Tense and dynamic algebras related to **GFA**

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Abstract

The present paper is an attempt to suggest and scrutinize tense operators in the dynamic logic **B** which is regarded as a set of propositions about the general fuzzy automaton $\tilde{F}$, in which its underlying structure has been a bounded poset. Here, the operators $T_\delta, P_\delta, H_\delta$ and F_δ are proposed regardless of what propositional connectives the logic comprises. For this purpose, the axiomatization of universal quantifiers is applied as a starting point and these axioms are modified. In this study, firstly, we demonstrate that the operators can be identified as modal operators and the pairs (T_δ, P_δ) are examined as the so-called dynamic pairs. In addition, constructions of these operators are attained in the corresponding algebra and in the following a transition frame is suggested. Besides, the problem of finding a transition frame is solved in the case when the tense operators are given. Specifically, this study shows that the tense algebra **B** is representable in its Dedekind-MacNeille completion. Representation theorems for dynamic and tense algebra are explicated in details in the related given theorems.

1 Introduction

The ever increasing concern to apply fuzzy logic in different areas and for different purposes has emphasized that the construction of fuzzy automata be more developed and better characterized to accomplish implementational conditions in a well-established manner. This requirement is because despite lots of research being done on fuzzy automata and also long history of its application in different fields, there are still some areas which have not been well-defined and issues which need

much more modification.

The notion of general fuzzy automata (GFA) was suggested by M. Doostfatemeh and S.C. Kremer [15] in order to underline the inadequacy of the existing literature and to manage the applications which depend on fuzzy automata as a modeling tool. Further, Abolpour and Zahedi [2, 1, 3] extended this concept as BL-general fuzzy automata, and explicated it based on the complete residuated lattice-valued (L-GFAs). In addition, bisimulation for BL-general fuzzy automata was defined and explained in more details by Shamsizadeh, Zahedi, and Abolpour [20, 21]. They investigated the minimal BL-general fuzzy automata on the basis of Myhill-Nerode's theory and introduced derivatives of the given fuzzy behavior [4].

In their studies, they also demonstrated that for every given GFA, which is regarded as an acceptor, a definite dynamic logic **B** can be assigned. The dynamic nature of a general fuzzy automaton is stated through its transition relation which is labeled by inputs. The logic comprises propositions on the given GFA and its dynamic nature is stated through the so-called transition functor. On the other hand, this logic makes them capable of driving again a definite relation on the set of states labeled by inputs. Indeed, they have demonstrated that if their set of propositions is large enough, this recovering of the transition relation is possible. On the contrary, through holding a set **B** of propositions that explains the behavior of their intended GFA and the transition functor, as stating the dynamicity of this process along with the set Σ of inputs, they suggested a construction of a set of states Q and the state-transition $\triangle$ on Q. Such constructed general fuzzy automaton $(\Sigma, Q, \triangle)$ identifies the description given by the propositions [5].

It is said that propositional logic, whether classic or non-classic; does not include the dimension of time. To attain a tense logic, we will enrich the given propositional logic through the use of new unary operators which are generally indicated by G, H, F and P (for more examples see [7], [13], [14] and [9]).The aforementioned G, H, F and P are usually called tense operators. They are in certain sense which quantify over the time dimension of the logic under consideration. However, these tense operators can be given axiomatically and this rises a natural question on the existence of a time frame for which these operators can be derived as mentioned above. This is called a representation problem. It is well known that this problem is easily solvable for Boolean algebras but is not solvable, for example, for MV-algebras or effect algebras in general. Hence, we are encouraged to find certain restrictions that are still in accordance with physical reality. These restrictions will enable us to solve the restricted problem for those algebras that are useful for axiomatization of many-valued and/ or quantum logics.

It is important to say that the operators G and H can be regarded as specific types of modal operators which are already examined for intuitionistic calculus by Wijesekera [21], and also in the De Morgan framework by cattaneo, Ciucci and Dubois [8]. This issue is also studied it is studied in a general setting by Ewald [17]; for the logic of quantum mechanics (also see [16] for the details on the so-called quantum structures). For instance, the underling algebraic structure is an orthomodular lattice or the so called effect algebra (for more examples see [16], [18] and the corresponding tense logic which is studied in [10, 11, 19], and in a bit more general setting in [6] as well.

The abovementioned operators G, H, F and P are generally tense operators. In particular sense, they are quantifiers which quantify over the time dimension of the logic under consideration. The semantical interpretation of these tense operators G and H is specified in the following: Consider a pair $(T, \leq)$ where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of a given logical calculus. We declare that $G(f(t))$ is valid if for any $s \geq t$ the formula $f(s)$ is valid. Analogously, $H(f(t))$ is valid if $f(s)$ is valid for each $s \leq t$. Therefore, the unary operators G and H constitute an algebraic counterpart of the tense operations "it is always going

to be the case that" and "it has always been the case that", respectively. These tense operators were first suggested as operators on Boolean algebras (see also [7] for an overview). In the current study, after proposing several necessary algebraic concepts and the notion of general fuzzy automaton, we present tense operators in the dynamic logic **B** with a set of propositions about GFA regardless of what propositional connectives the logic comprises. In addition, we demonstrate a simple construction of tense operators which makes use of lattice theoretical properties of the underlying ordered set **B**. When the underlying ordered set is not a complete lattice, we prove how to use the lattice completion for this construction. This study also tries to solve the problem of a representation of dynamic or tense algebra **B**. In other words, we suggest a procedure how to construct a corresponding transition frame to be in line with the proposed construction. Particularly, we explain that the dynamic or tense algebra **B** is representable in its Dedekind-MacNeille completion.

2   Preliminaries

The basic underlying definitions and theorems used for the elucidation of the proposed concepts in this study are presented in details in this section.

Definition 2.1. [15] *A general fuzzy automaton (GFA) is considered as:*

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2),$$

where (i) $Q = \{q_1, q_2, \ldots, q_n\}$ is a finite set of states, (ii) $\Sigma = \{a_1, a_2, \ldots, a_m\}$ is a finite set of input symbols, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) $Z = \{b_1, b_2, \ldots, b_k\}$ is a finite set of output symbols, (v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function. (vii) Function $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition.
With this definition, the process that occurs upon the transition from state q_i to q_j on input a_k is characterized by:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

It means that membership value (mv) of the state q_j at time $t + 1$ is calculated by function F_1, utilizing both the membership value of q_i at time t and the weight of the transition.
There have been many options for the function $F_1(\mu, \delta)$. For instance, it can be $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$, $\dfrac{\mu + \delta}{2}$, or any other pertinent mathematical functions.
As it can be observed in the above mentioned definition, associated with each fuzzy transition, there exists a membership value (mv) in unit interval $[0, 1]$. We identify this membership value the weight of the transition. The transition from state q_i (current state) to state q_j (next state) upon input a_k is designated as $\delta(q_i, a_k, q_j)$. Hereafter, we apply this notation to refer both to a transition and its weight.
Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition. Otherwise, it identifies the transition itself. The set of all transitions of a general fuzzy automaton $\tilde{F}$ is denoted as $\Delta_{\tilde{F}}$. However, whenever it is understood, we remove the subscript, and write simply Δ.
Concerning this, we say that Δ is a state-transition relation and it is regarded as a dynamics of $\tilde{F}$. On the other hand, we regularly formulate certain propositions on an automaton $\tilde{F}$ and draw conclusions from the behavior of $\tilde{F}$ in the present or in the future.

