Journal of Algebraic Hyperstructures and Logical Algebras

Volume 2, Number 1, (2021), pp. 47-67

On hyper BI-algebras

S. Niazian[1]

[1] *Faculty of Medicine, Tehran Medical Sciences, Islamic Azad University, Tehran, Iran*

s.niazian@iautmu.ac.ir

Abstract

In this paper, we introduce the notion of hyper BI-algebra and investigate some properties of it. Also, we state and prove some theorems which determine the relationship among $R/C/D/T$ and V-hyper BI-algebras under some conditions. Then we study the relation among hyper BI-algebra with some of other hyper logical algebras such as hyper BCI/BCK/K/B/BCC-algebras and show that under which condition these hyper structures coincide. In addition, we define hyper subalgebra and (weak) ideal of a hyper BI-algebra and obtain some results and the relation between them. Finally, we construct the quotient structure of hyper BI-algebra and examine the isomorphism theorems.

Article Information

Corresponding Author:
S. Niazian;
Received: December 2020;
Accepted: February 2021;
Papertype: Original.

Keywords:

BI-algebra, hyper BI-algebra, hyper K-algebra, hyper BCK/BCI-algebra.

1 Introduction

The notion of BCK-algebra and BCI-algebra were introduced by Imai and Iseki, in [11], as two classes of abstract algebras. It is known that the class of BCK-algebras is a proper subclass of the BCI-algebras.

Iseki posed an interesting problem (solved by Wronski in [26]) whether the class of BCK-algebras is a variety. In connection with this problem Komori introduced a notion of BCC-algebra which is a generalization of a BCK-algebra and proved that the class of all BCC-algebras is not a variety, but the variety generated by BCC-algebras is finitely based [9], [17], [18].

The notion of B-algebras was introduced by Neggers and Kim in 2002. They showed that the B-algebra is equivalent in some senses to a group [21], [22]. Many researchers generalized B-algebras and introduced new notions as: D-algebras, BM/BG/BO/BN/BP/BF-algebras and so on. For more details, the reader can study references [2] [13], [14], [15], [16], [20] and [25].

Borumand Saeid et al. introduced a new algebra, called BI-algebra [3]. These algebras are interesting and important in that they are an extension of both a/an (dual) implication algebra

https://doi.org/10.52547/HATEF.JAHLA.2.1.4

and an implicative BCK-algebra. Recently, in [23], the concepts of a Neutro-BI-algebra and Anti-BI-algebra are introduced, and some related properties are investigated. Also, was shown that the class of Neutro-BI-algebra is an alternative of the class of BI-algebras. The study of hyperstructures, started in 1934 by *Marty's* paper at the 8th Congress of Scandinavian Mathematicians [19] where hypergroups were introduced. Since then, many researchers have worked on and developed hyperstructure theory from the theoretical point of view and for their applications to many subjects of pure and applied mathematics. There are extensive applications in many branches such as Euclidian and non-Euclidian geometries, graphs and hyper-graphs, binary relations, lattices, fuzzy and rough sets, automata, cryptography, codes, probabilities, information sciences and so on. Some interesting applications of hyper structures can be found in the book [8].

In [12], Jun et al. applied the hyper structures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra and investigated some related properties. Borzooei et al. introduced and studied hyper K-algebras [6]. Further in (2006), Xin initiated the concept of hyper BCI-algebras which is basically a generalization of hyper BCK-algebras, and he proved that every hyper BCK-algebra is a hyper BCI-algebra [27].

Borzooei, Dudek and Kouhestani, have introduced the concept of hyper BCC-algebra as a common generalization of BCC-algebras and hyper BCK-algebras. In particular, they have investigated different types of hyper BCC-ideals and have described the relationship among them [5].

Endam et al. applied the concept of hyperstructure to B-algebras and proved that hyper B-algebra is a natural extension of B-algebra and presented some basic properties. Their study characterized the closed and invertible subhyper B-algebras [10]. A. L. O. Vicedo and J. P. Vilela [24] introduced the notions of (weak, strong) hyper B-ideals and investigated the relationship among these hyper B-ideals. Moreover, they studied relations between hyper B-ideals and subhyper B-algebras and some relations between hyper B-algebras and hypergroups.

This paper is organized as: In Section 2, we provide some definitions and preliminary conclusions about BI-algebras and some other algebras which will be used in next sections. In Section 3, we introduce the notion of hyper BI-algebra and investigate some properties of it. Finally, in Section 4, we study the relation among hyper BI-algebra with some of other hyper logical algebras and show that under which condition these hyper structures coincide.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1. [1] *A groupoid $(X;*)$ is called an implication algebra if for all $x, y, z \in X$, it satisfies the following identities:*
$(I1)$ $(x * y) * x = x$,
$(I2)$ $(x * y) * y = (y * x) * x$,
$(I3)$ $x * (y * z) = y * (x * z)$.

Definition 2.2. [7] *Let $(X;*)$ be an implication algebra and for any $x, y \in X$, define a binary operation " $\circ$ " on X by, $x \circ y = y * x$. Then $(X, \circ)$ is called a dual implication algebra and for all $x, y, z \in X$, it satisfies the following axioms:*
$(DI1)$ $x \circ (y \circ x) = x$,
$(DI2)$ $x \circ (x \circ y) = y \circ (y \circ x)$,
$(DI3)$ $(z \circ y) \circ x = (z \circ x) \circ y$.

Chen and Olivia in [7] proved that in any implication algebra $(X;*)$ for all $x, y \in X$, the identity $x * x = y * y$ holds which is denoted by the constant 0. BI-algebra is a generalization of the notion of dual implication algebra.

Definition 2.3. [3] *An algebraic structure $(X; *, 0)$ of type $(2, 0)$ is called BI-algebra if for all $x, y \in X$, it satisfies the following conditions:*
$(BI1)$ $x * x = 0$,
$(BI2)$ $x * (y * x) = x$.

Let $(X; *, 0)$ be a BI-algebra. For any $x, y \in X$, define the relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$. Notice that $(X, \leq)$ is not a poset and the relation $\leq$ is only reflexive.

Proposition 2.4. [3] *Let $(X, *, 0)$ be a BI-algebra. Then for any $x, y \in X$, the following conditions hold:*
(i) $x * 0 = x$,
(ii) $0 * x = 0$,
(iii) $x * y = (x * y) * y$.

Proposition 2.5. [3] (i) *Every implicative BCK-algebra is a BI-algebra.*
(ii) *Any dual implication algebra is a BI-algebra.*

Definition 2.6. [8] *A hyperoperation on a non-empty set H is a map $\circ : H \times H \to P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of H. Moreover, H with a hyperoperation is called a hypergroupoid. An element $a \in H$ is called scalar if $|a \odot x| = 1$, for any $x \in H$. In this definition, if A and B are two non-empty subsets of H, then we define $A \circ B$, $a \circ B$ and $A \circ b$ as follows, for any $a \in A$ and $b \in B$:*

$$A \circ B = \bigcup_{a \in A, b \in B} (a \circ b), \quad a \circ B = \{a\} \circ B \quad \textit{and} \quad A \circ b = A \circ \{b\}.$$

3 Hyper BI-algebras

In this section, we introduce the notion of hyper BI-algebra and investigate some properties of it. Then we show that any hyper BI-algebra of order n can be extend to a hyper BI-algebra of order $n + 1$. In addition, we construct a hyper BI-algebra by two (hyper) BI-algebras. Also, we present some special types of hyper BI-algebras and investigate relation of between them. Finally, we introduce and study the notion of (strong) hyper subalgebra of a hyper BI-algebra and check them under BI-homomorphism.

Definition 3.1. *Let H be a non-empty set endowed with a hyper operation "$\circ$". Then $(H, \circ, 0)$ is called a hyper BI-algebra if for any $x, y \in H$, the following conditions hold:*
$(HBI1)$ $x \leq x$,
$(HBI2)$ $x \in x \circ (y \circ x)$.

Where $x \leq y$ if $0 \in x \circ y$. For any $A, B \subseteq H$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \leq b$. We denote $A \ll \{y\}$ $(\{x\} \ll B)$ by $A \ll y$ $(x \ll B)$.

Example 3.2. *Let $H = \{0, a, b, c\}$. Define the hyper operation "$\circ$" on H as the following table:*

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a,c\}$	$\{a,b\}$
b	$\{b\}$	$\{b,c\}$	$\{0,c\}$	$\{b,c\}$
c	$\{c\}$	$\{b,c\}$	$\{b,c\}$	$\{0,b\}$

Then we can see that $(H;\circ,0)$ *is a hyper BI-algebra.*

Note. In the following proposition, we study the relation between BI-algebra and hyper BI-algebra

Proposition 3.3. (i) *Let* $(H,\star,0)$ *be a BI-algebra. Define the hyper operation* $x\circ y := \{x\star y\}$*, then* $(H,\circ,0)$ *is a hyper BI-algebra.*
(ii) *If* $(H,\circ,0)$ *is a hyper BI-algebra such that, for all* $x,y\in H$, $|x\circ y|=1$*, then* $(H,\circ,0)$ *is a BI-algebra.*

Proof. (i) By (B1), $x\circ x=\{x\star x\}=\{0\}$, then $x\leq x$. Also, since $x\circ(y\circ x)=\{x\star(y\star x)\}=\{x\}$, we get that $x\in x\circ(y\circ x)$.
(ii) Since $x\leq x$, we get $0\in x\circ x$. In addition, from $|x\circ x|=1$, we have $x\circ x=\{0\}$. Also, since $x\in x\circ(y\circ x)$ and $|x\circ(y\circ x)|=1$, we get $x\circ(y\circ x)=\{x\}$. Therefore $(H,\circ,0)$ is a BI-algebra. □

Note. The above proposition shows that any hyper BI-algebra is a generalization of BI-algebra.

