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The Belluce lattice associated with a bounded BCK-algebra

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Abstract

In this paper, we introduce the notions of Belluce lattice associated with a bounded BCK-algebra and reticulation of a bounded BCK-algebra. To do this, first, we define the operations $\curlywedge$, $\curlyvee$ and $\sqcup$ on BCK-algebras and we study some algebraic properties of them. Also, for a bounded BCK-algebra A we define the Zariski topology on $Spec(A)$ and the induced topology $\tau_{A,Max(A)}$ on $Max(A)$. We prove $(Max(A), \tau_{A,Max(A)})$ is a compact topological space if A has Glivenko property. Using the open and the closed sets of $Max(A)$, we define a congruence relation on a bounded BCK-algebra A and we show L_A, the quotient set, is a bounded distributive lattice. We call this lattice the Belluce lattice associated with A. Finally, we show (L_A, p_A) is a reticulation of A (in the sense of Definition 5.1) and the lattices L_A and S_A are isomorphic.

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1 Introduction

In [4], Belluce defined the reticulation for non-commutative rings (for commutative rings see [17]). Using this model, the reticulation was defined for others classes of universal algebras: MV-algebras ([3]), BL-algebras ([13]), residuated lattices ([14], [15]), Hilbert algebras ([5]) and quantales ([8]). Generally speaking, the reticulation for an algebra A of types mentioned above is a pair (L_A, λ) consisting of a bounded distributive lattice L_A and a surjection $\lambda : A \to L_A$ such that the function given by the inverse image of λ induces (by reticulation) a homeomorphism of topological spaces between the prime spectrum of L_A and that of A. Using this construction many properties can be transferred between L_A and A.

In this paper, we construct the Belluce lattice associated with a bounded BCK-algebra and we define the reticulation of a bounded BCK-algebra (in the sense of Definition 5.1). Also we prove several properties of it.

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The paper is organized as follows: In Section 2, we review some relevant concepts relative to BCK-algebras. Also, we define the new operations $\curlywedge$, $\curlyvee$ and $\sqcup$ on BCK-algebras and we study the algebraic properties of them.

For a bounded BCK-algebra A, in Section 3, we study the topological spaces $Spec(A)$, the prime spectrum of A, and $Max(A)$, the maximal spectrum of A, using a standard method ([1]). The family $\tau_A = \{D(S) : S \subseteq A\}$ is a topology on $Spec(A)$ having $\{D(x) : x \in A\}$ as basis. The topology τ_A is called the Zariski topology on $Spec(A)$ and the topological space $(Spec(A), \tau_A)$ is called the prime spectrum of A. Since $Max(A) \subseteq Spec(A)$ we can consider on $Max(A)$ the topology induced by Zariski topology. So, we obtain a topological space $(Max(A), \tau_{A,Max(A)})$ called the *maximal spectrum* of A.

If BCK-algebra A has Glivenko property, then $Max(A)$ is a compact topological space (Theorem 3.10).

Using the open and the closed sets of $Max(A)$, in Section 4, we construct and study the *Belluce lattice* L_A *associated with a bounded* BCK*-algebra* A (Theorems 4.4, 4.9, 4.11 and 4.13).

In Section 5, we introduce the notion of *reticulation* of a bounded BCK-algebra and prove that the uniqueness of this reticulation (Theorem 5.2). Finally, we show that (L_A, p_A) and (S_A, V_{Max}) are reticulations of A and L_A and S_A are isomorphic (Corollaries 5.4 and 5.5).

2 Preliminaries

Definition 2.1. ([11], [12]) A *BCK-algebra* is an algebra $(A, \to, 1)$ of type (2,0) such that the following axioms are fulfilled for every $x, y, z \in A$:

(a_1) $x \to x = 1$;

(a_2) if $x \to y = y \to x = 1$, then $x = y$;

(B) $(x \to y) \to [(y \to z) \to (x \to z)] = 1$;

(C) $x \to (y \to z) = y \to (x \to z)$;

(K) $x \to (y \to x) = 1$.

For examples of BCK-algebras, see [11] and [12].

If A is a BCK-algebra, then the relation $x \le y$ iff $x \to y = 1$ is a partial order on A; with respect to this order 1 is the largest element of A. A *bounded* BCK-algebra is a BCK-algebra A with the smallest element 0; in this case for $x \in A$ we denote $x^* = x \to 0$.

A bounded BCK-algebra A has *Glivenko property* (see [7]) if it satisfies the following condition:

(G) $(x \to y)^{**} = x \to y^{**}$, for every $x, y \in A$.

For a BCK-algebra A and $x_1, ..., x_n, x \in A$ $(n \ge 1)$ we define $(x_1, ..., x_n; x) = x_1 \to (x_2 \to ...(x_n \to x)...)$.

From [6] and [12] we have the following rules of calculus:

(c_1) $x \to 1 = 1, 1 \to x = x, x \le y \to x, x \le (x \to y) \to y$;

(c_2) $((x \to y) \to y) \to y = x \to y$;

(c_3) if $x \le y$, then $z \to x \le z \to y$ and $y \to z \le x \to z$;

(c_4) $x \to y \le (z \to x) \to (z \to y) \le z \to (x \to y)$, for every $x, y, z \in A$.

In a bounded BCK-algebra A, for $x, y, z \in A$ we have the following rules of calculus (see [7], [10], [11] and [12]):

(c_5) $0^* = 1, 1^* = 0, x \to y^* = y \to x^*, x \le x^{**}, x^{***} = x^*$;

(c_6) $x^{**} \le x^* \to x, x \to y \le y^* \to x^*$ and if $x \le y$, then $y^* \le x^*$.