(viii) $F_2 : [0,1]^* \to [0,1]$ *is called multi-membership resolution function. The multi-membership resolution function determines the multi-membership active states and allocates a single membership value to them.*
We let $Q_{act}(t_i)$ be the set of all active state at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) | \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\},$$

$\forall i \geq 1$. *Since $Q_{act}(t_i)$ is a fuzzy set, to demonstrate that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subseteq Domain(Q_{act}(t_i))$; henceforth, we simply specify them by: $q \in Q_{act}(t_i)$ and $T \subseteq Q_{act}(t_i)$.*

Definition 2.2. [12] *Let S be a non-empty set. Every subset $R \subseteq S \times S$ is called a relation on S and we declare that the couple (S, R) is a transition frame. The converse of R means the relation $R^{op} = \{(x, y) \in S \times S | (y, x) \in R\}$.*

Definition 2.3. [12] *A mapping f is called order-preserving or monotone if $a, b \in A$ and $a \leq b$ imply $f(a) \leq f(b)$ and order-reflecting if $a, b \in A$ and $f(a) \leq f(b)$ imply $a \leq b$. An objective order-preserving and order-reflecting mapping $f : A \to B$ is called an isomorphism and then we state that the partially ordered sets $(A; \leq)$ and $(B; \leq)$ are isomorphic.*

Definition 2.4. [12] *If a partially ordered set $\mathbf{A}$ has both a bottom and a top element, then it will be called bounded; the pertinent notation for a bounded partially ordered set is $(A; \leq, 0, 1)$. Let $(A; \leq, 0, 1)$ and $(B; \leq, 0, 1)$ be bounded partially ordered sets. A morphism $f : A \to B$ of bounded partially ordered sets is an order, top element and bottom element preserving map.*

Observation 2.5. [12] *Let $\mathbf{A}$ and $\mathbf{M}$ be bounded partially ordered sets, S be an arbitrary index set, and $h_s : A \to M$, $s \in S$, morphisms of bounded partially ordered sets. Then, the following conditions are equivalent:*
(i) $\forall s \in S$ $h_s(a) \leq h_s(b) \Rightarrow a \leq b$ for any elements $a, b \in A$;
(ii) The map $i_A^S : A \to M^S$ defined by $i_A^S(a) = (h_s(a))_{s \in S}$ for all $a \in A$ is order-reflecting.
Then, we declare that $\{h_s : A \to M; s \in S\}$ is a full set of order-preserving mappings concerning M. Note that in this situation, we may specify $\mathbf{A}$ with a bounded subposets of $\mathbf{M}^S$ because i_A^S is an order reflecting morphism alias embedding of bounded partially ordered sets. For any $s \in S$ and any $p = (p_s)_{s \in S}$, we indicate it by $p(s)$ the s-th projection p_s. Note that $i_A^S(a)(s) = h_s(a)$ for all $a \in A$ and all $s \in S$.

Definition 2.6. [9] *Consider an algebra $\mathcal{A} = (A; \leq, 0, 1)$. A couple (P, G) of partial mappings $P, G : A \to A$ is called dynamic pair on $\mathcal{A}$ if the following conditions hold:*
(P1) $G(0) = 0, G(1) = 1, P(0) = 0$ and $P(1) = 1$,
(P2) $x \leq y$ implies $G(x) \leq G(y)$, whenever $G(x), G(y)$ exist and $P(x) \leq P(y)$, whenever $P(x), P(y)$ exist,
(P3) $x \leq GP(x)$, where $P(x)$ and $GP(x)$ exist, $PG(y) \leq y$, where $G(y)$ and $PG(y)$ exist.
The operator P is called a weak dynamic operator and the operator G is called a strong dynamic operator (shortly dynamic operators). The triple $(\mathcal{A}; G, P)$ is called a partial dynamic algebra. If both G and P are total mappings, then we speak about a dynamic algebra.
We say that a partial map $G : A \to A$ is contractive (transitive) if $G(x) \leq x$ ($G(x) \leq GG(x)$) for all $x \in A$ such that $G(x)$ is defined (for all $x \in A$ such that $G(x)$ and $GG(x)$ are defined). A partial map G that is both contractive and transitive is called a conucleus. A dynamic pair (P, G) is called modal if G is a conucleus, then we speak about a partial modal algebra. If both G and P are total mappings, we speak about a modal algebra.

Definition 2.7. [9] *If* $(\mathcal{A}_1; G_1, P_1)$ *and* $(\mathcal{A}_2; G_2, P_2)$ *are partial dynamic algebras, then a morphism of partial dynamic algebras* $f : (\mathcal{A}_1; G_1, P_1) \to (\mathcal{A}_2; G_2, P_2)$ *is a morphism of bounded posets such that* $f(G_1(x)) = G_2(f(x))$, *and for any* $x \in A_1$ *such* $G_1(x)$ *is defined and* $G_2(f(x))$ *exists and* $f(P_1(y)) = P_2(f(y))$, *for any* $y \in A_1$ *such* $P_1(y)$ *is defined and* $P_2(f(y))$ *exists.*

Remark 2.8. [9] *If* $(A; \leq)$ *is an ordered set, then there always exists a lattice* $\mathcal{L} = (L; \vee, \wedge)$ *with the induced order* $\leq$ *such that* $A \subseteq L$ *and* $x \leq y$ *in* $(A; \leq)$ *implies* $x \leq y$ *in* $\mathcal{L}$. *One of possible constructions of* $\mathcal{L}$ *for a given ordered set* $(A; \leq)$ *is the use of so-called cuts. Then,* $\mathcal{L}$ *is called the DedeKind-MacNeille completion of* $(A; \leq)$.
Due to these reasons, it is an important fact that the DedeKind-MacNeille completion of any ordered set is a complete lattice; in particular, for each subset $S \subseteq A$, *there exists* $\vee S = sup S$ *and* $\wedge S = inf S$ *in* $\mathcal{L}$.

3   The construction of dynamic algebra on GFA

This section is an attempt to derive the logic **B**, which is a set of propositions about the general fuzzy automaton $\tilde{F}$ formulated by the observer, and to construct an ordered algebra structure on B. If we fix an input $a_k \in \Sigma$ at time t_i, then the proposition $\alpha|_{a_k}$ can be computed by $\mu^{t_i}(q_i)$ if the general fuzzy automaton $\tilde{F}$ is in the state q_i at time t_i, otherwise $\alpha|_{a_k}$ is 0 if $\tilde{F}$ is not in the active state q_i.