Proposition 3.4. *Let* $(H,\circ,0)$ *be a hyper BI-algebra. Then for all* $x,y\in H$ *and non-empty subsets* A,B,C *of* H *the following statements hold:*
(i) $A\ll A$.
(ii) $A\cap B\neq\emptyset$ *implies* $A\ll B$ *and* $B\ll A$.
(iii) $A\subseteq B$ *and* $A\ll C$ *imply* $B\ll C$*, specially* $A\ll x$, $A\subseteq B$ *imply* $B\leq x$.
(iv) $A\ll B$ *and* $B\subseteq C$ *imply* $A\ll C$.
(v) $0\in 0\circ(x\circ 0)$.
(vi) $0\ll x\circ 0$.
(vii) $x\in x\circ(A\circ x)$.
$(viii)$ $x\ll x\circ(y\circ x)\ll x$.
(ix) $x\ll x\circ(A\circ x)\ll x$.
(x) *If there exists* $y\in H$ *such that* $y\circ x=\{0\}$*, then* $x\in x\circ 0$.
(xi) *There is at least one* $t\in H$ *such that* $x\in x\circ t$*, particularly,* $x\in x\circ H$.

Proof. (i) Let A be a non-empty subset of H such that $x\in A$. Since $x\leq x$, we have $A\ll A$.
(ii) Let $A\cap B\neq\emptyset$. Then there exists $t\in A\cap B$. Since $t\in A$, $t\in B$ and $t\leq t$, we get $A\ll B$ and $B\ll A$.
(iii) Let $A\ll C$ and $A\subseteq B$. Then there exist $a\in A$ and $c\in C$ such that $a\leq c$. Since $a\in A\subseteq B$, we get $B\ll C$. Particularly, it is enough to let $C=\{x\}$.
(iv) Let $A\ll B$ and $B\subseteq C$. Then there exist $a\in A$ and $b\in B$ such that $a\leq b$. Since $b\in B\subseteq C$, we have $A\ll C$.
(v) It is the result of (HBI2).
(vi) It is the result of (v).
(vii) Since for all $a\in A$, $x\in x\circ(a\circ x)$, we have $x\in x\circ(A\circ x)$.
$(viii)$ Since $x\in x\circ(y\circ x)$ and $x\leq x$, we get $x\ll x\circ(y\circ x)\ll x$.
(ix) By (vii) and (HBI1), the proof is clear.
(x) Suppose $y\in H$ such that $y\circ x=\{0\}$. Then by (HBI2) $x\in x\circ(y\circ x)=x\circ 0$.
(xi) Since $x\in x\circ(y\circ x)=\bigcup\limits_{t\in y\circ x}(x\circ t)$, there exists $t\in y\circ x\subseteq H$ such that $x\in x\circ t$. □

In the following theorem, we prove that extend of a hyper BI-algebra, is a hyper BI-algebra, too.

Theorem 3.5. *Any hyper BI-algebra of order n can be extend to a hyper BI-algebra of order $n+1$.*

Proof. Let $(H, \circ, 0)$ be a hyper BI-algebra of order n and $\bar{H} = H \cup \{e\}$ for $e \notin H$. Then define the hyper operation $\bar{\circ}$ on $\bar{H}$ as follows:

$$x \bar{\circ} y \quad = \quad \begin{cases} x \circ y, & x, y \in H \\ \{0\}, & x \in \bar{H}, y = e \\ H, & x = e, y \in H - \{0\} \\ \{e\}, & x = e, y = 0. \end{cases}$$

Let $x \in \bar{H}$. If $x \in H$, since $(H, \circ, 0)$ is a hyper BI-algebra, then $x \leq x$. Also, if $x = e$, then by definition of hyper operation $\bar{\circ}$, we have $e \bar{\circ} e = \{0\}$. Hence, (HBI1) holds.
Let $x, y \in \bar{H}$. Then we prove that $x \in x \bar{\circ} (y \bar{\circ} x)$. For this, we suppose 5 cases as follows:
Case 1. If $x, y \in H$, then since $(H, \circ, 0)$ is a hyper BI-algebra, it is clear that (HBI2) holds.
Case 2. Let $x = y = e$. Since $e \bar{\circ} e = \{0\}$ and $e \bar{\circ} 0 = \{e\}$, we get that $e \in \{e\} = e \bar{\circ} 0 = e \bar{\circ} (e \bar{\circ} e)$. Therefore, (HBI2) holds.
Case 3. If $x \in H - \{0\}$ and $y = e$, then $x \bar{\circ} (y \bar{\circ} x) = x \bar{\circ} (e \bar{\circ} x) = x \bar{\circ} H = x \circ H$. Also, by Proposition 3.4(xi), $x \in x \circ t$ for some $t \in H$. Hence $x \in \bigcup_{t \in H} (x \circ t) = x \circ H$. So, in this case, the proof is completed.
Case 4. If $x = \{0\}$ and $y = e$, then $x \bar{\circ} (y \bar{\circ} x) = 0 \bar{\circ} (e \bar{\circ} 0) = 0 \bar{\circ} \{e\} = \{0\}$. Hence, $0 \in 0 \bar{\circ} (e \bar{\circ} 0)$ and so (HBI2) holds.
Case 5. If $x = e$ and $y \in H$, then $x \bar{\circ} (y \bar{\circ} x) = e \bar{\circ} (y \bar{\circ} e) = e \bar{\circ} \{0\} = \{e\}$. Hence, (HBI2) holds.
Therefore, $(\bar{H}, \bar{\circ}, 0)$ is a hyper BI-algebra as order $n+1$. □

Corollary 3.6. *For any $n \in \mathbb{N}$, there exists a hyper BI-algebra of order n.*

Proof. Let $H = \{0, a\}$ be a set. Define the hyper operation "$\circ$" on H as the following table:

$\circ$	0	a
0	$\{0\}$	$\{0\}$
a	$\{a\}$	H

Then $(H; \circ, 0)$ is a hyper BI-algebra of order 2. Therefore by Theorem 3.5, we can construct a hyper BI-algebra of any order n. □

In the following proposition, we construct a hyper BI-algebra by two hyper BI-algebras.

Proposition 3.7. *Let $(H_1, *, 0_1)$ and $(H_2, \star, 0_2)$ be two hyper BI-algebras. Define the hyperoperation "$\circ$" on $H_1 \times H_2$, for any $(x, y), (z, w) \in H_1 \times H_2$, as follows:*
- $(x, y) \circ (z, w) := \{(t, s) | \ t \in x * z, \ s \in y \star w\}$,
- $(x, y) \leq (z, w)$ *if and only if* $x \leq z$ *and* $y \leq w$.

Then $(H_1 \times H_2; \circ, 0)$ is a hyper BI-algebra where $0 =: (0_1, 0_2)$ witch is called Cartesian product of hyper BI-algebras H_1 and H_2.

Proof. Since $x \leq x$ and $y \leq y$, we have $(0_1, 0_2) \in \{(a, b) | \ a \in x * x, \ b \in y \star y\} = (x, y) \circ (x, y)$. Thus, $(x, y) \leq (x, y)$ and so (HBI1) holds. Since H_1 and H_2 are two hyper BI-algebras, for any

$x, z \in H_1$ and $y, w \in H_2$, we have $x \in x * (z * x)$ and $y \in y \star (w \star y)$. Then

$$\begin{aligned}
(x,y) &\in \{(t,s)|\ t \in x*(z*x),\ s \in y \star (w \star y)\} \\
&= \bigcup_{a \in z*x} \bigcup_{b \in w \star y} [\{(t,s)|\ t \in x * a,\ s \in y \star b\}] \\
&= \bigcup_{a \in z*x} \bigcup_{b \in w \star y} [(x,y) \circ (a,b)] \\
&= (x,y) \circ \{(a,b)|\ a \in z * x,\ b \in w \star y\} \\
&= (x,y) \circ ((z,w) \circ (x,y)).
\end{aligned}$$

Hence (HBI2) holds. Therefore, $(H_1 \times H_2; \circ, 0)$ is a hyper BI-algebra. □

In the following proposition we prove that we can construct a hyper BI-algebra by Cartesian product of two BI-algebras, too.