Remark 2.2. *Using (c_5) we deduce that a bounded BCK-algebra A has Glivenko property iff $(x \to y)^{**} = x^{**} \to y^{**}$, for every $x, y \in A$.*

If A is a bounded BCK-algebra, then for $x, y \in A$ we denote $x \curlyvee y = x^* \to y$ and $x \curlywedge y = (x \to y^*)^*$.

Proposition 2.3. *Let A be a bounded BCK-algebra and $x, y, z \in A$. Then:*

(c_7) $x \curlywedge 0 = 0,\ x \curlywedge 1 = x^{**}$ *and* $x \curlywedge x^* = 0$;

(c_8) $x \curlywedge y = y \curlywedge x \le x^{**},\ y^{**}$;

(c_9) *if* $x \le y$, *then* $x \curlywedge z \le y \curlywedge z$;

(c_{10}) $x, y \le x \curlyvee y,\ x \curlyvee 0 = x^{**},\ x \curlyvee 1 = 1,\ x \curlyvee x^* = 1$;

(c_{11}) $x \curlyvee (y \curlyvee z) = y \curlyvee (x \curlyvee z)$ *and* $(x \curlyvee y) \curlyvee z \le x \curlyvee (y \curlyvee z)$;

(c_{12}) $x \curlywedge (x \to y) \le y^{**},\ x^{**} \curlywedge y^{**} = x \curlywedge y$.

Proof. (c_7). $x \curlywedge 0 = (x \to 0^*)^* = (x \to 1)^* = 1^* = 0$, $x \curlywedge 1 = (x \to 1^*)^* = x^{**}$ and $x \curlywedge x^* = (x \to x^{**})^* = 1^* = 0$.

(c_8). $x \curlywedge y = (x \to y^*)^* \overset{(c_5)}{=} (y \to x^*)^* = y \curlywedge x$ and since $0 \le y^*$, by (c_3), $x^* \le x \to y^*$, so $x \curlywedge y \le x^{**}$. Similarly, $x \curlywedge y \le y^{**}$.

(c_9). Using (c_3), from $x \le y$ we deduce $y \to z^* \le x \to z^*$, so, $(x \to z^*)^* \le (y \to z^*)^*$. Hence $x \curlywedge z \le y \curlywedge z$.

(c_{10}). From (c_1) and (c_3), $x, y \le x \curlyvee y = x^* \to y$. Also, $x \curlyvee 0 = x^* \to 0 = x^{**}$, $x \curlyvee 1 = x^* \to 1 = 1$ and $x \curlyvee x^* = x^* \to x^* = 1$.

(c_{11}). We have $x^* \le (x^* \to y) \to y \le y^* \to (x^* \to y)^* \le ((x^* \to y)^* \to z) \to (y^* \to z)$. Therefore,

$$1 = x^* \to [((x^* \to y)^* \to z) \to (y^* \to z)] = ((x^* \to y)^* \to z) \to (x^* \to (y^* \to z)).$$

Thus, $(x^* \to y)^* \to z \le x^* \to (y^* \to z)$. We deduce that

$$x \curlyvee (y \curlyvee z) = x^* \to (y^* \to z) \ge (x^* \to y)^* \to z = (x \curlyvee y) \curlyvee z.$$

Also, $x \curlyvee (y \curlyvee z) = x^* \to (y^* \to z) \overset{(C)}{=} y^* \to (x^* \to z) = y \curlyvee (x \curlyvee z)$.

(c_{12}). Since $x \to y \le y^* \to x^*$, by (C), we have $y^* \le (x \to y) \to x^*$. So by (c_5) and (c_6), $y^* \le x \to (x \to y)^*$, thus, $x \curlywedge (x \to y) = [x \to (x \to y)^*]^* \le y^{**}$.

Also, $x^{**} \curlywedge y^{**} = (x^{**} \to y^{***})^* = (x^{**} \to y^*)^* = (y \to x^{***})^* = (y \to x^*)^* = y \curlywedge x = x \curlywedge y$. $\square$

Proposition 2.4. *Let A be a bounded BCK-algebra with Glivenko property and $x, y, z, x_1, x_2, ..., x_n \in A$, $n \geq 2$. Then:*

(c_{13}) $(x \curlywedge y)^* = x^* \curlyvee y^*$ *and* $(x \curlyvee y)^* = x^* \curlywedge y^*$;

(c_{14}) $x \curlywedge (y \curlywedge z) = (x \curlywedge y) \curlywedge z$;

(c_{15}) $x_1 \curlywedge x_2 \curlywedge ... \curlywedge x_n = (x_1, x_2, ..., x_{n-1}; x_n^*)^*$;

(c_{16}) *if* $x \curlywedge z \leq y$, *then* $x \leq z \to y^{**}$;

(c_{17}) $x \curlywedge z \leq y^{**}$ *iff* $x \leq z \to y^{**}$;

(c_{18}) *if* $(x_1, x_2, ..., x_n; y) = 1$, *then* $x_1 \curlywedge x_2 \curlywedge ... \curlywedge x_n \leq y^{**}$.

Proof. (c_{13}). We have $x^* \curlyvee y^* = x^{**} \to y^* = y \to x^{***} = y \to x^*$ and $(x \curlywedge y)^* = (x \to y^*)^{**} \overset{(G)}{=} x \to y^{***} = x \to y^*$, hence $(x \curlywedge y)^* = x^* \curlyvee y^*$.

Also, $x^* \curlywedge y^* = (x^* \to y^{**})^* \overset{(G)}{=} ((x^* \to y)^{**})^* = (x^* \to y)^* = (x \curlyvee y)^*$.