Therefore, for each state $q_i \in Q$ we can examine the truth value of $\alpha|_{a_k}$, and it is indicated by $\alpha|_{a_k}(q_i)$. As explained for above, $\alpha|_{a_k}(q_i) \in [0,1]$, we can establish the order $\leq$ on B as follows: for; $\alpha, \beta \in B$, $\alpha \leq \beta$ if and only if $\alpha(q_i) \leq \beta(q_i)$ for all $q_i \in Q$. One can instantly check that the contradiction, i.e., the proposition with the constant truth value 0, is the least element and the tautology, i.e., the proposition with the constant truth value 1 is the greatest component of the partially order set $(B; \leq)$. Note that any component ith of 1 is the maximum membership values of active states at time t_i, for any $i \geq 0$. This fact will be stated by the notation $\mathbf{B} = (B; \leq, 0, 1)$ for the bounded partially order set of proposition about the general fuzzy automaton $\tilde{F}$. Every automaton $\tilde{F}$ will be identified with the triple (B, Σ, Q). Where B is the set of propositions about $\tilde{F}$, Σ is the set of possible inputs and Q is the set of states on $\tilde{F}$. In what follows, the truth-values of our logic **B** will be considered to be from the complete lattice $\mathbf{M} = ([0,1]; \leq, 0, 1)$. Thus, **B** will be bounded subposet of $\mathbf{M}^Q$ for the complete lattice **M** of truth-values.

Definition 3.1. *As in the following, let* $\boldsymbol{M} = ([0,1]; \leq, 0, 1)$ *be a bounded partially ordered set and the bounded subposets* $\boldsymbol{A} = (A; \leq, 0, 1)$ *and* $\boldsymbol{B} = (B; \leq, 0, 1)$ *of* $\boldsymbol{M}^Q$ *will stand for the possibly different logics of propositions pertaining to our automaton* $\tilde{F}$, *a corresponding set of states* Q, *and a state-transition relation* $\triangle$ *on* Q. *Define mappings* $P_\delta : A \to (M^Q)^\Sigma$ *and* $T_\delta : B \to (M^Q)^\Sigma$ *as follows: for all* $b \in B, q_m \in Q$ *and* $a_k \in \Sigma$,

$$T_{\delta_{a_k}}(b)(q_m) = \bigwedge \{b(q_j) | q_j \in Q_{succ}(q_m, a_k)\}, \qquad (*)$$

where

$$Q_{succ}(q_m, a_k) = \{q_j | \delta(q_m, a_k, q_j) \in \triangle\},$$

and for all $a \in A$,

$$P_{\delta_{a_k}}(a)(q_m) = \bigvee \{a(q_j) | q_j \in Q_{pred}(q_m, a_k)\}, \qquad (**)$$

where

$$Q_{pred}(q_m, a_k) = \{q_j | \delta(q_j, a_k, q_m) \in \triangle\}.$$

Then, we state that $T_\delta(P_\delta)$ is an upper transition functor (lower transition functor) constructed through the transition frame $(Q, \triangle)$, respectively. We signify that T_δ is an order-preserving map such that $T_\delta(1) = 1, T_\delta(0) = 0$ and correspondingly, P_δ is an order-preserving map such that $P_\delta(0) = 0$ and $P_\delta(1) = 1$.

Example 3.2. *Consider the GFA in Figure 1, it is specified as:* $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, *where* $Q = \{q_0, q_1, q_2\}$ *is the set of states,* $\Sigma = \{a, b\}$ *is the set of input symbols,* $\tilde{R} = \{(q_0, 1)\}$, $Z = \emptyset$ *and* ω *is not applicable.*
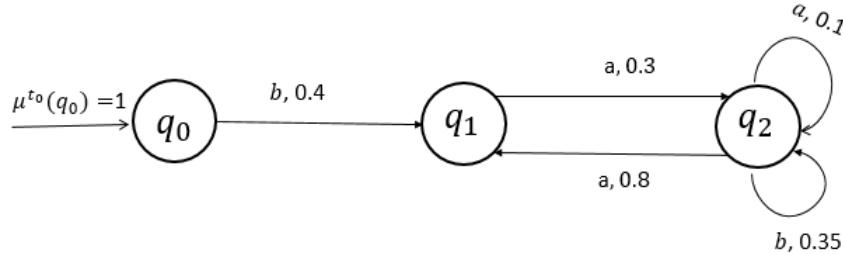

Figure 1: The GFA of Example 3.2

If we choose $F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \bigwedge\limits_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$, then we have:

$$
\begin{aligned}
\mu^{t_0}(q_0) &= 1 \\
\mu^{t_1}(q_1) &= F_1(\mu^{t_0}(q_0), \delta(q_0, b, q_1)) = \delta(q_0, b, q_1) = 0.4, \\
\mu^{t_2}(q_2) &= F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = \delta(q_1, a, q_2) = 0.3, \\
\mu^{t_3}(q_1) &= F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_1)) = \delta(q_2, a, q_1) = 0.8, \\
\mu^{t_3}(q_2) &= F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_2)) = \delta(q_2, a, q_2) = 0.1, \\
\mu^{t_4}(q_2) &= F_1(\mu^{t_3}(q_2), \delta(q_2, b, q_2)) = \delta(q_2, b, q_2) = 0.35.
\end{aligned}
$$

Table 1: Active states and their membership values (mv) at different times in Example 3.2 upon input string $"ba^2b"$

time	t_0	t_1	t_2	t_3		t_4
input	$\wedge$	b	a	a		b
$Q_{act}(t_i)$	q_0	q_1	q_2	q_1	q_2	q_2
mv	1	0.4	0.3	0.8	0.1	0.35

The set $B = \{0, s_0, s_1, s_2, s_0', s_1', s_2', 1\}$ of possible propositions B about the automaton $\tilde{F}$ is as follows:

-0   means that the GFA is not in active states of Q,

-s_0 means that the GFA is in active state q_0,

-s_1 means that the GFA is in active state q_1,

-s_2 means that the GFA is in active state q_2,

-s_0' means that the GFA is either in active state q_1 or in the active state q_2,

-s_1' means that the GFA is either in active state q_0 or in the active state q_2,
-s_2' means that the GFA is either in active state q_0 or in the active state q_1,
-1 means that the GFA is in at least one active state of Q.
We may B with the algebra $[0,1]^Q$ as follows:

$$0 = (0,0,0), \quad s_0 = (1,0,0), \quad s_1 = (0,0.8,0), \quad s_2 = (0,0,0.35),$$
$$s_0' = (0,0.8,0.35), \quad s_1' = (1,0,0.35), \quad s_2' = (1,0.8,0), \quad 1 = (1,0.8,0.35).$$