Theorem 3.8. *Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two BI-algebras such that 0_2 is the least element of H_2. Define the hyperoperation "$*$" on $H_1 \times H_2$, as the following:*

$$\bar{x} * \bar{y} =: \{(x_1 \circ_1 y_1,\ x_2),\ (x_1 \circ_1 y_1,\ x_2 \circ_2 y_2)\},$$

where $0 =: (0_1, 0_2)$ and $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2) \in H_1 \times H_2$. Then $(H_1 \times H_2; *, 0)$ is a hyper BI-algebra which is called strong hyper BI-algebra.

Proof. It is clear that the hyper operation "$*$" is well-defined. Let $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in H_1 \times H_2$. Then, we prove that

$$\bar{x} \leq \bar{y} \quad \text{if and only if} \quad x_1 \leq y_1,\ x_2 \leq y_2.$$

Let $\bar{x} \leq \bar{y}$. Then $0 \in \bar{x} * \bar{y}$ i.e. $(0_1, 0_2) \in \{(x_1 \circ_1 y_1,\ x_2),\ (x_1 \circ_1 y_1,\ x_2 \circ_2 y_2)\}$, so we have two following cases:
Case 1. If $(0_1, 0_2) = (x_1 \circ_1 y_1,\ x_2)$, then $0_1 = x_1 \circ_1 y_1$ and $0_2 = x_2$. Hence, $x_1 \leq y_1$ and $x_2 = 0_2 \leq y_2$.
Case 2. If $(0_1, 0_2) = (x_1 \circ_1 y_1,\ x_2 \circ_2 y_2)$, then $0_1 = x_1 \circ_1 y_1$ and $0_2 = x_2 \circ_2 y_2$. Hence, $x_1 \leq y_1$ and $x_2 \leq y_2$.

Conversely, let $x_1 \leq y_1$ and $x_2 \leq y_2$. Since H_1 and H_2 are BI-algebras, we get $x_1 \circ_1 y_1 = 0_1$ and $x_2 \circ_2 y_2 = 0_2$. Then

$$0 = (0_1, 0_2) = (x_1 \circ_1 y_1,\ x_2 \circ_2 y_2) \in \bar{x} * \bar{y}$$

Therefore, $\bar{x} \leq \bar{y}$.

Now, let $\bar{x} = (x_1, x_2) \in H_1 \times H_2$. Since $x_1 \leq x_1$ and $x_2 \leq x_2$, we have $(x_1, x_2) \leq (x_1, x_2)$, and so $\bar{x} \leq \bar{x}$. Hence, (HBI1) holds.

Let $\bar{x}, \bar{y} \in H_1 \times H_2$ such that $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$. Since H_1 and H_2 are BI-algebras, we have $x_1 = x_1 \circ_1 (y_1 \circ_1 x_1)$ and $x_2 = x_2 \circ_2 (y_2 \circ_2 x_2)$. Then

$$\begin{aligned}
& \bar{x} * (\bar{y} * \bar{x}) \\
= \; & (x_1, x_2) * ((y_1, y_2) * (x_1, x_2)) = (x_1, x_2) * \{(y_1 \circ_1 x_1,\ y_2),\ (y_1 \circ_1 x_1,\ y_2 \circ_2 x_2)\} \\
= \; & \{(x_1 \circ_1 (y_1 \circ_1 x_1),\ x_2),\ (x_1 \circ_1 (y_1 \circ_1 x_1),\ x_2 \circ_2 y_2)\} \\
& \bigcup \{(x_1 \circ_1 (y_1 \circ_1 x_1), x_2),\ (x_1 \circ_1 (y_1 \circ_1 x_1), x_2 \circ_2 (y_2 \circ_2 x_2))\} \\
= \; & \{(x_1, x_2),\ (x_1,\ x_2 \circ_2 y_2)\} \cup \{(x_1, x_2)\} \\
= \; & \{(x_1, x_2),\ (x_1, x_2 \circ_2 y_2)\} \\
= \; & \{\bar{x}, \bar{z}\},
\end{aligned}$$

where $\bar{z} = (x_1, x_2 \circ_2 y_2)$. Hence $\bar{x} \in \bar{x} * (\bar{y} * \bar{x})$. Therefore, (HBI2) holds and $(H_1 \times H_2; \circ, 0)$ is a hyper BI-algebra. □

Definition 3.9. *Let $(H, \circ, 0)$ be a hyper BI-algebra and $x \in H$. Then H is called:*

- *Row hyper (or briefly R-hyper) BI-algebra if $0 \circ x = \{0\}$.*
- *Column hyper (or briefly C-hyper) BI-algebra if $x \circ 0 = \{x\}$.*
- *Diagonal hyper (or briefly D-hyper) BI-algebra if $x \circ x = \{0\}$.*
- *Thin hyper (or briefly T-hyper) BI-algebra if it is R-hyper and C-hyper BI-algebra.*
- *Very thin hyper (or briefly V-hyper) BI-algebra if it is an R-hyper, C-hyper and D-hyper BI-algebra.*

Example 3.10. (*i*) *Let $H = \{0, a, b\}$ be a set. Define the hyperoperation $\circ$ on H as follows:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0,a\}$	$\{0,b\}$	$\{a,b\}$
b	$\{0,b\}$	$\{a,b\}$	$\{0,a\}$

Then $(H, \circ, 0)$ is an R-hyper BI-algebra. In addition, it is clear that $(H, \circ, 0)$ is not a C-hyper BI-algebra and D-hyper BI-algebra. Because $x \in H$, $x \circ 0 \neq \{x\}$ and $x \circ x \neq \{0\}$, for some $x \in H$.
(*ii*) *Let $H = \{0, a, b\}$ be a set. Define the hyperoperation $\circ$ on H as follows:*

$\circ$	0	a	b
0	$\{0\}$	$\{0,a\}$	$\{0,b\}$
a	$\{a\}$	$\{0,b\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a\}$

Then $(H, \circ, 0)$ is a C-hyper BI-algebra. In addition, it is clear that $(H, \circ, 0)$ is not an R-hyper BI-algebra and D-hyper BI-algebra. Because for some $x \in H$, $0 \circ x \neq \{0\}$ and $x \circ x \neq \{0\}$.
(*iii*) *Let $H = \{0, a, b\}$ be a set. Define the hyperoperation $\circ$ on H as follows:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0,a\}$
a	$\{a\}$	$\{0\}$	$\{0,a\}$
b	$\{0,b\}$	$\{0,b\}$	$\{0\}$

Then $(H, \circ, 0)$ is a D-hyper BI-algebra. In addition, it is clear that $(H, \circ, 0)$ is not a C-hyper BI-algebra and R-hyper BI-algebra. Because for some $x \in H$, $x \circ 0 \neq \{x\}$ and $0 \circ x \neq \{0\}$.

(iv) Let $H = \{0, a, b\}$ be a set. Define the hyperoperation $\circ$ on H as follows:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

Then $(H, \circ, 0)$ is a T-hyper BI-algebra. Clearly, $(H, \circ, 0)$ is not a V-hyper BI-algebra. Because it is not a diagonal hyper BI-algebra.
(v) Let $H = \{0, a, b\}$ be a set. Define the hyperoperation "$\circ$" on H as follows:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0\}$

Then $(H, \circ, 0)$ is a V-hyper BI-algebra.

Note. From now on, $\mathcal{H} = (H; \circ, 0)$ or simply $\mathcal{H}$ is a hyper BI-algebra unless otherwise state.

Definition 3.11. *(i) Let S be a subset of H containing "0". If $(S, \circ, 0)$ is a hyper BI-algebra, then S is called a hyper subalgebra of $\mathcal{H}$.*
(ii) Let S be a hyper subalgebra of $\mathcal{H}$. If for all $x \in S$, $x \circ S = S = S \circ x$, then S is called a strong hyper subalgebra of $\mathcal{H}$.

Theorem 3.12. *Let S be a non-empty subset of H. Then S is a hyper subalgebra of $\mathcal{H}$ if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Proof. ($\Rightarrow$) If S is a hyper subalgebra of $\mathcal{H}$, then S is closed under hyper operation "$\circ$".
($\Leftarrow$) Let $x, y \in S$ such that $x \circ y \subseteq S$. Since $0 \in x \circ x \subseteq S$, we get $0 \in S$. Hence, $x \leq x$, and so (HBI1) holds. Now, it is enough to show that S satisfies the condition (HBI2). For this, by assumption, for any $x, y \in S$, $y \circ x \subseteq S$. Then $x \in x \circ (y \circ x) \subseteq x \circ S = \bigcup_{s \in S} (x \circ s) \subseteq S$. Hence, S is a hyper subalgebra of $\mathcal{H}$. □

Example 3.13. *(i) Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.10(i). Then $S = \{0, a\}$ is not a hyper subalgebra of $\mathcal{H}$. Because $a \circ a = \{0, b\} \nsubseteq S$.*
(ii) Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.10(iii). Then $S = \{0, a\}$ is a hyper subalgebra of $\mathcal{H}$. But it is not a strong hyper subalgebra, because $S \circ a = \{0\} \neq S$.
(iii) Let $H = \{0, a, b\}$. Then define the hyper operations "$\circ$" on H as follows:

$\circ$	0	a	b
0	$\{0\}$	$\{0,a\}$	$\{0,b\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0,b\}$
b	$\{b\}$	H	$\{0\}$

Routine calculations show that $(H; \circ, 0)$ is a hyper BI-algebra. Then $S = \{0\}$, $S = \{0, a\}$ and $S = \{0, b\}$ are strong hyper subalgebras of $\mathcal{H}$.

