R^{\alpha}(S)$. Therefore, a is adjacent to at least one element of S. Therefore, S is a dominating set. $\square$

Corollary 4.14. *Consider a non-commutative polygroup P with a non-empty set X. If $P = \langle X \rangle$, then $X \setminus \zeta^{\alpha}(P)$ is a dominating set of T_P^{α}.*

Proof. Consider $Y = X \setminus \zeta^{\alpha}(P)$. Since P is non-commutative, we have $Y \neq \phi$. We show $R^{\alpha}(Y) \cap C_P^{\alpha}(Y) \subseteq \zeta^{\alpha}(P) \cup Y$. Suppose $r \in R^{\alpha}(Y) \cap C_P^{\alpha}(Y)$. Then

$$r \in C_P^{\alpha}(Y) = C_P^{\alpha}(X \setminus \zeta^{\alpha}(P)) = C_P^{\alpha}(\langle X \rangle \setminus \zeta^{\alpha}(P)) = C_P^{\alpha}(P \setminus \zeta^{\alpha}(P)).$$

Clearly, $r \in C_P^{\alpha}(\zeta^{\alpha}(P))$ and so $r \in \zeta^{\alpha}(P) \subseteq \zeta^{\alpha}(P) \cup Y$. Thus, by Theorem 4.13, $X \setminus \zeta^{\alpha}(P)$ is a dominating set. $\square$

Lemma 4.15. *Let P be a non-commutative finite polygroup. The girth of the graph Γ_P^α is at most 4.*

Proof. We have two cases
(*i*) $\zeta(P) \neq fix(\alpha)$.
(*ii*) $\zeta(P) = fix(\alpha)$.
Case (i) there exists $x \in \zeta(P) \setminus fix(\alpha)$ or $x \in fix(\alpha) \setminus \zeta(P)$. Firstly suppose that there exists a vertex $x \in \zeta(P) \setminus fix(\alpha)$, then for any $t \in V(P)$, if $tx^\alpha\omega = xt\omega$, then by definition $\zeta(P)$ we have $tx^\alpha\omega = xt\omega = tx\omega$. Thus $x^\alpha\omega = x\omega$ and so $\alpha \in fix(\alpha)$. Hence $tx^\alpha\omega \neq xt\omega$. So t, x are adjusent. Basically, x, y are adjucent. Let y and z be two arbitrary vertices in $V(P)$. If y is adjacent to z, then three elements x, y, z induce a cycle of length 3. If y is not adjacent to z, then we show that there exists $r \in V(P)$ such that r is adjusent with y. Since $yz\omega \neq zy^\alpha\omega$ or $zy\omega \neq yz^\alpha\omega$, then for the case $y\omega \neq y^\alpha\omega$, we have $[\overline{y}, \overline{y} \otimes \overline{z}]_{\overline{\alpha}} = \overline{z^{-\alpha}} \otimes \overline{y^{-1}} \otimes \overline{y^\alpha} \otimes \overline{z^\alpha} \neq 1$ and so for any $r \in yz$ we have $[\overline{y}, \overline{r}]_{\overline{\alpha}} \neq 1$, then $\overline{y} \otimes \overline{r^\alpha} \neq \overline{r} \otimes \overline{y}$ and so by Theorem 3.3, y and r are adjusent. Also, for the case $y^\alpha\omega = y\omega$ there exists $r \in V(P)$ such that $r^\alpha\omega \neq r\omega$ (since α is not the identity map) and so if $yr^\alpha\omega = ry\omega$ and $ry^\alpha\omega = yr\omega$, then by $y^\alpha\omega = y\omega$ we have $yr^\alpha\omega = ry\omega = ry^\alpha\omega = yr\omega$ implies that $r^\alpha\omega = r\omega$, that is a contradiction. Hence, y and r are adjusent. Therefore, elements x, y, r induce a cycle of length 3. Secondly, assume that $x \in fix(\alpha) \setminus \zeta(P)$ and y and z are two vertices in $V(P)$ such that $y^\alpha\omega \neq y\omega$ and $z^\alpha\omega \neq z\omega$. Then we have two cases;
Case (I) $xy\omega = yx\omega$.
Case (II) $xy\omega \neq yx\omega$.
Case (I) If $xy^\alpha\omega = yx\omega$, then $y^\alpha\omega = y\omega$ that is a contradiction.
Case (II) If $yx^\alpha\omega = xy\omega$, then by $x \in fix(\alpha)$ we have $yx\omega = yx^\alpha\omega = xy\omega$, that is a contradiction. Therefore, if $x \in fix(\alpha) \setminus \zeta(P)$ and $y^\alpha\omega \neq y\omega$, then x and y are adjasent. Similarly, x and z are adjasent. Now, if y and z are adjasent, then x, y, z induce a cycle of length 3. If y is not adjacent to z, then we can see that y and z are adjacent to r. Hence elements x, y, z, r induce a cycle of length 4.
(ii) Finally, $\zeta(P) = fix(\alpha)$. Then $\zeta^\alpha(P) = \zeta(P) = fix(\alpha)$. Since P is non-commuting we conclude that $\overline{P}$ is a non-abelian group. If for any $x \in V(P)$, $\overline{x} \otimes \overline{x} = 1$, then $\overline{x} = \overline{x^{-1}}$ and so for any $y \in V(P)$, we have $\overline{x} \otimes \overline{y} = \overline{x^{-1}} \otimes \overline{y^{-1}} = (\overline{y} \otimes \overline{x})^{-1} = \overline{y} \otimes \overline{x}$. Then $xy\omega = yx\omega$ and so $x \in \zeta^\alpha(P) = \zeta^\alpha(P)$, that is a contradiction. Therefore, there exists $x \in V(P)$ such that $\overline{x}^2 \neq 1$, i.e $xx\omega \neq \omega$, and so $x\omega \neq x^{-1}\omega$. Now, if $x^{-1}x^\alpha\omega = xx^{-1}\omega$, then $x^{-1}x^\alpha\omega = \omega$. Thus $x \in fix(\alpha) = \zeta^\alpha(P)$, that is a contradiction and so x is adjacent to x^{-1}. Also, if $y \in V(P)$ such that y is adjacent to x, then $yx^\alpha\omega \neq xy\omega$. Now, if $yx^{-\alpha}\omega = x^{-1}y\omega$, then $xyx^{-\alpha}x^\alpha\omega = xx^{-1}yx^\alpha\omega$ and so $xy\omega = yx^\alpha\omega$, which is a contradiction. Hence, y is adjacent to x^{-1}, too. Thus, elements x, y, x^{-1} induce a cycle of length 3. Therefore, the girth of P is at most 4. □