Here, $\alpha(q_i)$ is the maximum membership values of active states at time t_i for any $i \geq 0$. We have $\delta_a = \{(q_1,q_2),(q_2,q_1),(q_2,q_2)\}$ and $\delta_b = \{(q_0,q_1),(q_2,q_2)\}$, then $\Delta = \{(q_0,q_1),(q_1,q_2),(q_2,q_1),(q_2,q_2)\}$. Using our formulas (∗) and (∗∗), we can obtain the upper transition functors $T_{\delta_a}, T_{\delta_b} : B \to [0,1]^Q$ and the lower transition functors $P_{\delta_a}, P_{\delta_b} : B \to [0,1]^Q$ as follows:

$$
\begin{array}{llll}
T_{\delta_a}(0) = 0, & P_{\delta_a}(0) = 0, & T_{\delta_b}(0) = 0, & P_{\delta_b}(0) = 0, \\
T_{\delta_a}(0) = s_0, & P_{\delta_a}(s_0) = 0, & T_{\delta_b}(s_1) = s_0, & P_{\delta_b}(s_0) = s_1, \\
T_{\delta_a}(s_2) = s_1, & P_{\delta_a}(s_1) = s_2, & T_{\delta_b}(0) = s_1, & P_{\delta_b}(s_1) = 0, \\
T_{\delta_a}(s_0') = s_2, & P_{\delta_a}(s_2) = s_0', & T_{\delta_b}(s_2) = s_2, & P_{\delta_b}(s_2) = s_2, \\
T_{\delta_a}(s_0') = s_0', & P_{\delta_a}(s_0') = s_0', & T_{\delta_b}(s_2) = s_0', & P_{\delta_b}(s_0') = s_2. \\
T_{\delta_a}(s_0') = s_1', & P_{\delta_a}(s_1') = s_0', & T_{\delta_b}(s_0') = s_1', & P_{\delta_b}(s_1') = s_0', \\
T_{\delta_a}(s_2) = s_2', & P_{\delta_a}(s_2') = s_2, & T_{\delta_b}(s_1) = s_2', & P_{\delta_b}(s_2') = s_1, \\
T_{\delta_a}(1) = 1, & P_{\delta_a}(1) = 1, & T_{\delta_b}(1) = 1, & P_{\delta_b}(1) = 1.
\end{array}
$$

E.g. $T_{\delta_a}(s_2) = s_1$ means that if the GFA is in active state q_1, when it enters input a, it will change to q_2 and $P_{\delta_a}(s_2) = s_0'$ means that if the GFA is in active state q_2, when it enters a, it will change to q_1 or q_2.

Theorem 3.3. Let $\tilde{F} = (Q,\Sigma,\tilde{R},Z,\tilde{\delta},\omega,F_1,F_2)$ be a general fuzzy automaton, $\boldsymbol{B} = (B;\leq,0,1)$ be a bounded poset of proposition about the $\tilde{F}$ and $T_\delta, P_\delta : B \to (M^Q)^\Sigma$ be labaled transition functors constructed through the transition frame $(Q,\triangle)$. Also, let for all $a_k \in \Sigma$, $b \in B$ and $q_m \in Q$,

$$H_{\delta_{a_k}}(b)(q_m) = \bigwedge\{b(q_j)|q_j \in Q_{pred}(q_m,a_k)\},$$

$$F_{\delta_{a_k}}(b)(q_m) = \bigvee\{b(q_j)|q_j \in Q_{succ}(q_m,a_k)\}.$$

If this is the case, then T_δ, P_δ, H_δ and F_δ are dynamic operators on $\boldsymbol{B}$ such that for all $\alpha \in B$,

$$T_\delta(\alpha) \leq F_\delta(\alpha) \quad and \quad H_\delta(\alpha) \leq P_\delta(\alpha),$$

whenever the respective sides of the relation $\leq$ are defined, i.e., both $\mathcal{D} = (\boldsymbol{B}; T_\delta, P_\delta)$ and $\mathcal{E} = (\boldsymbol{B}; H_\delta, F_\delta)$ are partial daynamic algebras. Moreover, the following holds:
(a) If $\triangle$ is reflexive, then T_δ and H_δ are contractive.
(b) If $\triangle$ is transitive, then T_δ and H_δ are transitive.
(c) If $\triangle$ is both reflexive and transitive, then both $\mathcal{D} = (\boldsymbol{B}; T_\delta, P_\delta)$ and $\mathcal{E} = (\boldsymbol{B}; H_\delta, F_\delta)$ are partial modal algebras.

Proof. By the definition of T_δ and P_δ, we have $T_\delta(0) = 0, T_\delta(1) = 1$, $P_\delta(0) = 0$ and $P_\delta(1) = 1$; thus, (P_1) holds. Let us check (P_2). Assume that $T_\delta(\alpha)$ and $T_\delta(\beta)$ exist and $\alpha \leq \beta$ for $\alpha, \beta \in B$. Then for all $q_j \in Q$, $\alpha(q_j) \leq \beta(q_j)$. This yields that for $a_k \in \Sigma, q_m \in Q$,

$$T_{\delta_{a_k}}(\alpha)(q_m) = \bigwedge\{\alpha(q_j)|q_j \in Q_{succ}(q_m,a_k)\} \leq \bigwedge\{\beta(q_j)|q_j \in Q_{succ}(q_m,a_k)\} = T_{\delta_{a_k}}(\beta)(q_m).$$

Therefore, $T_\delta(\alpha) \leq T_\delta(\beta)$. Similarly, $P_\delta(\alpha) \leq P_\delta(\beta)$, whenever $P_\delta(\alpha)$ and $P_\delta(\beta)$ exist and $\alpha \leq \beta$. It remains to prove (P_3). Assume that $P_\delta(\alpha)$ and $T_\delta P_\delta(\alpha)$ exist. We have for $a_k \in \Sigma$, $q_m \in Q$,

$$P_{\delta_{a_k}}(\alpha)(q_m) = \bigvee\{\alpha(q_j)|q_j \in Q_{pred}(q_m, a_k)\},$$

and

$$T_{\delta_{a_k}} P_{\delta_{a_k}}(\alpha)(q_i) = \bigwedge\{\bigvee\{\alpha(q_j)|q_j \in Q_{pred}(q_m, a_k)\}; q_m \in Q_{succ}(q_i, a_k)\}.$$

Since every member of the infimum is greater or equal to $\alpha(q_i)$, we conclude $T_{\delta_{a_k}} P_{\delta_{a_k}}(\alpha)(q_i) \geq \alpha(q_i)$ for each $q_i \in Q, a_k \in \Sigma$, i.e., $\alpha \leq T_\delta P_\delta$. Analogously, it can be shown $P_\delta T_\delta(\alpha) \leq \alpha$. Now, assume that $\alpha \in B$ and both $T_\delta(\alpha)$ and $F_\delta(\alpha)$ are defined. Let us verify that $T_\delta(\alpha) \leq F_\delta(\alpha)$. Let $q_i \in Q$. Then, there is $q_j \in Q_{succ}(q_i, a_k)$ such that $(q_i, q_j) \in \triangle$. It follows that $T_\delta(\alpha)(q_i) \leq \alpha(q_i) \leq F_\delta(\alpha)(q_i)$. Let us check (a). Assume that $\alpha \in B$ and $T_\delta(\alpha)$ is defined. Since $\triangle$ is reflexive; then from $(q_i, q_i) \in \triangle$, we obtain that

$$T_{\delta_{a_k}}(\alpha)(q_i) = \bigwedge\{\alpha(q_j)|q_j \in Q_{succ}(q_i, a_k)\} \leq \alpha(q_i),$$

for $a_k \in \Sigma$. Let us proceed similarly for (b).
Assume that $\alpha \in B$ and both $T_\delta(\alpha)$ and $T_\delta T_\delta(\alpha)$ are defined. We have for $q_i \in Q, a_k \in \Sigma$,

$$\begin{aligned}
T_{\delta_{a_k}}(\alpha)(q_i) &= \bigwedge\{\alpha(q_j)|q_j \in Q_{succ}(q_i, a_k)\} \\
&\leq \bigwedge\{\alpha(q_j)|q_k \in Q_{succ}(q_i, a_k), q_k \in Q_{pred}(q_j, a_k)\} \\
&= \bigwedge\{\bigwedge\{\alpha(q_j)|q_j \in Q_{succ}(q_k, a_k)\}; q_k \in Q_{succ}(q_i, a_k)\} \\
&= \bigwedge\{T_{\delta_{a_k}}(\alpha)(q_k)|q_k \in Q_{succ}(q_i, a_k)\} = T_{\delta_{a_k}} T_{\delta_{a_k}}(\alpha)(q_i).
\end{aligned}$$