In the following example, we show that $S = \{0\}$ is not a (strong) hyper subalgebra, in general.

Example 3.14. *Let $H = \{0, a, b\}$. Then define the hyper operations "$\circ$" on H as follows:*

$\circ$	0	a	b
0	H	$\{0,a\}$	$\{0,b\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0,b\}$
b	$\{b\}$	H	$\{0\}$

Routine calculations show that $(H; \circ, 0)$ is a hyper BI-algebra. Then $S = \{0\}$ is not a (strong) hyper subalgebra of H, because $0 \circ 0 = H \not\subseteq \{0\}$.

Proposition 3.15. *If $\mathcal{H}$ is a/an (C-hyper, D-hyper) R-hyper BI-algebra, then $S = \{0\}$ is a strong hyper subalgebra of $\mathcal{H}$.*

Proof. The proof is straightforward. $\square$

Theorem 3.16. *There is not any non-trivial proper strong hyper subalgebra of R-hyper BI-algebra.*

Proof. Let $\mathcal{H}$ be an R-hyper BI-algebra and $S \neq \{0\}$ be a proper strong hyper subalgebra of $\mathcal{H}$. Then for all $x \in \mathcal{H}$, $0 \circ x = \{0\}$ and $x \circ S = S$ for all $x \in S$. In particularly, if $x = 0$, then $0 \circ S = \bigcup_{s \in S} (0 \circ s) = \{0\}$, which is a contradiction with $S \neq \{0\}$. Hence an R-hyper BI-algebra does not have any non-trivial strong hyper subalgebra. $\square$

In the following, we define a homomorphism on hyper BI-algebras and investigate some properties of it.

Definition 3.17. *Let $\mathcal{H}_1 = (H_1, \circ_1, 0_1)$ and $\mathcal{H}_2 = (H_2, \circ_2, 0_2)$ be two hyper BI-algebras. A mapping $f : \mathcal{H}_1 \to \mathcal{H}_2$ is called a hyper BI-homomorphism if for any $x, y \in X$ we have*
(i) $f(0_1) = 0_2$,
(ii) $f(x \circ_1 y) = f(x) \circ_2 f(y)$.
If f is one to one and onto, then we say that f is a hyper BI-isomorphism.

Example 3.18. *Let $H_1 = \{0_1, a, b\}$ and $H_2 = \{0_2, x, y, z\}$. Define two hyper operations "$\circ$" and "$*$" on H_1 and H_2, respectively, as follows:*

$\circ$	0_1	a	b
0_1	$\{0_1\}$	$\{0_1\}$	$\{0_1\}$
a	$\{a\}$	$\{0_1, a\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b\}$	$\{0_1,b\}$

$*$	0_2	x	y	z
0_2	$\{0_2\}$	$\{0_2\}$	$\{0_2\}$	$\{0_2\}$
x	$\{x\}$	$\{0_2, x\}$	$\{x, y\}$	$\{x, z\}$
y	$\{y\}$	$\{x, y\}$	$\{0_2, y\}$	$\{y, z\}$
z	$\{z\}$	$\{x, z\}$	$\{y, z\}$	$\{0_2\}$

Then routine calculations show that $(H_1, \circ, 0_1)$ and $(H_2; *, 0_2)$ are hyper BI-algebras. Now, if we define $f : H_1 \to H_2$ such that:

$$f(0_1) := 0_2, \quad f(a) := x, \quad \text{and } f(b) := y,$$

then f is a hyper BI-homomorphism.

Suppose f is a BI-homomorphism of H_1 to H_2. We denote $Kerf = \{x \in H_1 : f(x) = 0_2\}$. Let A and B be two non-empty subsets of H_1 and H_2, respectively. Then $f(A) = \{f(a) : a \in A\}$ is called the image of A under f and $f^{-1}(B) = \{x \in H_1 : f(x) \in B\}$ is called the inverse image of B under f.

Proposition 3.19. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hyper BI-algebras. If $f : \mathcal{H}_1 \to \mathcal{H}_2$ is a hyper BI-homomorphism, then f is an order preserving map.*

Proof. Let $x, y \in H_1$ such that $x \leq y$. Then $0_1 \in x \circ y$ and $0_2 = f(0_1) \in f(x \circ y) = f(x) \circ f(y)$. Hence, $f(x) \leq f(y)$. $\square$

Proposition 3.20. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a hyper BI-homomorphism.*
(i) If S is a (strong) hyper subalgebra of $\mathcal{H}_1$, then $f(S)$ is a (strong) hyper subalgebra of $\mathcal{H}_2$.
(ii) If S is a hyper subalgebra of $\mathcal{H}_2$, then $f^{-1}(S)$ is a hyper subalgebra of $\mathcal{H}_1$.
(iii) If S is a strong hyper subalgebra of $\mathcal{H}_2$ and f is onto, then $f^{-1}(S)$ is a strong hyper subalgebra of $\mathcal{H}_1$.

Proof. (i) Let S be a hyper subalgebra of $\mathcal{H}_1$ and $y_1, y_2 \in f(S)$. Then there exist $x_1, x_2 \in S$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. By Theorem 3.12, we have $x_1 \circ x_2 \subseteq S$, then

$$y_1 \circ y_2 = f(x_1) \circ f(x_2) = f(x_1 \circ x_2) \subseteq f(S).$$

Hence, $f(S)$ is a hyper subalgebra of $\mathcal{H}_2$. Now, let S be a strong hyper subalgebra of $\mathcal{H}_1$ and $y \in f(S)$. Then there exists $x \in S$ such that $y = f(x)$. Thus,

$$y \circ f(S) = f(x) \circ f(S) = f(x \circ S) = f(S) = f(S \circ x) = f(S) \circ f(x) = f(S) \circ y.$$

Therefore, $f(S)$ is a (strong) hyper subalgebra of $\mathcal{H}_2$.

(ii) Let S be a hyper subalgebra of $\mathcal{H}_2$. If $x_1, x_2 \in f^{-1}(S)$, then $f(x_1), f(x_2) \in S$ and we get $f(x_1 \circ x_2) = f(x_1) \circ f(x_2) \subseteq S$. So $x_1 \circ x_2 \subseteq f^{-1}(S)$. Hence $f^{-1}(S)$ is a hyper subalgebra of $\mathcal{H}_1$.

(iii) By (ii), we know that $f^{-1}(S)$ is a hyper subalgebra of $\mathcal{H}$. Then it is enough to prove that $f^{-1}(S)$ is strong. For this, since f is onto, we have $f(f^{-1}(S)) = S$. Let $x \in f^{-1}(S)$. Since $f(x) \in S$ and S is strong, we get

$$f(x \circ f^{-1}(S)) = f(x) \circ f(f^{-1}(S)) = f(x) \circ S = S.$$

Hence $x \circ f^{-1}(S) = f^{-1}(S)$. Similarly $f^{-1}(S) \circ x = f^{-1}(S)$. Therefore, $x \circ f^{-1}(S) = f^{-1}(S) = f^{-1}(S) \circ x$. $\square$

Proposition 3.21. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a hyper BI-homomorphism. If $0_2 \circ 0_2 = \{0_2\}$, then $Kerf$ is a hyper subalgebra of $\mathcal{H}_1$.*

Proof. Let $x_1, x_2 \in Kerf$. Then $f(x_1) = 0_2 = f(x_2)$ and for all $t \in x_1 \circ x_2$, we have

$$f(t) \in f(x_1 \circ x_2) = f(x_1) \circ f(x_2) = 0_2 \circ 0_2 = \{0_2\}.$$

Hence, $t \in Kerf$, and so $x_1 \circ x_2 \subseteq Kerf$. Therefore, $Kerf$ is a hyper subalgebra of $\mathcal{H}_1$. $\square$

Proposition 3.22. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be an onto hyper BI-homomorphism. Then:*
(i) If $\mathcal{H}_1$ is an R-hyper BI-algebra, then $\mathcal{H}_2$ is an R-hyper BI-algebra, too.
(ii) If $\mathcal{H}_1$ is a C-hyper BI-algebra, then $\mathcal{H}_2$ is a C-hyper BI-algebra, too.
(iii) If $\mathcal{H}_1$ is a D-hyper BI-algebra, then $\mathcal{H}_2$ is, too.
(iv) If $\mathcal{H}_1$ is a T-hyper BI-algebra, then $\mathcal{H}_2$ is, too.
(v) If $\mathcal{H}_1$ is a V-hyper BI-algebra, then $\mathcal{H}_2$ is, too.

Proof. We only proof (i) and the proof of other cases is similar.