Lemma 4.16. *Let P be a polygroup and $\alpha \in Aut(P)$. Then $\omega^\alpha = \omega$.*

Proof. Let $x \in \omega$. Then $\overline{x} = \overline{e}$. i.e $x \in \Pi_{i=1}^n z_i$ and so $x^{\alpha^{-1}} \in \Pi_{i=1}^n z_i^{\alpha^{-1}}$. Put $y = x^{\alpha^{-1}}$ and so $x = y^\alpha$ and $\overline{y} = \overline{e}$ i.e $x \in \omega^\alpha$. By the similar way, we have $\omega^\alpha \subseteq \omega$. Thus $\omega^\alpha = \omega$. □

Theorem 4.17. *Consider $\Gamma_P^\alpha = (V, E)$ is a graph of P, $W = \overline{p} \setminus Z(\overline{P})$ and $x \in V$. Then $x\omega \subseteq V$. Also, $V = \bigcup_{\overline{x} \in W} x\omega$.*

Proof. Since $x \in V$ there exists $y \in V$ such that $xy\omega \neq yx^\alpha\omega$. Take $z \in x\omega$. Then by Lemma 4.16, $z^\alpha \in x^\alpha\omega^\alpha = x^\alpha\omega$ and so we have $z\omega = x^\alpha\omega$ and $z\omega = x\omega$. Thus $zy\omega = xy\omega \neq yx^\alpha\omega = yz^\alpha\omega$. Hence $z \in V$ and so $x\omega \subseteq V$. Now, since $x \in V$ if and only if $\overline{x} \in W$ we have $V = \bigcup_{\overline{x} \in W} x\omega$.