Since $\{q_j \in Q|q_k \in Q, q_k \in Q_{succ}(q_i, a_k), q_k \in Q_{pred}(q_j, a_k)\} \subseteq \{q_j \in Q|q_j \in Q_{succ}(q_i, a_k)\}$ by transitivity the proof is complete.
The validity of (c) follows immediately from (a) and (b).
The remaining part of the proof for operators H_δ and F_δ follows from the preceding, since it is enough to work with the transition frame $(Q, \triangle^{op})$. $\qquad\square$

Corollary 3.4. *Let $\boldsymbol{M} = ([0, 1]; \leq, 0, 1)$ be a non-trivial complete lattice, $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton and $(Q, \triangle)$ be a transition frame. Define mapping $\hat{T}_\delta$, $\hat{P}_\delta$, $\hat{H}_\delta$, $\hat{F}_\delta : [0, 1]^Q \to ([0, 1]^Q)^\Sigma$ as follows:*
for $\alpha \in [0, 1]^Q$, $a_k \in \Sigma$,

$$\hat{T}_{\delta_{a_k}}(\alpha)(p) = \bigwedge\{\alpha(q)|q \in Q_{succ}(p, a_k)\},$$

$$\hat{P}_{\delta_{a_k}}(\alpha)(p) = \bigvee\{\alpha(q)|q \in Q_{pred}(p, a_k)\},$$

$$\hat{H}_{\delta_{a_k}}(\alpha)(p) = \bigwedge\{\alpha(q)|q \in Q_{pred}(p, a_k)\},$$

$$\hat{F}_{\delta_{a_k}}(\alpha)(p) = \bigvee\{\alpha(q)|q \in Q_{succ}(p, a_k)\}.$$

Then $\hat{T}_\delta$, $\hat{P}_\delta$, $\hat{H}_\delta$, $\hat{F}_\delta$ are dynamic operators on $\boldsymbol{M}^Q$ such that

$$\hat{T}_\delta \leq \hat{F}_\delta \quad and \quad \hat{H}_\delta \leq \hat{P}_\delta,$$

and both $(\boldsymbol{M}^Q; \hat{T}_\delta, \hat{P}_\delta)$ and $(\boldsymbol{M}^Q; \hat{H}_\delta, \hat{F}_\delta)$ are dynamic algebras (modal algebras).

4 Representation of dynamic algebras induced by GFA

We introduced a construction of natural dynamic operators in Section 3, when a bounded **B** and a transition frame $(Q, \triangle)$ of $\tilde{F}$ are given. However, we can ask, for a given dynamic algebra $(\mathbf{B}; T_\delta, P_\delta)$, whether there exists a transition frame $(Q, \triangle)$ and a bounded poset $\mathbf{M} = ([0,1]; \leq, 0, 1)$ such that the dynamic operators T_δ, P_δ can be derived by this construction, where **B** is embedded into the power algebra $\mathbf{M}^Q$. Hence, we ask, if every element b of B is in the form $(b(q))_{q \in Q}$ in $\mathbf{M}^Q$,

$$T_{\delta_{a_k}}(b)(q) = \bigwedge \{b(p) | \delta(q, a_k, p) \in \triangle \text{ or } p \in Q_{succ}(q, a_k)\},$$

and

$$P_{\delta_{a_k}}(b)(q) = \bigvee \{b(p) | \delta(p, a_k, q) \in \triangle \text{ or } p \in Q_{pred}(q, a_k)\},$$

for $a_k \in \Sigma$. If such a representation exists, then one can recognize the time variability of elements of B expressed as time dependent functions $b : Q \to M$.

From Corollary 3.4, we immediately see that $(\mathbf{M}^Q; \hat{T}_\delta, \hat{P}_\delta)$ is automatically a dynamic algebra.

Proposition 4.1. *Let $\boldsymbol{B} = (B; \leq, 0, 1)$ be a bounded poset equipped with a full set S_B of morphisms into a complete lattice $\boldsymbol{M}$,*

$$S_B = \{h_D : B \to [0,1]; D \subseteq B\},$$

such that for $a_k \in \Sigma$, $h_{D_{a_k}}(b) = \begin{cases} T_{\delta_{a_k}}(b) & \text{if } b \notin D \\ 0 & \text{if } b \in D \end{cases}$. Then, the map $i_B^{S_B} : B \to M^{S_B}$ given by $i_B^{S_B}(\alpha)(h_D) = h_D(\alpha)$ for $\alpha \in B, h_D \in S_B$ is an order reflecting morphism of bounded posets such that $i_B^{S_B}(B)$ is a subposet of $\boldsymbol{M}^{S_B}$.

Proof. Since S_B is a full set of morphisms, we have from Observation 2.5 that $i_B^{S_B}$ is an injective order-reflecting morphism of bounded posets. It follows that $i_B^{S_B}(B)$ is a bounded subposet of $\mathbf{M}^{S_B}$. □

We immediately obtain from Theorem 3.3 and from Proposition 4.1 the following:

Corollary 4.2. *Let $\boldsymbol{B} = (B; \leq, 0, 1)$ be a bounded poset equipped with a full set S_B of morphisms into a complete lattice $\boldsymbol{M}$ and let $(Q, \triangle)$ be a transition frame. Define partial mappings $T_\delta, P_\delta : B \to (M^Q)^\Sigma$ as follows:*

$$T_{\delta_{a_k}}(\alpha)(p) = \bigwedge \{\alpha(q) | q \in Q_{succ}(p, a_k)\},$$

and

$$P_{\delta_{a_k}}(\alpha)(p) = \bigvee \{\alpha(q) | q \in Q_{pred}(p, a_k)\},$$

for $\alpha \in B$, $a_k \in \Sigma$. Then T_δ, P_δ are dynamic operators on B, i.e., $\mathcal{D} = (\boldsymbol{B}; T_\delta, P_\delta)$ is a partial dynamic algebra and $i_B^{S_B, Q} : B \to M^Q$ defined by $i_B^{S_B, Q}((\alpha(p))_{p \in Q}) = i_B^{S_B}((\alpha(p))_{p \in Q})$ is an order reflecting morphism of bounded posets into the dynamic complete lattice algebra $(\boldsymbol{M}^Q; \hat{T}_\delta, \hat{P}_\delta)$ given by Theorem 3.3.