(i) Let $\mathcal{H}_1$ be an R-hyper BI-algebra and $y \in H_2$. Then there exists $x \in H_1$ such that $y = f(x)$. Thus

$$0_2 \circ y = f(0_1) \circ f(x) = f(0_1 \circ x) = \{f(t) |\ t \in 0_1 \circ x = \{0_1\}\} = \{f(0_1)\} = \{0_2\}.$$

Therefore, $\mathcal{H}_2$ is an R-hyper BI-algebra. □

Proposition 3.23. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a one-to-one hyper BI-homomorphism. If $\mathcal{H}_2$ is a/an (D/V/T/C) R-hyper BI-algebra, then $\mathcal{H}_1$ is, too.*

Proof. Let $\mathcal{H}_2$ be an R-hyper BI-algebra and $x \in H_1$. Then

$$\{f(t) |\ t \in 0_1 \circ x\} = f(0_1 \circ x) = f(0_1) \circ f(x) = 0_2 \circ f(x) = \{0_2\} = \{f(0_1)\}.$$

Since f is one-to-one, we get that $0_1 \circ x = \{0_1\}$. Therefore, $\mathcal{H}_1$ is an R-hyper BI-algebra. The proof of other cases is similar. □

4 Relation among hyper BI-algebra and some of other hyper algebars

In this section, we study the relation among hyper BI-algebra with some of other hyper logical algebras such as hyper (BCK/ BCI/ BCC/ K) B-algebra and show that under which condition these hyper structures coincide.

Definition 4.1. [12] *An algebraic structure $(H; \circ, 0)$ with hyper operation "$\circ$" and a constant "0" is called a hyper BCK-algebra if for all $x, y, z \in H$ it satisfies the following conditions:*

(HBCK1) $(x \circ y) \circ (y \circ z) \ll x \circ z$,

(HBCK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HBCK3) $x \circ H \ll \{x\}$,

(HBCK4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$.

Where $x \leq y$ if and only if $0 \in x \circ y$. Also, suppose A and B are two non-empty subsets of H. Then $A \ll B$ means that for all $a \in A$, there exists an element $b \in B$ such that $a \leq b$.

Theorem 4.2. [12] *If $(H, \circ, 0)$ is a hyper BCK-algebra, then for any $x \in H$ we have $x \leq x$.*

Example 4.3. (*i*) *Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.13(iii). Since for $a \in \mathcal{H}$, we have, $0 \in 0 \circ a$ and $0 \in a \circ 0$, then $0 \leq a$ and $a \leq 0$, but $a \neq 0$. Thus $\mathcal{H}$ does not satisfy the condition (HBK4). Hence, $\mathcal{H}$ it is not a hyper BCK-algebra.*

(*ii*) *Let $H = \{0, a, b\}$. Define the hyper operation "$\circ$" on H as the following table:*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a, b\}$	$\{0,a,b\}$

Then $(H, \circ, 0)$ is a hyper BCK-algebra. But it is not a hyper BI-algebra, because

$$a \notin a \circ (b \circ a) = a \circ \{a, b\} = (a \circ a) \cup (a \circ b) = \{0\}.$$

By the above examples we can see that hyper BI-algebras and hyper BCK-algebras are not same.

Definition 4.4. [27] *An algebraic structure* $(H; \circ, 0)$ *with hyper operation "*$\circ$*" and constant "0" is called a hyper BCI-algebra if for all* $x, y, z \in H$ *it satisfies the following conditions:*

(HBCI1) $(x \circ y) \circ (y \circ z) \ll x \circ z$,
(HBCI2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HBCI3) $x \leq x$,
(HBCI4) $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
(HBCI5) $0 \circ (0 \circ x) \ll x$.

Where $x \leq y$ *if and only if* $0 \in x \circ y$. *Also* $A \ll B$ *means that for all* $a \in A$, *there exists an element* $b \in B$ *such that* $a \leq b$.

According to Example 4.3(i), clearly, $\mathcal{H}$ is not a hyper BCI-algebra.

Example 4.5. *Let* $H = \{0, a, b\}$. *Define the hyperoperation "*$\circ$*" on* H *as the following table:*

$\circ$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0,a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Then $(H, \circ, 0)$ *is a hyper BCI-algebra. But it is not a hyper BI-algebra, because* $a \notin a \circ (b \circ a) = \{b\}$.

By the above example, we can see that hyper BI-algebras and hyper BCI-algebras are not same.

Definition 4.6. [6] *An algebraic structure* $(H; \circ, 0)$ *with hyper operation "*$\circ$*" and constant "0" is called a hyper K-algebra if for all* $x, y, z \in H$, *it satisfies the following axioms:*

(HK1) $(x \circ y) \circ (y \circ z) \ll x \circ z$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \leq x$,
(HK4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
(HK5) $0 \leq x$.

Where $x \leq y$ *if and only if* $0 \in x \circ y$. *For any* $A, B \subseteq H$, $A \ll B$ *if and only if there exist* $a \in A$ *and* $b \in B$ *such that* $a \leq b$.

According to Example 4.3(i), obviously, $\mathcal{H}$ is not a hyper K-algebra.

Example 4.7. *Let* $H = \{0, a, b, c\}$. *Define the hyperoperation "*$\circ$*" on* H *as follows:*

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{b\}$	$\{a\}$	$\{0, a\}$

Then $(H; \circ, 0)$ *is a hyper K-algebra. But it is not a hyper BI-algebra, since* $a \notin a \circ (b \circ a) = \{0\}$.

By the above example, we can see that hyper BI-algebras and hyper K-algebras are not same.

Definition 4.8. [10] *An algebraic structure* $(H;\circ,0)$ *with a binary hyper operation "$\circ$" and a constant "0" is called a hyper B-algebra if for all* $x,y,z\in H$, *it satisfies the following axioms:*

(HB1) $0\in x\circ x$,

(HB2) $x\circ H=H=H\circ x$,

(HB3) $(x\circ y)\circ z=x\circ(z\circ(0\circ y))$.

Where $x\leq y$ *if and only if* $0\in x\circ y$ *and* $A\ll B$ *means that for all* $a\in A$, *there exists an element* $b\in B$ *such that* $a\leq b$.

Example 4.9. (i) *Hyper BI-algebra* $\mathcal{H}$ *in Example 3.13(iii) is not a hyper B-algebra. Because*

$$\{0,b\}=(a\circ b)\circ 0\neq a\circ(0\circ(0\circ b))=\{0,a,b\}.$$

(ii) *Let* $H=\{0,a,b\}$ *and define the hyperoperation "$\circ$" on* H *as follows:*

$\circ$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{0,a\}$	$\{0,a\}$

Then routine calculations show that $(H,\circ,0)$ *is a hyper B-algebra. But it is not a hyper BI-algebra, because* $0\notin 0\circ(a\circ 0)=\{a\}$.

By the above examples we can see that hyper BI-algebras and hyper B-algebras are not same.

Definition 4.10. [5] *An algebraic structure* $(H;\circ,0)$ *with a binary hyper operation "$\circ$" and a constant "0" is called a hyper BCC-algebra if for all* $x,y,z\in H$, *it satisfies the following axioms:*

(HBCC1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,

(HBCC2) $0\circ x=0$,

(HBCC3) $x\circ 0=x$,

(HBCC4) $x\leq y$ *and* $y\leq x$ *imply* $x=y$.

Where $x\leq y$ *if and only if* $0\in x\circ y$. *Also* $A\ll B$ *means that for all* $a\in A$, *there exists an element* $b\in B$ *such that* $a\leq b$.

Theorem 4.11. [5] *In any hyper BCC-algebra* H, *for all* $x\in H$, *we have* $x\leq x$.

Example 4.12. (i) *Let* $\mathcal{H}$ *be the hyper BI-algebra as in Example 3.13(iii). Since* $0\leq a$ *and* $a\leq 0$ *but* $a\neq 0$, *we get* $\mathcal{H}$ *does not satisfy the condition (HBCC4). Hence,* $\mathcal{H}$ *it is not a hyper BCC-algebra.*

(ii) *Let* $H=\{0,a,b,c\}$. *Define the hyperoperation "$\circ$" on* H *as follows:*

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$	$\{b\}$
c	$\{c\}$	$\{a, c\}$	$\{0, a, c\}$	$\{0, a, c\}$

Then $(H;\circ,0)$ *is a hyper BCC-algebra. But it is not a hyper BI-algebra, because* $a\notin a\circ(b\circ a)=\{0\}$.

By the above examples we can see that hyper BI-algebras and hyper BCC-algebras are not same.

Theorem 4.13. *Let $(H; \circ, 0)$ be a hyper (BCK/ BCI/ BCC/ K) B-algebra. If H satisfies the condition $x \in x \circ (y \circ x)$, then it is a hyper BI-algebra.*

Proof. For any hyper (BCK/ BCI/ BCC/ K) B-algebra, we have $x \leq x$, in axioms or properties. So (HBI1) holds in all of them. Hence if these hyper algebras satisfy the condition (HBI2), then these hyper algebraic structures can be a hyper BI-algebra. $\square$

5 Hyper ideals of hyper BI-algebras

In this section, we introduce the notion of (weak) ideals on hyper BI-algebra and investigate the relation between hyper subalgebras and ideals of a hyper BI-algebra. Also, by defining the notion of normal subset, we construct the quotient hyper BI-algebra. Finally, we survey the isomorphism theorems on structures of hyper BI-algebras.