(c_{14}). Let $x, y, z \in A$. Then

$$(x \curlywedge y) \curlywedge z \overset{(c_8)}{=} z \curlywedge (x \curlywedge y) = [z \to (x \curlywedge y)^*]^* \overset{(c_{13})}{=} [z \to (x^* \curlyvee y^*)]^*$$
$$= [z \to (x^{**} \to y^*)]^* \overset{(c_5)}{=} [z \to (y \to x^*)]^* \overset{(C)}{=} [y \to (z \to x^*)]^*.$$

Similarly, $x \curlywedge (y \curlywedge z) = [y \to (x \to z^*)]^*$. Using ($c_5$) we deduce that $x \curlywedge (y \curlywedge z) = (x \curlywedge y) \curlywedge z$.

(c_{15}). By induction on n, using the associativity of $\curlywedge$ we can write

$$x_1 \curlywedge x_2 \curlywedge ... \curlywedge x_n = x_1 \curlywedge (x_2 \curlywedge ... \curlywedge x_n) = [x_1 \to (x_2 \curlywedge ... \curlywedge x_n)^*]^* = [x_1 \to (x_2, ..., x_{n-1}; x_n^*)^{**}]^*$$
$$\overset{(G)}{=} [x_1 \to (x_2, ..., x_{n-1}; x_n^*)]^{***} = [x_1 \to (x_2, ..., x_{n-1}; x_n^*)]^* = (x_1, x_2, ..., x_{n-1}; x_n^*)^*.$$

(c_{16}). If $x \curlywedge z \leq y$, then $(x \to z^*)^* \leq y$, so $y^* \leq (x \to z^*)^{**} \overset{(G)}{=} x \to z^{***} = x \to z^*$, hence $x \leq y^* \to z^* = z \to y^{**}$.

(c_{17}). Suppose that $x \curlywedge z \leq y^{**}$. From ($c_{16}$) we deduce that $x \leq z \to (y^{**})^{**} = z \to y^{**}$. Conversely, if $x \leq z \to y^{**}$, then $x \leq y^* \to z^*$. Thus, $y^* \leq x \to z^* = x \to z^{***} \overset{(G)}{=} (x \to z^*)^{**}$. We deduce that $(x \to z^*)^* \leq y^{**}$, so $x \curlywedge z \leq y^{**}$.

(c_{18}). Mathematical induction on n.

Consider $n = 2$ and $(x_1, x_2; y) = 1$, that is, $x_1 \to (x_2 \to y) = 1$. From $y \leq y^{**}$ we deduce that $1 = x_1 \to (x_2 \to y) \leq x_1 \to (x_2 \to y^{**})$, hence $x_1 \to (x_2 \to y^{**}) = 1$, that is, $x_1 \leq x_2 \to y^{**} = y^* \to x_2^*$. Then $y^* \leq x_1 \to x_2^*$, hence $(x_1 \to x_2^*)^* \leq y^{**}$, that is, $x_1 \curlywedge x_2 \leq y^{**}$.

Suppose that the assertion is true for $n-1$ and let $(x_1, x_2, ..., x_n; y) = 1$. Since $1 = (x_1, x_2, ..., x_n; y) = (x_1, x_2, ..., x_{n-1}; x_n \to y)$ then $x_1 \curlywedge x_2 \curlywedge ... \curlywedge x_{n-1} \leq (x_n \to y)^{**} \overset{(G)}{=} x_n \to y^{**}$. From ($c_{17}$), we obtain $x_1 \curlywedge x_2 \curlywedge ... \curlywedge x_n \leq y^{**}$. □

Definition 2.5. [6] *Let A be a BCK-algebra. A subset D of A is called a deductive system (or filter) of A if $1 \in D$ and for every $x, y \in A$ if $x, x \to y \in D$, then $y \in D$.*

A deductive system D is called *proper* if $D \neq A$. We denote by $Ds(A)$ the set of all deductive systems of A. If A is bounded, then a deductive system D is proper iff $0 \notin D$.

Lemma 2.6. *Let A be a bounded BCK-algebra and $D \in Ds(A)$. If $x, y \in D$, then $x \curlywedge y \in D$.*

Proof. We have $y \to (x \curlywedge y) = y \to (x \to y^*)^* = (x \to y^*) \to y^* \in D$, since by (c_1), $x \leq (x \to y^*) \to y^*$. Because $y \in D$, we deduce that $x \curlywedge y \in D$. □

If A is a BCK-algebra and $S \subseteq A$ is a nonempty subset of A, we denote by $\langle S \rangle$ the lowest deductive system of A (relative to inclusion) which contains S; $\langle S \rangle$ is called the deductive system of A *generated* by S.

For two elements $x, y \in A$ and a natural number $n \geq 1$ we define $x \to_n y = x \to (x \to ...(x \to y)...)$, where n indicates the number of occurrences of x.

Theorem 2.7. [6], [12] *Let A be a BCK-algebra and $S \subseteq A$ be a nonempty subset of A, $D \in Ds(A)$ and $a \in A$. Then:*

- (i) *$\langle S \rangle = \{x \in A :$ there are $n \geq 1$ and $a_1, a_2, ..., a_n \in S$ such that $(a_1, a_2, ..., a_n; x) = 1\}$; In particular, $\langle a \rangle = \langle \{a\} \rangle = \{x \in A : a \to_n x = 1$, for some $n \geq 1\}$;*
- (ii) *$(Ds(A), \subseteq)$ is a complete distributive lattice, where for D_1, $D_2 \in Ds(A)$, $D_1 \wedge D_2 = D_1 \cap D_2$ and $D_1 \vee D_2 = \langle D_1 \cup D_2 \rangle$.*

A proper deductive system P of a BCK-algebra A is called *irreducible (prime)* if it is a meet-irreducible (meet-prime) element of the lattice $Ds(A)$. Since $(Ds(A), \subseteq)$ is distributive, then the notions of irreducible and prime coincide. We denote by $Spec(A)$ the set of all prime deductive systems of A.