The previous corollary allows us to introduce, for any poset **B** of proposition about the general fuzzy automaton $\tilde{F}$ with the MacNeille completion $MC(\mathbf{B})$ and for any transiton frame $(Q, \triangle)$, a partial dynamic structure on **B** that can be fully reconstructed from $(MC(\mathbf{B}), \hat{T}_\delta, \hat{P}_\delta)$.

Lemma 4.3. *Let $(\boldsymbol{B}; T_\delta, P_\delta)$ be a dynamic algebra and let S_B be a set of morphisms from $\boldsymbol{B}$ into a bounded poset $\boldsymbol{M}$. Let us put $\triangle_{T_\delta} = \{(s,t) \in S_B \times S_B | \forall \alpha \in B, \ (s(T_\delta(\alpha))) \le t(\alpha)\}$. Then:*
(i) If T_δ is contractive, then $\triangle_{T_\delta}$ is reflexive.
(ii) If T_δ is transitive, then $\triangle_{T_\delta}$ is transitive.
(iii) Let $s, soT_\delta \in S_B$. Then $(s, soT_\delta) \in \triangle_{T_\delta}$ and for $\alpha \in B$,

$$s(T_\delta(\alpha)) = \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\}.$$

(iv) Let $s, soP_\delta \in S_B$. Then $(soP_\delta, s) \in \triangle_{T_\delta}$ and for $\alpha \in B$,

$$s(P_\delta(\alpha)) = \bigvee \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\}.$$

Proof. (i): $T_\delta(\alpha) \le \alpha$ yields $s(T_\delta(\alpha)) \le s(\alpha)$ for $\alpha \in B$ and $s \in S_B$. Hence $(s,s) \in \triangle_{T_\delta}$.
(ii): Let $s, t, u \in S_B$, $(s,t) \in \triangle_{T_\delta}$ and $(t,u) \in \triangle_{T_\delta}$. Suppose $\alpha \in B$. $T_\delta(\alpha) \le T_\delta T_\delta(\alpha)$ yields $s(T_\delta(\alpha)) \le s(T_\delta T_\delta(\alpha)) \le t(T_\delta(\alpha)) \le u(\alpha)$. Hence $(s,u) \in \triangle_{T_\delta}$.
(iii): Since for $\alpha \in B$, $s(T_\delta(\alpha)) = (soT_\delta)(\alpha)$ we have $(s, soT_\delta) \in \triangle_{T_\delta}$ and clearly, for $\alpha \in B$,

$$(soT_\delta)(\alpha) \ge \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\} \ge s(T_\delta(\alpha)).$$

(iv): From the definition of a dynamic algebra, we get $s(P_\delta T_\delta(\alpha)) \le s(\alpha)$ for $\alpha \in B$. Hence $(soP_\delta, s) \in \triangle_{T_\delta}$.
Evidently, for $\alpha \in B$, $(s,t) \in \triangle_{T_\delta}$ yields $t(\alpha) \le t(T_\delta P_\delta(\alpha)) \le s(P_\delta(\alpha))$. It follows that $(soP_\delta)(\alpha) \le \bigvee \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\} \le s(P_\delta(\alpha))$. □

The relation $\triangle_{T_\delta}$ introduced in Lemma 4.3 will be called the T_δ-induced relation.

Theorem 4.4. *Let $(\boldsymbol{B}; T_\delta, P_\delta)$ be a dynamic algebra and S_B be a set of morphisms from $\boldsymbol{B}$ into a complete lattic $\boldsymbol{M}$ such that*
(1) for all $\alpha \in B$ and for all $s \in S_B$, $s(T_\delta(\alpha)) = \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\}$,
(2) for all $\alpha \in B$ and for all $s \in S_B$, $s(P_\delta(\alpha)) = \bigvee \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\}$.
Then the map $i_B^{S_B}$ is an order reflecting morphism of dynamic algebras into the complete lattice dynamic algebra $(\boldsymbol{M}^{S_B}; \hat{T}_\delta, \hat{P}_\delta)$ given by the transition frame $(S_B, \triangle_{T_\delta})$. The dynamic algebra $(\boldsymbol{B}; T_\delta, P_\delta)$ is said to be representable in $\boldsymbol{M}$ with respect to S_B. Moreover, if T_δ is contractive (transitive), then $\hat{T}_\delta$ is contractive (transitive) and if $(\boldsymbol{B}; T_\delta, P_\delta)$ is a modal algebra, then $(\boldsymbol{M}^{S_B}; \hat{T}_\delta, \hat{P}_\delta)$ is a complete lattice modal algebra.

Proof. Recall first that since $\mathbf{M}$ is a complete lattice we have from Corollary 4.2 that $(\mathbf{M}^{S_B}; \hat{T}_\delta, \hat{P}_\delta)$ is a complete dynamic algebra. Here $\hat{T}_\delta(\alpha)(s) = \bigwedge \{(\alpha)(t) | (s,t) \in \triangle_{T_\delta}\}$ and $\hat{P}_\delta(\alpha)(s) = \bigvee \{(\alpha)(t) | (s,t) \in \triangle_{T_\delta}\}$ for all $\alpha \in \mathbf{M}$ and for all $s \in S_B$. Therefore by Proposition 4.1, $i_B^{S_B}$ is an order reflecting morphisms into $\mathbf{M}^{S_B}$. Since $s(T_\delta(\alpha)) = \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\}$ we get $i_B^{S_B}(T_\delta(\alpha))(s) = \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta}\}$. Then $i_B^{S_B}(T_\delta(\alpha)) = \hat{T}_\delta(i_B^{S_B}(\alpha))$. Similary $i_B^{S_B}(P_\delta(\alpha)) = \hat{P}_\delta(i_B^{S_B}(\alpha))$. Now, let T_δ is contractive (transitive). Then by Lemma 4.3 $\triangle_{T_\delta}$ is reflexive and transitive. Therefore, by Theorem 3.3 $\hat{T}_\delta$ is contractive (transitive). Also, let $(\mathbf{B}; T_\delta, P_\delta)$ is a modal algebra. Then by Corollary 3.4 $(\mathbf{M}^{S_B}; \hat{T}_\delta, \hat{P}_\delta)$ is a complete lattice modal algebra. □

Now, let us prove a representation theorem for dynamic algebras with a full set of morphisms.

Theorem 4.5. *(Representation theorem for dynamic algebras) For any dynamic algebra $(\boldsymbol{B}; T_\delta, P_\delta)$ with a full set S_B of morphisms into a complete lattice $\boldsymbol{M}$, there exists a full set T of morphisms into $\boldsymbol{M}$ containing S_B such that $(\boldsymbol{B}; T_\delta, P_\delta)$ is representable in $\boldsymbol{M}$ with respect to T. In particular, i_B^T is an order reflecting morphism of dynamic algebras.*

Proof. Let T be the smallest set of morphisms into **M** containing S_B such that $s \in T$ implies $soT_\delta, soP_\delta \in T$. Since T contains S_B, it is again a full set of morphisms.