Definition 5.1. *Let I be a non-empty subset of a hyper BI-algebra $\mathcal{H}$ containing 0. Then I is called*

- *a weak ideal if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$;*
- *an ideal if $x \circ y \ll I$ and $y \in I$ imply $x \in I$, for any $x, y \in H$.*

Example 5.2. (i) *Let $(H, \circ, 0)$ be the hyper BI-algebra as in Example 3.13(iii). Then $I = \{0, b\}$ is not a weak ideal of H. Because $a \circ b = \{0, b\} \subseteq I$ and $b \in I$ but $a \notin I$. Also, $I = \{0, a\}$ is a weak ideal of H.*
(ii) *Let H be the hyper BI-algebra as in Example 3.2. Then $I = \{0, a\}$ is an ideal of H.*

Proposition 5.3. *If I is an ideal of $\mathcal{H}$, then I is a weak ideal of $\mathcal{H}$.*

Proof. Let $x \circ y \subseteq I$ and $y \in I$. Then by Proposition 3.4(ii), $x \circ y \ll I$ and $y \in I$. Since I is an ideal of $\mathcal{H}$, we get $x \in I$. Hence I is a weak hyper ideal of $\mathcal{H}$. $\square$

Note: In the following example, we can see that the converse of above proposition is not true, in general.

Example 5.4. *Let $I = \{0, a\}$ be weak ideal of hyper BI-algebra as in Example 3.13(iii). Then it is not an ideal of $\mathcal{H}$, because $(b \circ a)I = H \cap I \neq \emptyset$ and $b \notin I$.*

Proposition 5.5. *Let I be a non-empty subset of H containing 0. Then I is an ideal of $\mathcal{H}$ if and only if $(x \circ y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$.*

Proof. $(\Rightarrow)$ Let I be an ideal of $\mathcal{H}$ and $(x \circ y) \cap I \neq \emptyset$. Then by Proposition 3.4(ii), $x \circ y \ll I$. Since $y \in I$ and I is an ideal of $\mathcal{H}$, we have $x \in I$.
$(\Leftarrow)$ Let $x, y \in H$. If $x \circ y \ll I$ and $y \in I$, then there exists $a \in x \circ y$ and $b \in I$ such that $a \leq b$. Thus $0 \in (a \circ b) \cap I \neq \emptyset$. Since $b \in I$, by assumption, we have $a \in I$. Hence $a \in (x \circ y) \cap I \neq \emptyset$. Again, since $y \in I$, by assumption, we get $x \in I$. Therefore, I is an ideal of $\mathcal{H}$. $\square$

The following example shows that hyper subalgebra and a/an (weak) ideal of a hyper BI-algebra are different notions:

Example 5.6. *(i) Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.13(iii). Then $S = \{0, b\}$ is a hyper subalgebra. Since $a \circ b = \{0, b\} \subseteq S$, $b \in S$ and $a \notin S$, then S is not a weak ideal of $\mathcal{H}$. Consequently, S is not an ideal of $\mathcal{H}$.*
(ii) Let $H = \{0, a, b\}$ and define the hyperoperation "$\circ$" on H as follows:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then $(H; \circ, 0)$ is a hyper BI-algebra and $I = \{0, b\}$ is an ideal of it. But I is not a hyper subalgebra of $\mathcal{H}$ because of $b \circ b = \{0, a\} \nsubseteq I$.

Proposition 5.7. *Let $\mathcal{H}$ be a hyper BI-algebra and S be a hyper subalgebra of $\mathcal{H}$. Then*
(i) S is a weak ideal of $\mathcal{H}$ if and only if for all $x \in H - S$ and $y \in S$, we have $x \circ y \nsubseteq S$.
(ii) S is an ideal of $\mathcal{H}$ if and only if for all $x \in H - S$ and $y \in S$, we get $(x \circ y) \cap S = \emptyset$.

Proof. (i) Let S be a weak ideal of $\mathcal{H}$, $x \in H - S$, and $y \in S$. Suppose $x \circ y \subseteq S$. Since S is a weak hyper ideal of $\mathcal{H}$, we have $x \in S$, which is a contradiction. Thus $x \circ y \nsubseteq S$. Conversely, let $x \circ y \subseteq S$ and $y \in S$. Suppose $x \notin S$. Then by assumption $x \circ y \nsubseteq S$, which is a contradiction. Thus $x \in S$ and so S is a weak ideal of $\mathcal{H}$.
(ii) Let S be an ideal of $\mathcal{H}$, $x \in H - S$ and $y \in S$. Suppose $(x \circ y) \cap S \neq \emptyset$. Since $y \in S$ and S is an ideal of $\mathcal{H}$, we have $x \in S$, which is a contradiction. Thus $(x \circ y) \cap S = \emptyset$.
Conversely, let for any $x \in H - S$ and $y \in S$ we have $(x \circ y) \cap S = \emptyset$. If $(x \circ y) \cap S \neq \emptyset$ and $y \in S$, then $x \in S$, which is a contradiction. Hence, S is an ideal of $\mathcal{H}$. □

Definition 5.8. *A non-empty subset I of H is called a downset if $x \leq y$ and $y \in I$, then $x \in I$, for any $x, y \in H$.*

Proposition 5.9. *If I is an ideal of $\mathcal{H}$, then I is a downset.*

Proof. Let $x \leq y$ and $y \in I$. Then $0 \in (x \circ y) \cap I$. Since $y \in I$ and I is an ideal of $\mathcal{H}$, we get $x \in I$. □

Let $\mathcal{H}$ be a hyper BI-algebra. Then 0 is called *minimal* element of $\mathcal{H}$ if for any $x \in H$, $x \leq 0$, then $x = 0$.

Example 5.10. *(i) Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.2. Then $\mathcal{H}$ has the least element* 0.
(ii) Let $(H, \circ, 0)$ be the BI-algebra as in Example 3.13(iii). Then "0" is not minimal element of H. Because $0 \in a \circ 0$ and so $a \leq 0$ but $a \neq 0$.

Proposition 5.11. *If $\mathcal{H}$ is a hyper BI-algebra with minimal element 0, then $I = \{0\}$ is an ideal of $\mathcal{H}$.*

Proof. Let $y \in I = \{0\}$ and $(x \circ y) \cap I \neq \emptyset$. Then $y = 0$ and $0 \in x \circ 0$. Thus $x \leq 0$ and so $x = 0 \in \{0\}$. Therefore $\{0\}$ is an ideal of $\mathcal{H}$. □

Note: In the following example we show that existing of minimal element is necessary in Proposition 5.11.

Example 5.12. *Let $(H, \circ, 0)$ be the BI-algebra as in Example 3.13(iii). Then $I = \{0\}$ is not an ideal of H. Because $(a \circ 0) \cap I \neq \emptyset$ but $a \notin I$.*

Proposition 5.13. (i) *If $\mathcal{H}$ is a C-hyper BI-algebra, then 0 is minimal element of $\mathcal{H}$.*
(ii) *In any C-hyper BI-algebra, $I = \{0\}$ is an ideal of $\mathcal{H}$.*

Proof. (i) Let $\mathcal{H}$ be a C-hyper BI-algebra and $x \leq 0$. Then $0 \in x \circ 0 = \{x\}$ and so $x = 0$.
(ii) It follows by Proposition 5.11. □

The set of all (weak) ideals of $\mathcal{H}$ is denoted by $(WI(\mathcal{H}))$ $I(\mathcal{H})$. Let for any $i \in \Delta$, I_i is a/an (weak) ideal of $\mathcal{H}$, then $\bigcap\limits_{i\in\Delta} I_i$ is a/an (weak) ideal of $\mathcal{H}$ but this is not true for $\bigcup\limits_{i\in\Delta} I_i$. If $\{I_i : i \in \Delta\}$ is a chain, then $\bigcup\limits_{i\in\Delta} I_i$ is a/an (weak) ideal of $\mathcal{H}$. Also, if I_1 and I_2 are (weak) ideals of hyper BI-algebras H_1 and H_2, respectively, then $I_1 \times I_2$ is a/an (weak) ideal of $H_1 \times H_2$.