Theorem 2.8. [6], [12] *Let A be a BCK-algebra and $P \in Ds(A)$ such that $P \neq A$. Then the following statements are equivalent:*

- (i) *$P \in Spec(A)$;*
- (ii) *if $D_1 \cap D_2 \subseteq P$ with $D_1, D_2 \in Ds(A)$, then $D_1 \subseteq P$ or $D_2 \subseteq P$;*
- (iii) *for every $x, y \in A$, if $U(x, y) = \{z \in A : z \geq x$ and $z \geq y\} \subseteq P$, then $x \in P$ or $y \in P$.*

For a BCK-algebra A, a subset $I \subseteq A$ is called an *ideal* of A (see [6]) if:

- (i_1) $y \in I$ and $x \leq y$ imply $x \in I$;
- (i_2) for every $x, y \in I$ there exists $z \in I$ such that $x, y \leq z$.

Theorem 2.9. ([6]) *Let A be a BCK-algebra and $D \in Ds(A)$.*

- (i) *If I is an ideal of A such that $D \cap I = \emptyset$, then there exists $P \in Spec(A)$ such that $D \subseteq P$ and $I \cap P = \emptyset$;*
- (ii) *For each $a \notin D$ there exists $P \in Spec(A)$ such that $a \notin P$ and $D \subseteq P$;*
- (iii) *$D = \cap\{P \in Spec(A) : D \subseteq P\}$.*

A proper deductive system M of a BCK-algebra A is called *maximal* if it is a maximal element in the lattice $(Ds(A), \subseteq)$. We denote by $Max(A)$ the set of all maximal deductive systems of A. Obviously, $Max(A) \subseteq Spec(A)$.

In a BCK-algebra A, for $x, y \in A$ we denote $x \sqcup y = (x \to y) \to y$. Using (c_1) and (c_2), we deduce that $x, y \leq x \sqcup y$ and $(x \sqcup y) \to y = x \to y$.

Theorem 2.10. ([9]) *Let M be a proper deductive system of a bounded BCK-algebra A. Then the following are equivalent:*

(i) $M \in Max(A)$;

(ii) *if* $x \notin M$, *then there exists* $n \geq 1$ *such that* $x \rightarrow_n 0 \in M$.

Theorem 2.11. ([9], Corollary 6.7) *Let A be a BCK-algebra and $M \in Max(A)$. For $x, y \in A$, if $x \sqcup y \in M$, then $x \in M$ or $y \in M$.*

Lemma 2.12. *Let A be a bounded BCK-algebra, $x \in A$ and $M \in Max(A)$. Then $x \in M$ iff $x^{**} \in M$.*

Proof. If $x \in M$, then since $x \leq x^{**}$ we deduce that $x^{**} \in M$.

Conversely, suppose that $x^{**} \in M$. If $x \notin M$, then by Theorem 2.10 *(ii)*, we deduce that $x \rightarrow_n 0 \in M$, for some $n \geq 1$.

If $n = 1$, then $x^*, x^{**} \in M$ imply that $0 \in M$, which is a contradiction.

If $n \geq 2$, then $x \rightarrow_n 0 \in M$ and $x^{**} = (x \rightarrow 0) \rightarrow 0 \in M$ implies $x \rightarrow_{n-1} 0 \in M$, hence $x \rightarrow 0 \in M$. Since $x^{**} \in M$ we obtain $0 \in M$, a contradiction. We conclude that $x \in M$. □

3 The topological spaces Spec(A) and Max(A)

Let A be a bounded BCK-algebra, $S \subseteq A$ and $x \in A$. We denote $D(S) = \{P \in Spec(A) : S \nsubseteq P\}$ and $D(x) = \{P \in Spec(A) : x \notin P\}$.

Proposition 3.1. *Let A be a bounded BCK-algebra and $S, S_1, S_2 \subseteq A$. Then the following hold:*

(i) $D(\emptyset) = \emptyset$ *and* $D(A) = Spec(A)$;

(ii) *if* $S_1 \subseteq S_2$, *then* $D(S_1) \subseteq D(S_2)$;

(iii) $D(S) = D(\langle S \rangle)$;

(iv) $D(S_1) = D(S_2)$ *iff* $\langle S_1 \rangle = \langle S_2 \rangle$;

(v) *if* $F, G \in Ds(A)$, *then* $F = G$ *iff* $D(F) = D(G)$;

(vi) *if* $S_i \subseteq A$, $i \in I$, *then* $D(\underset{i \in I}{\cup} S_i) = \underset{i \in I}{\cup} D(S_i)$;

(vii) *if* $F_i \in Ds(A)$, $i \in I$, *then* $D(\underset{i \in I}{\vee} F_i) = \underset{i \in I}{\cup} D(F_i)$;

(viii) $D(\langle S_1 \rangle) \cap D(\langle S_2 \rangle) = D(\langle S_1 \rangle \cap \langle S_2 \rangle)$.

Proof. *(i)*, *(ii)*. Obviously.

(iii). A deductive system of A that includes S also includes $\langle S \rangle$, so, $D(S) = D(\langle S \rangle)$.