It is enough to verify that for all $\alpha \in B$ and for all $s \in T$, $s(T_\delta(\alpha)) = \bigwedge\{t(\alpha)|(s,t) \in \triangle_{T_\delta}\}$ and $s(P_\delta(\alpha)) = \bigvee\{t(\alpha)|(s,t) \in \triangle_{T_\delta}\}$ where $\triangle_{T_\delta}$ is the T_δ-induced relation. But this is immediate since, for any $s \in T$, we have a morphism $t_s = soT_\delta \in T$ such that $s(T_\delta(\alpha)) = t_s(\alpha)$ for all $\alpha \in B$. Therefore, $s(T_\delta(\alpha)) = t_s(\alpha) \geq \bigwedge\{t(\alpha)|(s,t) \in \triangle_{T_\delta}\} \geq s(T_\delta(\alpha))$. Similarly, we obtain from Lemma 4.3 the remaining case. □

Corollary 4.6. *Let* $(\boldsymbol{B}; T_\delta, P_\delta)$ *be a dynamic algebra and* $MC(\boldsymbol{B})$ *be the MacNeille completion of* $\boldsymbol{B}$. *Then there is a countable frame* $(Q, \triangle)$ *such that* $T_\delta = \hat{T}_\delta|_B$ *and* $P_\delta = \hat{P}_\delta|_B$ *with* $(MC(\boldsymbol{B}), \hat{T}_\delta, \hat{P}_\delta)$ *given as in Theorem 3.3.*

Proof. Let $i_B : B \to MC(B)$ be the corresponding embedding of bounded poset. Let T be the smallest set of morphisms containing i_B that is closed under composition with T_δ and P_δ. Evidently, $T = \{i_BoT_\delta^{n_1}oP_\delta^{m_1}o\ldots oT_\delta^{n_k}oP_\delta^{m_k}; k, n_1, m_1, \ldots, n_k, m_k \in N\}$ is countable and we have an order reflecting morphism $i_B^T : B \to (MC(B))^T$ of bounded posets. By $\triangle$ can be taken the T_δ-induced relation. □

5   Representation of tense algebras induced by GFA

Definition 5.1. *Let* $\tilde{F}$ *be a general fuzzy automaton,* $(\boldsymbol{B}; T_\delta, P_\delta)$ *and* $(\boldsymbol{B}; H_\delta, F_\delta)$ *be a partial dynamic algebras such that, for all* $\alpha \in B, T_\delta(\alpha) \leq F_\delta(\alpha)$ *and* $H_\delta(\alpha) \leq P_\delta(\alpha)$, *whenever the respective sides of the relation* $\leq$ *are defined. The quintuple* $\tau(B) = (\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ *is called a partial tense algebra. If all* $T_\delta, P_\delta, H_\delta, F_\delta$ *are total maps, then we speak about a tense algebra.*

If $(\boldsymbol{B}; T_\delta, P_\delta)$ and $(\boldsymbol{B}; H_\delta, F_\delta)$ are partial modal algebras, then $\tau(B)$ is called a partial tense modal algebra. If $(\boldsymbol{B}_1; T_{\delta_1}, P_{\delta_1}, H_{\delta_1}, F_{\delta_1})$ and $(\boldsymbol{B}_2; T_{\delta_2}, P_{\delta_2}, H_{\delta_2}, F_{\delta_2})$ are partial modal algebras and $f : B_1 \to B_2$ a map such that
(i) $f : (\boldsymbol{B}_1; T_{\delta_1}, P_{\delta_1}) \to (\boldsymbol{B}_2; T_{\delta_2}, P_{\delta_2})$ is a morphism of partial dynamic algebras,
(ii) $f : (\boldsymbol{B}_1; H_{\delta_1}, F_{\delta_1}) \to (\boldsymbol{B}_2; H_{\delta_2}, F_{\delta_2})$ is a morphism of partial dynamic algebras,
then we say that $f : (\boldsymbol{B}_1; T_{\delta_1}, P_{\delta_1}, H_{\delta_1}, F_{\delta_1}) \to (\boldsymbol{B}_2; T_{\delta_2}, P_{\delta_2}, H_{\delta_2}, F_{\delta_2})$ is a morphism of partial tense algebras.

Lemma 5.2. *Let* $\tilde{F}$ *be a general fuzzy automaton,* $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ *be a tense algebra on B and let* S_B *be a set of morphisms from* $\boldsymbol{B}$ *into a bounded poset* $\boldsymbol{M}$. *Let us put*

$$\triangle_{T_\delta, H_\delta} = \triangle_{T_\delta} \cap \triangle_{H_\delta}^{op} = \{(s,t) \in S_B \times S_B | \forall \alpha \in B \; s(T_\delta(\alpha)) \leq t(\alpha) \;\; and \;\; t(H_\delta(\alpha)) \leq s(\alpha)\},$$

and

$$\triangle_{H_\delta, T_\delta} = \triangle_{H_\delta} \cap \triangle_{T_\delta}^{op} = \{(s,t) \in S_B \times S_B | \forall \alpha \in B \; s(H_\delta(\alpha)) \leq t(\alpha) \;\; and \;\; t(T_\delta(\alpha)) \leq s(\alpha)\},$$

where $\triangle_{T_\delta}$ *is the T_δ-induced relation and* $\triangle_{H_\delta}$ *is the H_δ-induced relation.*
Then $\triangle_{T_\delta, H_\delta} = \triangle_{H_\delta, T_\delta}^{op}$ *and*
(i) If T_δ *is contractive, then* $\triangle_{T_\delta, H_\delta}$ *and* $\triangle_{H_\delta, T_\delta}$ *are reflexive.*
(ii) If T_δ *is transitive, then* $\triangle_{T_\delta, H_\delta}$ *and* $\triangle_{H_\delta, T_\delta}$ *are transitive.*
(iii) Let $s, soT_\delta \in S_B$. *Then* $(s, soT_\delta) \in \triangle_{T_\delta, H_\delta}$ *and for all* $\alpha \in B$,

$$s(T_\delta(\alpha)) = \bigwedge\{t(\alpha)|(s,t) \in \triangle_{T_\delta, H_\delta}\}.$$

(iv) Let $s, soP_\delta \in S_B$. Then $(soP_\delta, s) \in \triangle_{T_\delta, H_\delta}$ and for all $\alpha \in B$,

$$s(P_\delta(\alpha)) = \bigvee \{t(\alpha) | (s,t) \in \triangle_{T_\delta, H_\delta}\}.$$

(v) Let $s, soH_\delta \in S_B$. Then $(s, soH_\delta) \in \triangle_{T_\delta, H_\delta}$ and for all $\alpha \in B$,

$$s(H_\delta(\alpha)) = \bigwedge \{t(\alpha) | (s,t) \in \triangle_{H_\delta, T_\delta}\}.$$

(vi) Let $s, soF_\delta \in S_B$. Then $(soF_\delta, s) \in \triangle_{T_\delta, H_\delta}$ and for all $\alpha \in B$,

$$s(F_\delta(\alpha)) = \bigvee \{t(\alpha) | (s,t) \in \triangle_{H_\delta, T_\delta}\}.$$

Proof. (i): From Lemma 4.3(i), we know that $\triangle_{T_\delta}$ and $\triangle_{H_\delta}$ are reflexive. It follows that $\triangle_{H_\delta, T_\delta}$ and $\triangle_{T_\delta, H_\delta}$ are reflexive.