Definition 5.14. *Let $x, y \in H$. Define $A(x, y) := \{t \in H |\ 0 \in (t \circ x) \circ y\}$.*

Example 5.15. *Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.2. Then $A(a, b) = \{0, a, b, c\}$ and $A(b, a) = \{0, a, b\}$. So $A(a, b) \neq A(b, a)$.*

Proposition 5.16. *Let $x, y \in H$. If $0 \leq x$, then $0, x \in A(x, y)$.*

Proof. Since $0 \in 0 \circ x \subseteq (0 \circ x) \circ y$, we have $0 \in A(x, y)$. Also, $0 \in 0 \circ y \subseteq (x \circ x) \circ y$, then $x \in A(x, y)$. □

Proposition 5.17. *Let $\mathcal{H}$ be a C-hyper BI-algebra. Then for all $x \in H$ and a non-empty subset $B \subseteq H$, the following statements hold:*
(i) *$t \in A(0, x)$ if and only if $t \leq x$. Particularly, $x \in A(0, x)$ and $H = \bigcup\limits_{x\in H} A(0, x)$.*
(ii) *$B \subseteq \bigcup\limits_{b\in B} A(0, b)$.*
(iii) *$A(x, 0) = A(0, x)$.*

Proof. (i) Let $x \in H$. By Definition 5.14 we have,

$$t \in A(0, x) \quad \Leftrightarrow \quad 0 \in (t \circ 0) \circ x \quad \Leftrightarrow \quad 0 \in t \circ x \quad \Leftrightarrow \quad t \leq x.$$

Particularly, by (HBI1), $x \leq x$ and so $x \in A(0, x)$, for all $x \in H$. Therefore, $H = \bigcup\limits_{x\in H} A(0, x)$.
(ii) It is straightforward by (i).
(iii) We note that

$$A(x, 0) = \{t \in H |\ 0 \in (t \circ x) \circ 0\} = \{t \in H |\ 0 \in t \circ x\} = \{t \in H |\ 0 \in (t \circ 0) \circ x\} = A(0, x).$$

□

Proposition 5.18. *Let $\mathcal{H}$ be a C-hyper BI-algebra such that $0 \leq x$ for all $x \in H$. Then for all $x, y \in H$, we have*
(i) *$A(0, x) \subseteq A(x, y)$,*
(ii) *$A(0, x) = \bigcup\limits_{x,y\in H} A(x, y)$.*

Proof. (i) Let $z \in A(0, x)$. Then $0 \in (z \circ 0) \circ x$. Since $\mathcal{H}$ is a C-hyper BI-algebra, we get $0 \in z \circ x$. On the other hand, $0 \in 0 \circ y \subseteq (z \circ x) \circ y$. Thus by definition, $z \in A(x, y)$ and so $A(0, x) \subseteq A(x, y)$.
(ii) The proof is clear. □

Theorem 5.19. *Let I be a non-empty subset of H. Then I is an ideal of $\mathcal{H}$ if and only if $A(x, y) \subseteq I$ for all $x, y \in I$.*

Proof. Let I be an ideal of $\mathcal{H}$ and $x, y \in I$. If $z \in A(x,y)$, then $0 \in (z \circ x) \circ y$ and so $((z \circ x) \circ y) \cap I \neq \emptyset$. Since I is an ideal of $\mathcal{H}$ and $y \in I$, we get $(z \circ x) \cap I \neq \emptyset$. Moreover, since $x \in I$ and I is an ideal of $\mathcal{H}$, we have $z \in I$. Therefore, $A(x,y) \subseteq I$.
Conversely, let $A(x,y) \subseteq I$ for all $x, y \in I$. Since $0 \in A(x,y) \subseteq I$ we have $0 \in I$. Now, if $(a \circ b) \cap I \neq \emptyset$ and $b \in I$, then there exists $c \in (a \circ b) \cap I$ such that $0 \in c \circ c \subseteq (a \circ b) \circ c$. So $a \in A(b,c) \subseteq I$ i.e. $a \in I$. Therefore, I is an ideal of $\mathcal{H}$. □

Theorem 5.20. *If I is an ideal of C-hyper BI-algebra $\mathcal{H}$, then $I = \bigcup\limits_{x \in I} A(0,x)$.*

Proof. By Proposition 5.17(ii), we know that $I \subseteq \bigcup\limits_{x \in I} A(0,x)$. On the other hand, if $t \in \bigcup_{x \in I} A(0,x)$, then there exists $a \in I$ such that $t \in A(0,a)$. By Proposition 5.17(i), $t \leq a$. Since $a \in I$ and I is an ideal of $\mathcal{H}$, we get I is downset. Hence, $t \in I$. Therefore, $\bigcup\limits_{x \in I} A(0,x) \subseteq I$. □

Definition 5.21. *Let $N \subseteq H$ such that $0 \in N$. Then N is called a normal subset if for any $x, y, a, b \in H$, $(x \circ y) \cap N \neq \emptyset$ and $(a \circ b) \cap N \neq \emptyset$ imply $((x \circ a) \circ (y \circ b)) \cap N \neq \emptyset$.*

Example 5.22. *Let $\mathcal{H}$ be the hyper BI-algebra as in Example 3.13(iii). Then $N = \{0, a\}$ is a normal subset of $\mathcal{H}$. But it is not an ideal of $\mathcal{H}$, because $(b \circ a) \cap N \neq \emptyset$ and $b \notin N$.*

Example 5.23. *Let $H = \{0, a, b, c\}$ and define the hyper operation "$\circ$" on H as follows:*

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0,a\}$	$\{a,b\}$
b	$\{b\}$	$\{0,b\}$	$\{0\}$	$\{b\}$
c	$\{c\}$	$\{a,b\}$	$\{c\}$	$\{0\}$

Then $(H; \circ, 0)$ is a V-hyper BI-algebra. Obviously, $I = \{0, c\}$ is an ideal of $\mathcal{H}$. But it is not a normal subset, because $(b \circ b) \cap I \neq \emptyset$, $(c \circ b) \cap I \neq \emptyset$ and $((b \circ c) \circ (b \circ b)) \cap I = \emptyset$.

Definition 5.24. *Let N be a normal subset of $\mathcal{H}$. Then for any $x, y \in H$, define the relation $\equiv_N$ on $\mathcal{H}$ as follows:*

$$x \equiv_N y \quad \Leftrightarrow \quad (x \circ y) \cap N \neq \emptyset \quad \textit{and} \quad (y \circ x) \cap N \neq \emptyset.$$

Proposition 5.25. *Let N be a normal subset of $\mathcal{H}$. Then the relation $\equiv_N$ is reflexive and symmetric.*

Proof. Let $x, y \in H$. Since $0 \in (x \circ x) \cap N \neq \emptyset$, we get $x \equiv_N x$ and the relation $\equiv_N$ is reflexive. The proof of symmetric is clear. □

Proposition 5.26. *Let $\mathcal{H}$ be a C-D-hyper BI-algebra and N be a normal subset of $\mathcal{H}$. Then for any $x, y \in H$, we have $(x \circ y) \cap N \neq \emptyset$ if and only if $(y \circ x) \cap N \neq \emptyset$.*

Proof. Let N be a normal subset of $\mathcal{H}$ and for $x, y \in H$, $(x \circ y) \cap N \neq \emptyset$. Since H is a C-D hyper BI-algebra, $y \circ x = (y \circ x) \circ 0$ and $y \circ y = \{0\}$, respectively. Since $0 \in (y \circ y) \cap N \neq \emptyset$ and N is normal, we get

$$(y \circ x) \cap N = ((y \circ x) \circ 0) \cap N = ((y \circ x) \circ (y \circ y)) \cap N \neq \emptyset.$$

Hence $(y \circ x) \cap N \neq \emptyset$. By the similar way, if $(y \circ x) \cap N \neq \emptyset$, then $(x \circ y) \cap N \neq \emptyset$. □

Corollary 5.27. *Let N be a normal subset of a C-D-hyper BI-algebra such as $\mathcal{H}$. Then for any $x, y \in H$, $x \equiv_N y$ if and only if $(x \circ y) \cap N \neq \emptyset$.*

Proof. Since N is a normal subset of $\mathcal{H}$ and $x, y \in H$, we have $(x \circ y) \cap N \neq \emptyset$. Conversely, if $(x \circ y) \cap N \neq \emptyset$, then by Proposition 5.26, $(y \circ x) \cap N \neq \emptyset$. Hence $x \equiv_N y$. □

Theorem 5.28. *Let $\mathcal{H}$ be a C-D-hyper BI-algebra. If N is a normal subset of $\mathcal{H}$, then $\equiv_N$ is a congruence relation on $\mathcal{H}$.*

Proof. Let $\mathcal{H}$ be a C-D-hyper BI-algebra, N be a normal subset of $\mathcal{H}$ and $x, y, z \in H$. If $x \equiv_N y$ and $y \equiv_N z$, then $(x \circ y) \cap N \neq \emptyset$ and $(z \circ y) \cap N \neq \emptyset$, respectively. Since H is a C-Diagonal hyper BI-algebra, $x \circ z = (x \circ z) \circ 0$ and $y \circ y = \{0\}$, respectively. Since N is normal, we get

$$(x \circ z) \cap N = ((x \circ z) \circ 0) \cap N = ((x \circ z) \circ (y \circ y)) \cap N \neq \emptyset.$$

Then $(x \circ z) \cap N \neq \emptyset$ and so by Corollary 5.27, $x \equiv_N z$. Hence $\equiv_N$ is an equivalence relation on H. Also, if $x \equiv_N y$ and $a \equiv_N b$, then $(x \circ y) \cap N \neq \emptyset$ and $(a \circ b) \cap N \neq \emptyset$, respectively. So by definition of normal subset, we get $((x \circ a) \circ (y \circ b)) \cap N \neq \emptyset$. Thus by Corollary 5.27, $(x \circ a) \equiv_N (y \circ b)$. Therefore, the relation $\equiv_N$ is a congruence on $\mathcal{H}$. □

Note. From now on, we suppose $\mathcal{H}$ is a C-D-hyper BI-algebra and N is a normal subset of $\mathcal{H}$ unless otherwise state.