(iv). First, we suppose that $\langle S_1 \rangle = \langle S_2 \rangle$. From *(iii)* we have $D(S_1) = D(\langle S_1 \rangle) = D(\langle S_2 \rangle) = D(S_2)$. Conversely, we suppose that $D(S_1) = D(S_2)$. If $\langle S_1 \rangle = A$, then $D(S_1) = D(\langle S_1 \rangle) = D(A) = Spec(A)$ and $D(S_2) = Spec(A)$ so, $\langle S_2 \rangle = A$. If we suppose that $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are proper filters of A, then applying Theorem 2.9 *(iii)*, we obtain

$$\begin{aligned} \langle S_1 \rangle &= \cap\{P \in Spec(A) : \langle S_1 \rangle \subseteq P\} = \cap\{P \in Spec(A) : P \notin D(\langle S_1 \rangle)\} \\ &= \cap\{P \in Spec(A) : P \notin D(\langle S_2 \rangle)\} = \cap\{P \in Spec(A) : \langle S_2 \rangle \subseteq P\} = \langle S_2 \rangle. \end{aligned}$$

(v). Follows from (iv) since $F, G \in Ds(A)$ implies $F = \langle F \rangle$ and $G = \langle G \rangle$.

(vi). Using (ii), we deduce that $\underset{i\in I}{\cup} D(S_i) \subseteq D(\underset{i\in I}{\cup} S_i)$. Conversely, let $P \in D(\underset{i\in I}{\cup} S_i)$. Then there exists $i \in I$ such that $S_i \nsubseteq P$. This is equivalent with $P \in D(S_i) \subseteq \underset{i\in I}{\cup} D(S_i)$. Thus $D(\underset{i\in I}{\cup} S_i) = \underset{i\in I}{\cup} D(S_i)$.

(vii). Follows from (iii) and (vi).

$(viii)$. Using (ii) we deduce that $D(\langle S_1 \rangle \cap \langle S_2 \rangle) \subseteq D(\langle S_1 \rangle) \cap D(\langle S_2 \rangle)$. Let $P \in D(\langle S_1 \rangle) \cap D(\langle S_2 \rangle)$. From Theorem 2.8(ii), $\langle S_1 \rangle \cap \langle S_2 \rangle \nsubseteq P$, so $P \in D(\langle S_1 \rangle \cap \langle S_2 \rangle)$. □

Theorem 3.2. *For a BCK-algebra A, the family $\tau_A = \{D(S) : S \subseteq A\}$ is a topology on $Spec(A)$ having $\{D(x) : x \in A\}$ as basis.*

Proof. Using Proposition 3.1 we deduce that τ_A is a topology on $Spec(A)$. For $S \subseteq A$, $S = \underset{x\in S}{\cup} \{x\}$, so $D(S) = D(\underset{x\in S}{\cup} \{x\}) = \underset{x\in S}{\cup} D(x)$. □

Definition 3.3. *The topology τ_A is called the Zariski topology on $Spec(A)$ and the topological space $(Spec(A), \tau_A)$ is called the prime spectrum of A.*

For $S \subseteq A$ and $x \in A$ we define $V(S) = Spec(A) \backslash D(S) = \{P \in Spec(A) : S \subseteq P\}$ and $V(x) = Spec(A) \backslash D(x) = \{P \in Spec(A) : x \in P\}$.

Proposition 3.4. *Let A be a bounded BCK-algebra and $S, S_1, S_2 \subseteq A$. Then the following assertions hold:*

(i) $V(0) = \emptyset$ *and* $V(\emptyset) = V(1) = Spec(A)$;

(ii) *if* $S_1 \subseteq S_2$, *then* $V(S_2) \subseteq V(S_1)$;

(iii) $V(S) = \emptyset$ *iff* $\langle S \rangle = A$;

(iv) $V(S) = Spec(A)$ *iff* $S = \emptyset$ *or* $S = \{1\}$;

(v) $V(S) = V(\langle S \rangle)$;

(vi) $V(S_1) = V(S_2)$ *iff* $\langle S_1 \rangle = \langle S_2 \rangle$;

(vii) *for* $F, G \in Ds(A)$, $V(F) = V(G)$ *iff* $F = G$;

$(viii)$ $V(S_1) \cup V(S_2) = V(\langle S_1 \rangle \cap \langle S_2 \rangle)$.

(ix) *if* $S_i \subseteq A, i \in I$, *then* $V(\underset{i\in I}{\cup} S_i) = \underset{i\in I}{\cap} V(S_i)$.

Proof. $(i), (ii), (v)$. Obviously.

(iii). Suppose that $V(S) = \emptyset$ and $\langle S \rangle \neq A$. By Theorem 2.9(i), there exists $P \in Spec(A)$ such that $S \subseteq \langle S \rangle \subseteq P$. We deduce that $P \in V(S)$, a contradiction. Conversely, we suppose that $\langle S \rangle = A$. If $V(S) \neq \emptyset$, then there is some $P \in Spec(A)$ such that $S \subseteq P$. Thus $\langle S \rangle \subseteq P \neq A$, a contradiction.

(iv). For $S = \emptyset$ or $S = \{1\}$, by (i), we deduce that $V(S) = Spec(A)$.

Conversely, we suppose that $V(S) = Spec(A)$ but $S \neq \emptyset$ and $S \neq \{1\}$. Then there is $s \in S, s \neq 1$. By Theorem 2.9(ii), there exists $P \in Spec(A)$ such that $s \notin P$. Thus, $S \nsubseteq P$, so $P \notin V(S)$. We conclude that $V(S) \neq Spec(A)$, a contradiction.

(vi). Let $S_1, S_2 \subseteq A$ such that $\langle S_1 \rangle = \langle S_2 \rangle$. Using (v), $V(S_1) = V(\langle S_1 \rangle) = V(\langle S_2 \rangle) = V(S_2)$. Conversely, let $S_1, S_2 \subseteq A$ such that $V(S_1) = V(S_2)$. Thus $D(S_1) = D(S_2)$, so by Proposition 3.1(iv), $\langle S_1 \rangle = \langle S_2 \rangle$.

(vii). Follows from (vi), since $F = \langle F \rangle$ and $G = \langle G \rangle$.