(ii): From Lemma 4.3(ii), we know that $\triangle_{T_\delta}$ and $\triangle_{H_\delta}$ are transitive. It follows that $\triangle_{H_\delta, T_\delta}$ and $\triangle_{T_\delta, H_\delta}$ are transitive.

(iii): Since, for all $\alpha \in B$, $(soT_\delta)(H_\delta(\alpha)) \le s(F_\delta H_\delta(\alpha)) \le s(\alpha)$ we have that $(s, soT_\delta) \in \triangle_{H_\delta}^{op}$. From Lemma 4.3(iii), we obtain that $(s, soT_\delta) \in \triangle_{T_\delta}$, i.e., $(s, soT_\delta) \in \triangle_{T_\delta, H_\delta}$. Cleary for all $\alpha \in B$,

$$(soT_\delta)(\alpha) \ge \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta, H_\delta}\} \ge s(T_\delta(\alpha)).$$

(iv): From the definition of a tense algebra, we get that $s(H_\delta(\alpha)) \le s(P_\delta(\alpha))$ for all $\alpha \in B$. Hence $(soP_\delta, s) \in \triangle_{H_\delta}^{op}$. From Lemma 4.3(iv), we know that $(soP_\delta, s) \in \triangle_{T_\delta}$. It follows that $(soP_\delta, s) \in \triangle_{T_\delta, H_\delta}$ and for all $\alpha \in B$,

$$(soP_\delta)(\alpha) \le \bigvee \{t(\alpha) | (t,s) \in \triangle_{T_\delta, H_\delta}\} \le s(P_\delta(\alpha)).$$

(v),(vi): By interchanging the role of T_δ with H_δ and of P_δ with F_δ, by (iii) and (iv) the proof is clear. $\qquad \square$

Theorem 5.3. *Let $\tilde{F}$ be a general fuzzy automaton, $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ be a tense algebra with a full set S_B of morphisms from $\boldsymbol{B}$ into a complete lattice $\boldsymbol{M}$ such that*
(1) for all $\alpha \in B$ and for all $s \in S_B$, $s(T_\delta(\alpha)) = \bigwedge \{t(\alpha) | (s,t) \in \triangle_{T_\delta, H_\delta}\}$,
(2) for all $\alpha \in B$ and for all $s \in S_B$, $s(P_\delta(\alpha)) = \bigvee \{t(\alpha) | (t,s) \in \triangle_{T_\delta, H_\delta}\}$,
(3) for all $\alpha \in B$ and for all $s \in S_B$, $s(H_\delta(\alpha)) = \bigwedge \{t(\alpha) | (t,s) \in \triangle_{T_\delta, H_\delta}\}$,
(4) for all $\alpha \in B$ and for all $s \in S_B$, $s(F_\delta(\alpha)) = \bigvee \{t(\alpha) | (s,t) \in \triangle_{T_\delta, H_\delta}\}$.
Then the map $i_B^{S_B}$ is an order reflecting morphism of tense algebra into the complete lattice tense algebra $(\boldsymbol{M}^{S_B}; \hat{T}_\delta, \hat{P}_\delta, \hat{H}_\delta, \hat{F}_\delta)$ given by the transition frame $(S_B, \triangle_{T_\delta, H_\delta})$. $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ is then said to be representable in $\boldsymbol{M}$ with respect to S_B.
Moreover, if T_δ and H_δ are contractive (transitive), then $\hat{T}_\delta$ and $\hat{H}_\delta$ are contractive (transitive) and if $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ is a tense model algebra, then $(\boldsymbol{M}^{S_B}; \hat{T}_\delta, \hat{P}_\delta, \hat{H}_\delta, \hat{F}_\delta)$ is a complete lattice tense modal algebra.

Proof. It follows by the same arguments as in Theorem 4.4. $\qquad \square$

Theorem 5.4. *(Representation theorem for tense algebra) Let $\tilde{F}$ be a general fuzzy automaton, for any tense (tense modal) algebra $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ with a full set S_B of morphisms into a complete lattice $\boldsymbol{M}$, there exists a full set T of morphism into $\boldsymbol{M}$ containing S_B such that $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ is reflecting morphism of tense (tense modal) algebras.*

Proof. Let T be the smallest set of morphisms into **M** containing S_B such that $s \in T$ implies that $soT_\delta, soP_\delta, soH_\delta, soF_\delta \in T$. Since T contains S_B, it is again a full set of morphisms. The remaining conditions on T from Theorem 5.3 are satisfied by the same arguments as in the proof of Theorem 4.5, using Lemma 5.2. $\square$

Corollary 5.5. *Let $\tilde{F}$ be a general fuzzy automaton, $(\boldsymbol{B}; T_\delta, P_\delta, H_\delta, F_\delta)$ be a tense (tense modal) algebra and $MC(\boldsymbol{B})$ be the MacNeille completion of $\boldsymbol{B}$. Then, there is a countable transition frame $(T, \triangle_{T_\delta, H_\delta})$ such that $T_\delta = \hat{T}_\delta|_B, P_\delta = \hat{P}_\delta|_B, H_\delta = \hat{H}_\delta|_B$ and $F_\delta = \hat{F}_\delta|_B$ with $((MC(\boldsymbol{B}))^T; \hat{T}_\delta, \hat{P}_\delta, \hat{H}_\delta, \hat{F}_\delta)$ given as in Theorem 3.3.*

Proof. Let $i_B : B \to MC(B)$ be the corresponding embedding of bounded posets. Let T be the smallest of morphisms containing i_B that is closed under composition with $T_\delta, P_\delta, H_\delta$ and F_δ. Evidently,

$$T = \{i_B o T_\delta^{n_1} o P_\delta^{m_1} o H_\delta^{p_1} o F_\delta^{q_1} o \ldots o T_\delta^{n_k} o P_\delta^{m_k} o H_\delta^{p_k} o F_\delta^{q_k}, k, n_1, m_1, p_1, q_1, \ldots, n_k, m_k, p_k, q_k \in N\},$$

is countable and we have an order reflecting morphism $i_B^T : B \to (MC(B))^T$ of bounded posets. $\square$

6 Conclusion

This study was an endeavor to suggest and examine tense operators in the dynamic logic B which has been regarded as a set of propositions about the general fuzzy automaton regardless of what propositional connectives the logic comprised. It also demonstrated a simple construction of tense operators which makes use of lattice theoretical properties of the underlying ordered set B. The study further proved that when the underlying ordered set was not a complete lattice, the lattice completion for this construction could be used. Further, the axiomatization of universal quantifiers was applied and the related axioms were modified. In this study, it was also shown that the operators could be identified as modal operators and the pairs (T_δ, P_δ) were examined as the so-called dynamic pairs. The constructions of these operators were attained in the corresponding algebra and then a transition frame was proposed. In addition, the problem of finding a transition frame was solved in the case when the tense operators were given. In particular, this study demonstrated that the tense algebra B was representable in its Dedekind-MacNeille completion. Representation theorems for dynamic and tense algebra were explicated in details through examples. In our future work, L^B-valued general fuzzy automata will be defined and examined in details based on topologies and topological characteristics.

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