□

A graph $\Gamma = (V, E)$ is *bipartite* if V can be partitioned into two sets V_1, V_2 such that every edge of E has one end vertice in V_1 and the other in V_2. Also, Γ is called a *complete bipartite graph* if it containes exactly all edges with one end vertex in V_1 and the other one in V_2.

Theorem 4.18. *Let P be a non-commutative polygroup and x, y be two edges of graph Γ_P^α. Then $A = \{(a, b) \in E | a \in x\omega, b \in y\omega\}$ is a complete bipartite graph.*

Proof. First we show that for any $z, x \in P$,

$$\text{if } z \in x\omega \text{ ,then } z\omega = x\omega. \quad (IV)$$

Since $z \in x\omega$ there exists $r \in \omega$ such that $z \in xr$ and so $\overline{z} = \overline{x} \otimes \overline{e} = \overline{x}$. Then $\overline{x^{-1}} \otimes \overline{z} = \overline{e}$ i.e $x^{-1}z \subseteq \omega$. Hence $z\omega = x\omega$.
Now, assume $a \in x\omega$ and $b \in y\omega$. Then by (IV) and Lemma 4.16, we have $a^\alpha \in x^\alpha\omega^\alpha = x^\alpha\omega$ and so $a\omega = x\omega$, $a^\alpha\omega = x^\alpha\omega$ and $b\omega = y\omega$. Sine x, y are two edges of graph Γ_P^α we have $xy\omega \neq yx^\alpha\omega$ and so

$$a\omega b\omega = x\omega y\omega \neq y\omega x^\alpha\omega = b\omega a^\alpha\omega$$

Therefore, $ab\omega \neq ba$ and so there exists an edge between a and b. Next A can be partitioned into sets $V_1 = x\omega$ and $V_2 = y\omega$. Then every edge (a, b) has end vertices in V_1 and V_2. Consequently, Γ is a complete bipartite graph. □

In what follows we study the identity graphs. Basically, by some properties of graph of groups we obtain results on order of polygroups and graph of polygroups.

Theorem 4.19. [10]
Let $(G, .)$ be a group. Then $(P_G, \circ, e,^{-1})$ is a polygroup, where $P_G = G \cup \{a\}$, $a \notin G$ and $\circ$ is defined as follows:
(1) $a \circ a = e$,
(2) $e \circ x = x \circ e = x$, $\forall x \in G$,
(3) $a \circ x = x \circ a = x$, $\forall x \in G - \{e, a\}$,
(4) $x \circ y = x.y$, $\forall (x, y) \in G^2; y \neq x^{-1}$,
(5) $x \circ x^{-1} = x^{-1} \circ x = \{e, a\}$, $\forall x \in G - \{e, a\}$.

Clearly, $\zeta(P_G) = Z(G) \cup \{e, a\}$ and $\omega_{P_G} = \{e, a\}$ and $|P_G| = |G| + 1$.

Theorem 4.20. *Let G and H be two groups, $a \notin G$, $b \notin H$, $P_H = H \cup \{b\}$, $P_G = G \cup \{a\}$ and i be the identity automorphism. Then $\Gamma_{P_G}^i \approx \Gamma_{P_H}^i$ if and only if $\Gamma_G \approx \Gamma_H$.*

Proof. ($\Rightarrow$) Consider $g_1, g_2 \in G \setminus Z(G)$ and $g_1 \cdot g_2 \neq g_2 \cdot g_1$ (*). Then by Theorem 4.19 (4), $g_1 \circ g_2 \circ \omega_{P_G} \neq g_2 \circ g_1 \circ \omega_{P_G}$. Since $\Gamma_{P_G}^i \approx \Gamma_{P_H}^i$ we have a bijection $f : P_G \setminus \zeta(P_G) \to P_H \setminus \zeta(P_H)$ and so by (*) we have

$$f(g_1) \circ f(g_2) \circ \omega_{P_H} \neq f(g_2) \circ f(g_1) \circ \omega_{P_H}. \ (**)$$