$(viii)$. From (ii) and (v), since $\langle S_1 \rangle \cap \langle S_2 \rangle \subseteq \langle S_1 \rangle, \langle S_2 \rangle$ we deduce that $V(S_1) = V(\langle S_1 \rangle) \subseteq V(\langle S_1 \rangle \cap \langle S_2 \rangle)$ and $V(S_2) \subseteq V(\langle S_1 \rangle \cap \langle S_2 \rangle)$. Thus, $V(S_1) \cup V(S_2) \subseteq V(\langle S_1 \rangle \cap \langle S_2 \rangle)$.

If $P \in V(\langle S_1 \rangle \cap \langle S_2 \rangle)$, then $P \in Spec(A)$ and $\langle S_1 \rangle \cap \langle S_2 \rangle \subseteq P$.

Using Theorem 2.8(ii), we deduce that $\langle S_1 \rangle \subseteq P$ or $\langle S_2 \rangle \subseteq P$. Hence $P \in V(\langle S_1 \rangle) \cup V(\langle S_2 \rangle) = V(S_1) \cup V(S_2)$. We conclude that, $V(\langle S_1 \rangle \cap \langle S_2 \rangle) = V(S_1) \cup V(S_2)$.

(ix). By duality from Proposition 3.1(vi). □

Proposition 3.5. *Let A be a bounded BCK-algebra and $x, y \in A$. Then the following hold:*

(i) *if $x \leq y$, then $D(y) \subseteq D(x)$;*

(ii) *$D(x) = \emptyset$ iff $x = 1$;*

(iii) *$D(x) = Spec(A)$ iff $\langle x \rangle = A$ iff $x \to_n 0 = 1$, for some $n \geq 1$;*

(iv) *$D(x^{**}) \cup D(y^{**}) = D(x \curlywedge y)$;*

(v) *$D(x) \cap D(y) = D(U(x, y))$;*

(vi) *$D(x) = D(y)$ iff $\langle x \rangle = \langle y \rangle$.*

Proof. (i). If $P \in D(y)$, then $y \notin P$. Clearly, $x \notin P$, since if $x \in P$, from $x \leq y$ we deduce that $y \in P$, a contradiction. So, $P \in D(x)$, that is, $D(y) \subseteq D(x)$.

(ii). $D(x) = \emptyset$ iff $V(x) = Spec(A)$ iff $x = 1$, by Proposition 3.4(iv).

(iii). $D(x) = Spec(A)$ iff $V(x) = \emptyset$ iff $\langle x \rangle = A$, by Proposition 3.4(iii), iff $0 \in \langle x \rangle$ iff $x \to_n 0 = 1$, for some $n \geq 1$.

(iv). Since $x \curlywedge y \leq x^{**}, y^{**}$, by (i), we deduce that $D(x^{**}), D(y^{**}) \subseteq D(x \curlywedge y)$, so, $D(x^{**}) \cup D(y^{**}) \subseteq D(x \curlywedge y)$. Let $P \in D(x \curlywedge y)$. Hence $x \curlywedge y \notin P$. Then $x^{**} \notin P$ or $y^{**} \notin P$ since if we suppose by contrary that $x^{**} \in P$ and $y^{**} \in P$, using Lemma 2.6 and (c_{12}) we deduce that $x^{**} \curlywedge y^{**} = x \curlywedge y \in P$, a contradiction. Thus, $P \in D(x^{**}) \cup D(y^{**})$ and $D(x \curlywedge y) \subseteq D(x^{**}) \cup D(y^{**})$. We conclude that $D(x^{**}) \cup D(y^{**}) = D(x \curlywedge y)$.

(v). Let $P \in D(x) \cap D(y)$. Thus, $x \notin P$ and $y \notin P$. If we suppose that $P \notin D(U(x, y))$, thus, $U(x, y) \subseteq P$, so by Theorem 2.8(iii), $x \in P$ or $y \in P$, a contradiction. Conversely, we suppose that $P \in D(U(x, y))$. Thus, $U(x, y) \nsubseteq P$, so there exists $z \in U(x, y)$ such that $z \geq x$, $z \geq y$ and $z \notin P$. If by contrary, $P \notin D(x) \cap D(y)$, then $x \in P$ or $y \in P$. Since $z \geq x$, y we deduce that $z \in P$, a contradiction. Hence $D(x) \cap D(y) = D(U(x, y))$.

(vi). Using Proposition 3.1(iv), $D(x) = D(y)$ iff $\langle x \rangle = \langle y \rangle$. □

Proposition 3.6. *Let A be a bounded BCK-algebra and $x, y \in A$. Then the following hold:*

(i) *if $x \leq y$, then $V(x) \subseteq V(y)$;*

(ii) *$V(x) = \emptyset$ iff $\langle x \rangle = A$ iff $x \to_n 0 = 1$, for some $n \geq 1$;*

(iii) *$V(x) = Spec(A)$ iff $x = 1$;*

(iv) *$V(x^{**}) \cap V(y^{**}) = V(x \curlywedge y)$;*

(v) $V(x) \cup V(y) = V(U(x, y))$;

(vi) $V(x) \subseteq D(x^*)$.

Proof. $(i) - (v)$. Follows from Proposition 3.5, $(i) - (vi)$.

(vi). If $P \in V(x)$, then $x \in P$. If by contrary, $x^* \in P$, then $0 \in P$, so, $P = A$, a contradiction. So, $x^* \notin P$, that is, $P \in D(x^*)$. Hence $V(x) \subseteq D(x^*)$. □

For a bounded BCK-algebra A, $Max(A) \subseteq Spec(A)$, so we can consider on $Max(A)$ the topology induced by the Zariski topology and we obtain a topological space called the *maximal spectrum* of A.