Given a normal subset N of hyper BI-algebra $\mathcal{H}$, we denote the equivalence class of the relation $\equiv_N$ containing x, by $[x]$ and the quotient set of $\mathcal{H}$ by $\dfrac{H}{N}$, where

$$[x] = \{y \in H |\ y \equiv_N x\}, \quad \frac{H}{N} = \{[x] |\ x \in H\}.$$

Theorem 5.29. *For any $x, y \in H$ define the operation $\star$ on $\dfrac{H}{N}$ as follows,*

$$[x] \star [y] := \bigcup_{t \in x \circ y} [t].$$

Then $(\dfrac{H}{N}; \star, [0])$ is a hyper BI-algebra which is called quotient algebra of $\mathcal{H}$ induced by N, where $[x] \leq' [y]$ if and only if $[0] \in [x] \star [y]$.

Proof. Let $x, y \in H$. Since $\mathcal{H}$ is a D-hyper BI-algebra, we have $x \circ x = \{0\}$ and so $[x] \star [x] = [x \circ x] = [0]$. Thus $[x] \leq' [x]$ and (HBI1) holds. Also, we have

$$[x] \star ([y] \star [x]) = \bigcup_{t \in x \circ (y \circ x)} [t].$$

By (HBI2), $x \in x \circ (y \circ x)$, and so $[x] \in [x] \star ([y] \star [x])$. Hence (HBI2) holds on $\dfrac{H}{N}$. Therefore, $(\dfrac{H}{N}; \star, [0])$ is a hyper BI-algebra. □

Proposition 5.30. *If $\pi : \mathcal{H} \to \dfrac{H}{N}$ is a natural homomorphism such that $\pi(x) := [x]$, then ϕ is an onto hyper BI-homomorphism, where $Ker\pi = N$.*

Proof. Let $x, y \in H$. Clearly, π is onto and $\pi(0) = [0]$. Since

$$\pi(x \circ y) = \pi(\bigcup_{t \in x \circ y} t) = \bigcup_{t \in x \circ y} \pi(t) = \bigcup_{t \in x \circ y} [t] = [x] \star [y],$$

we get π is an onto hyper BI-homomorphism. In addition,

$$\begin{aligned} Ker\pi &= \{x \in H |\ \pi(x) = [0]\} = \{x \in H |\ [x] = [0]\} \\ &= \{x \in H |\ x \equiv_N 0\} = \{x \in H |\ (x \circ 0) \cap N \neq \emptyset\} \\ &= \{x \in H |\ x \in N\} \\ &= N. \end{aligned}$$

Therefore, $Ker\pi = N$. □

Theorem 5.31. *Let $f : \mathcal{H} \to \mathcal{H}'$ be a hyper BI-homomorphism between two hyper BI-algebras, where N and N' are normal subsets of H and H', respectively. If $f(N) \subseteq N'$, then there exists a unique hyper BI-homomorphism $g : \frac{H}{N} \to \frac{H'}{N'}$ such that $go\pi = \pi' of$.*

Proof. Define $g : \frac{H}{N} \to \frac{H'}{N'}$ as $[x]_N \rightarrowtail [f(x)]_{N'}$. First we show that g is well-defined:
Let $x, y \in H$ and $[x] = [y]$. Then $(x \circ y) \cap N \neq \emptyset$ and so there exists $t \in N$ such that $t \in x \circ y$. Thus $f(t) \in f(N) \subseteq N'$ and $f(t) \in f(x \circ y) = f(x) \circ' f(y)$. Hence, $(f(x) \circ f(y)) \cap N' \neq \emptyset$. Therefore, $[f(x)]_{N'} = [f(y)]_{N'}$. Now, we show g is a hyper BI-homomorphism: $g([0]) = [f(0)] = [0']$. If $[t'] \in g([x] \star [y])$, then

$$[t'] \in g(\bigcup_{t \in x \circ y} [t]) = \{g([t]) |\ t \in x \circ y\} = \{[f(t)] |\ t \in x \circ y\}.$$

So there exists $t \in x \circ y$ such that $[t'] = [f(t)]$. Since $t \in x \circ y$, we have $f(t) \in f(x \circ y) = f(x) \circ' f(y)$ and so

$$[t'] = [f(t)] \in \{[z] |\ z \in f(x) \circ' f(y)\} = [f(x)] \star' [f(y)] = g([x]) \star' g([y]).$$

Therefore, $g([x] \star [y]) \subseteq g([x]) \star' g([y])$. By the similar way, we obtain $g([x]) \star' g([y]) \subseteq g([x] \star [y])$. Obviously, g is unique. □

6 Conclusion and future work

In this paper, a new hyper algebra was introduced which is a generalization of BI-algebras. Basic properties, hyper subalgebras, BI-homomorphism and some special types of hyper BI-algebras were discussed. Also, relationship between this new hyperstructure and some the other hyper algebras were investigated. Finally, isomorphism theorems were investigated on the constructed quotient structure of hyper BI-algebras. As future works, we shall define commutative hyper BI-algebras and some types of ideals in hyper BI-algebras.

References

[1] J.C. Abbott, *Semi-Boolean algebras*, Matematićki Vesnik, 4 (1967), 177–198.

[2] S.S. Ahn, J.S. Han, *On BP-algebras*, Hacettepe Journal of Mathematics and Statistics, 42 (2013), 551–557.

[3] A. Borumand Saeid, H.S. Kim, A. Rezaei, *On BI-algebras*, Analele Stiintifice ale Universitatii Ovidius Constanta, 25(1) (2017), 177–194.

[4] R.A. Borzooei, M. Aaly Kologani, *An overview of hyper logical algebras*, Journal of Algebraic Hyperstructures and Logical Algebras, 1(3) (2020), 31–50.

[5] R.A. Borzooei, W.A. Dudek, N. Kouhestani, *On hyper BCC-algebras*, International Journal of Mathematics and Mathematical Sciences, (2006), 1–18.

[6] R.A. Borzooei, A. Hasankhani, M.M. Zahedi, Y.B. Jun, *On hyper K-algebras*, Japanese Journal of Mathematics, 52(1) (2000), 113–121.

[7] W.Y. Chen, J.S. Oliveira, *Implication algebras and the metropolis rota axioms for cubic lattices*, Journal of Algebra, 171 (1995), 383–396.

[8] P. Corsini, *Prolegomena of hypergroup theory*, (Second Edition), Aviani Editor, 1993.

[9] W.A. Dudek, *On BCC-algebras*, Logique et Analyse. Nouvelle Series, 33(129-130) (1990), 103–111.

[10] J.C. Endam, *On JB-Semigroups and hyper B-Algebras*, Dissertation, MSU-IIT, 2014.

[11] K. Iséki, *On BCI-algebras*, Mathematics Seminar Notes, 8 (1980), 125–130.

[12] Y.B. Jun, M.M. Zahedi, X.L. Xin, R.A. Borzooei, *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematics, 8 (2000), 127–136.

[13] C.B. Kim, H.S. Kim, *On BM-algebras*, Scientiae Mathematicae Japonicae, 63(3) (2006), 421–427.

[14] C.B. Kim, H.S. Kim, *On BG-algebras*, Demonstratio Mathematica, 41 (2008), 497–505.

[15] C.B. Kim, H.S. Kim, *On BO-algebras*, Mathematica Slovaca, 62 (2012), 855–864.

[16] C.B. Kim, H.S. Kim, *On BN-algebras*, Kyungpook Mathematical Journal, 53 (2013), 175–184.

[17] Y. Komori, *The variety generated by BCC-algebras is finitely based*, Reports of Faculty of Science, Shizuoka University, 17 (1983), 13–16.

[18] Y. Komori, *The class of BCC-algebras is not a variety*, Mathematica Japonica, 29(3) (1984), 391–394.

[19] F. Marty, *Sur une generalization de la notion de group*, 8th Congress Math. Scandinaves, Stockholm, (1934), 45–49.

[20] J. Neggers, H.S. Kim, *On D-algebras*, Mathematica Slovaca, 49 (1999), 19–26.

[21] J. Neggers, H.S. Kim, *On B-algebras*, Matematiki Vesnik, 54 (2002), 21–29.

[22] J. Neggers, H.S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, International Journal of Mathematics, 2 (2002), 215–219.

[23] A. Rezaei, F. Smarandache *The neutrosophic triplet of BI-algebras*, Neutrosophic Sets and Systems, 33 (2020), 313–321.

[24] A.L.O. Vicedo, J.P. Vilela, *Hyper B-ideals in hyper B-algebra*, Journal of Ultra Scientist of Physical Sciences, 29(8) (2017), 339–347.

[25] A. Walendziak, *On BF-algebras*, Mathematica Slovaca, 57 (2007), 119–128.

[26] A. Wronski, *BCK-algebras do not form a variety*, Mathematica Japonica, 28 (1983), 211–213.

[27] X.L. Xin, *Hyper BCI-algebras*, Discuss Mathematics Society, 26 (2006), 5–19.