Note that $f(g) \circ b = b \circ f(g) = f(g)$ for any $g \in G$ and so $f(g_i) \neq b$ for $i = 1, 2$. By (**) and Theorem 4.19 (4), we have $f(g_1) \cdot f(g_2) \cdot \omega_{P_H} \neq f(g_2) \cdot f(g_1) \cdot \omega_{P_H}$. Then $f(g_1) \cdot f(g_2) \neq f(g_2) \cdot f(g_1)$. Therefore, $\Gamma_G \approx \Gamma_H$ with bijection $f|_{G \setminus Z(G)}$, where $f|_{G \setminus Z(G)} : G \setminus Z(G) \to H \setminus Z(H)$ is defined by $f|_{G \setminus Z(G)}(x) = f(x)$.
($\Leftarrow$) Assume $\Gamma_G \approx \Gamma_H$. Then there exists a bijection $f : G \setminus Z(G) \to H \setminus Z(H)$. Thus f is a

bijection from $P_G \setminus \zeta(P_G)$ to $P_H \setminus \zeta(P_H)$. Let $x, y \in P_G$ and $x \circ y \circ \omega_{P_G} \neq y \circ x \circ \omega_{P_G}$. Then $x, y \neq \{e, a\}$ and so by Theorem 4.19 (4), $x \cdot y \cdot \omega_{P_G} \neq y \cdot x \cdot \omega_{P_G}$. Thus, $x \cdot y \neq y \cdot x$. Since $\Gamma_G \approx \Gamma_H$ we have $f(x) \cdot f(y) \neq f(y) \cdot f(x)$. Then by Theorem 4.19 (4) and $x, y \neq \{e, a\}$ we conclude $f(x) \circ f(y) \circ \omega_{P_H} \neq f(y) \circ f(x) \circ \omega_{P_H}$. Consequently, $\Gamma^i_{P_G} \approx \Gamma^i_{P_H}$.

□

Theorem 4.21. [15] *Let G and H be two groups and $\Gamma_G \approx \Gamma_H$. Then $|G| \neq |H|$ in general.*

Corollary 4.22. *If $\Gamma_G \approx \Gamma_H$, then $|P_G| \neq |P_H|$ in general.*

Proof. Let $\Gamma_G \approx \Gamma_H$. By Theorem 4.21, $|G| \neq |H|$ in general and so $|P_G| \neq |P_H|$ in general. □

Definition 4.23. *A polygroup $(P, \circ)$ is called a polygroup of exponent n ($n \in \mathbb{N}$) if for each non-trivial element x of P we have $\underbrace{x \circ x \circ \ldots \circ x}_{n} = e$.*

Corollary 4.24. *Every polygroup of exponent 3 is a group. In addition, if P and H are two polygroups of exponent 3 and $\Gamma_P \approx \Gamma_H$, then $|P| \neq |H|$ in general.*

Proof. Let P be a polygroup of exponent 3 and $b \in P$ be an arbitrary element. Since $bbb = e$, then $bx = e$ for some $x \in bb$. Thus $b^{-1} = x$, which implies

$$bb^{-1} = e \;\text{ for any } b \in P. \;(V)$$

Let $x \in yz$. By Definition 2.1, we have $z \in y^{-1}x$. Thus by (V) $x \in yz \subseteq yy^{-1}x = x$. Therefore, $x = yz$ and so every polygroups of exponent 3 is a group.
In addition, consider P and H are two polygroups of exponent 3 and $\Gamma_P \approx \Gamma_H$. Then P and H are two groups. Now, we get the result by Theorem 4.21, i.e $|P| \neq |H|$ in general. □

It is proved that, for many groups G if H is a group with Γ_G isomorphic to Γ_H, then $|G| = |H|$ (see [17]). Now, by Theorem 4.20, we have the following corollary.

Corollary 4.25. *For many groups G if H is a group with Γ_{P_G} isomorphic to Γ_{P_H}, then $|P_G| = |P_H|$.*

5 Conclusion

In this paper, the notion of weak α-commutative polygroup was defined. Then a connection between a weak α-commutative polygroup with its β-commutative fundamental group was obtained. Espesially, the notion of an α-graph was introduced. Also, some properties of this graph, like girth and diameter, was stated. Moreover, α-isomorphic graphs were investigated. This paper would be useful to study polygroups (see Corollaries 4.22, 4.24).

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