For $S \subseteq A$ and $x \in A$, we define $D_{Max}(S) = D(S) \cap Max(A) = \{M \in Max(A) : S \nsubseteq M\}$, $D_{Max}(x) = D(x) \cap Max(A) = \{M \in Max(A) : x \notin M\}$ and $V_{Max}(x) = V(x) \cap Max(A) = \{M \in Max(A) : x \in M\}$. Obviously, $D_{Max}(x) = Max(A) \backslash V_{Max}(x)$.

Theorem 3.7. *The set $\tau_{A,Max(A)} = \{D_{Max}(S) : S \subseteq A\}$ is the family of open sets of the maximal spectrum of A and the family $\{D_{Max}(x) : x \in A\}$ is a basis for the topology $\tau_{A,Max(A)}$ of $Max(A)$.*

Proposition 3.8. *Let A be a bounded BCK-algebra and $x, y, z \in A$. Then the following hold:*

(i) $V_{Max}(0) = \emptyset$, $V_{Max}(1) = Max(A)$, $D_{Max}(0) = Max(A)$, $D_{Max}(1) = \emptyset$;

(ii) *if* $x \leq y$, *then* $V_{Max}(x) \subseteq V_{Max}(y)$ *and* $D_{Max}(y) \subseteq D_{Max}(x)$;

(iii) $V_{Max}(x^{**}) = V_{Max}(x)$ *and* $D_{Max}(x^{**}) = D_{Max}(x)$;

(iv) $V_{Max}(x \curlywedge (y \sqcup z)) = V_{Max}((x \curlywedge y) \sqcup (x \curlywedge z))$;

(v) $V_{Max}(x) \cap V_{Max}(y) = V_{Max}(x \curlywedge y)$ *and* $D_{Max}(x) \cup D_{Max}(y) = D_{Max}(x \curlywedge y)$;

(vi) $V_{Max}(x) \cup V_{Max}(y) = V_{Max}(x \sqcup y)$ *and* $D_{Max}(x) \cap D_{Max}(y) = D_{Max}(x \sqcup y)$.

Proof. (i) and (ii). Follows from Propositions 3.5 and 3.6.

(iii). For $M \in Max(A)$, using Lemma 2.12, $x \in M$ iff $x^{**} \in M$. Thus, $V_{Max}(x^{**}) = V_{Max}(x)$ and $D_{Max}(x^{**}) = D_{Max}(x)$.

(iv). Let $M \in V_{Max}(x \curlywedge (y \sqcup z))$. Then $x \curlywedge (y \sqcup z) \in M$. Since $x \curlywedge (y \sqcup z) \leq x^{**}, (y \sqcup z)^{**}$, from Lemma 2.12, $x, y \sqcup z \in M$. But $M \in Max(A)$, so, from Theorem 2.11, $y \in M$ or $z \in M$. If $x, y \in M$, by Lemma 2.6, $x \curlywedge y \in M$, so, $(x \curlywedge y) \sqcup (x \curlywedge z) \in M$. Analogous if $x, z \in M$. We deduce that $M \in V_{Max}((x \curlywedge y) \sqcup (x \curlywedge z))$, so $V_{Max}(x \curlywedge (y \sqcup z)) \subseteq V_{Max}((x \curlywedge y) \sqcup (x \curlywedge z))$.

Conversely, let $M \in V_{Max}((x \curlywedge y) \sqcup (x \curlywedge z))$. We deduce that $(x \curlywedge y) \sqcup (x \curlywedge z) \in M$. Using Theorem 2.11, $x \curlywedge y \in M$ or $x \curlywedge z \in M$. Thus, $x^{**} \in M$ and y^{**} or $z^{**} \in M$. By Lemma 2.12, we have $x \in M$ and y or $z \in M$. Since $y, z \leq y \sqcup z$ we obtain $y \sqcup z \in M$ and from Lemma 2.6, $x \curlywedge (y \sqcup z) \in M$, so $M \in V_{Max}(x \curlywedge (y \sqcup z))$. We deduce that $V_{Max}((x \curlywedge y) \sqcup (x \curlywedge z)) \subseteq V_{Max}(x \curlywedge (y \sqcup z))$.

(v). From Proposition 3.6, we deduce that

$$V_{Max}(x) \cap V_{Max}(y) = V_{Max}(x^{**}) \cap V_{Max}(y^{**}) = V_{Max}(x \curlywedge y).$$

Then, $D_{Max}(x) \cup D_{Max}(y) = D_{Max}(x \curlywedge y)$.

(vi). Since $x, y \leq x \sqcup y$, by (ii), we deduce that $V_{Max}(x), V_{Max}(y) \subseteq V_{Max}(x \sqcup y)$ so, $V_{Max}(x) \cup V_{Max}(y) \subseteq V_{Max}(x \sqcup y)$. Conversely, let $M \in V_{Max}(x \sqcup y)$. Using Theorem 2.11, we deduce that $x \in M$ or $y \in M$. Hence $M \in V_{Max}(x) \cup V_{Max}(y)$, so, $V_{Max}(x) \cup V_{Max}(y) = V_{Max}(x \sqcup y)$. We conclude that $D_{Max}(x) \cap D_{Max}(y) = D_{Max}(x \sqcup y)$. □

Proposition 3.9. *Let A be a bounded BCK-algebra with Glivenko property. Then $D_{Max}(x)$ is a compact set in $Max(A)$, for every $x \in A$.*

Proof. We prove that any cover of $D_{Max}(x)$ with basic open sets contains a finite cover of $D_{Max}(x)$. Let $D_{Max}(x) = \underset{i\in I}{\cup} D_{Max}(x_i)$. Using Proposition 3.1, (vi), $D_{Max}(x) = D_{Max}(\underset{i\in I}{\cup}\{x_i\})$. From Proposition 3.1(iv), we deduce that $\langle x\rangle = \langle\{x_i : i \in I\}\rangle$, so, $x \in \langle\{x_i : i \in I\}\rangle$. Using Theorem 2.7, there are $n \geq 1$ and $i_1, ..., i_n \in I$ such that $(x_{i_1}, x_{i_2}, ..., x_{i_n}; x) = 1$.

We prove that $D_{Max}(x) = D_{Max}(x_{i_i}) \cup ... \cup D_{Max}(x_{i_n})$.

From $(x_{i_1}, x_{i_2}, ..., x_{i_n}; x) = 1$, using (c_{18}) we deduce that $x_{i_i} \curlywedge ... \curlywedge x_{i_n} \leq x^{**}$, so, by Proposition 3.8, we obtain

$$D_{Max}(x) = D_{Max}(x^{**}) \subseteq D_{Max}(x_{i_i} \curlywedge ... \curlywedge x_{i_n}) = D_{Max}(x_{i_i}) \cup ... \cup D_{Max}(x_{i_n}).$$

Since $D_{Max}(x_{i_i}) \cup ... \cup D_{Max}(x_{i_n}) \subseteq \underset{i\in I}{\cup} D_{Max}(x_i) = D_{Max}(x)$, the other inclusion is obvious. □

Theorem 3.10. *If A is a bounded BCK-algebra with Glivenko property, then $Max(A)$ is a compact topological space.*

Proof. Since $Max(A) = D_{Max}(0)$, by Proposition 3.9 we deduce that $Max(A)$ is compact. □

4 The Belluce lattice associated with a bounded BCK-algebra

Let L be a bounded lattice. A nonempty subset F of L is called *a filter* of L ([2]) if it satisfies:

(f_1) $1 \in F$;

(f_2) if $x, y \in F$, then $x \wedge y \in F$;

(f_3) if $x \in F, y \in L$, and $x \leq y$, then $y \in F$.

The set of all filters of L is denoted by $F(L)$; if L is a distributive lattice, then $(F(L), \subseteq)$ is also a distributive lattice, see [2]. A filter F of L is called *proper* if $F \neq L$.

For a distributive lattice L and $P \in F(L)$, $P \neq L$, the following are equivalent: [P is a meet-prime element in $F(L)$] iff [P is a meet-irreducible element in $F(L)$] iff [for every $x, y \in L$ if $x \vee y \in P$, then $x \in P$ or $y \in P$].

A proper filter P of a distributive lattice L is called *prime* if it verifies one of the above equivalent conditions, see [2]. The set of all prime filters of L is denoted by $Spec(L)$ and it is called the *prime spectrum* of L. For $S \subseteq L, x \in L$ we denote $D(S) = \{P \in Spec(L) : S \nsubseteq P\}$ and $D(x) = \{P \in Spec(L) : x \notin P\}$. It is known that the family $\{D(S) : S \subseteq L\}$ is a topology on $Spec(L)$ and the family $\{D(x) : x \in L\}$ is a basis for this topology.

Also, we recall that a proper filter M of a lattice L is called *maximal* (see [2]) if it is a maximal element of the set of all proper filters of L. The set of all maximal filters of L is called the *maximal spectrum* of L and it is denoted by $Max(L)$.

In a lattice L for $S \subseteq L$ and $x \in L$ we denote $D_{Max}(S) = \{M \in Max(L) : S \nsubseteq M\}$ and $D_{Max}(x) = \{M \in Max(L) : x \notin M\}$. If L is distributive, since $Max(L) \subseteq Spec(L)$, the family $\{D_{Max}(S) : S \subseteq L\}$ is a topology on $Max(L)$ having $\{D_{Max}(x) : x \in L\}$ as a basis.

Now let A be a bounded BCK-algebra. We define a binary relation $\equiv$ on A as follows: for $x, y \in A$, $x \equiv y$ iff for any $M \in Max(A)$, ($x \notin M$ iff $y \notin M$) iff for any $M \in Max(A)$, ($x \in M$ iff $y \in M$).

Remark 4.1. *From Proposition 3.5, for $x, y \in A$, $x \equiv y$ iff $V_{Max}(x) = V_{Max}(y)$ iff $D_{Max}(x) = D_{Max}(y)$ iff $\langle x \rangle = \langle y \rangle$.*

Proposition 4.2. *$\equiv$ is a congruence relation on A with respect to $\curlywedge$ and $\sqcup$.*

Proof. It is obvious that $\equiv$ is an equivalence relation on A. Let $x, y, z, t \in A$ such that $x \equiv y$ and $z \equiv t$. We prove that $x \curlywedge z \equiv y \curlywedge t$ and $x \sqcup z \equiv y \sqcup t$.

Let $M \in Max(A)$. If $x \curlywedge z \in M$, since by (c_8), $x \curlywedge z \leq x^{**}, z^{**}$ then $x^{**}, z^{**} \in M$. From Lemma 2.12, we deduce that $x, z \in M$. Since $x \equiv y$ and $z \equiv t$ we have $y, t \in M$. By Lemma 2.6 we obtain $y \curlywedge t \in M$.

If $x \sqcup z \in M$, by Theorem 2.11, $x \in M$ or $z \in M$. Since $x \equiv y$ and $z \equiv t$ we deduce that y or $t \in M$, hence $y \sqcup t \in M$, since $y, t \leq y \sqcup t$. □

For $x \in A$ we denote by $[x]$ the congruence class of x and by L_A the quotient set $L_A = A/\equiv$ $= \{[x] : x \in A\}$. Also, let $p_A : A \to L_A$ be the canonical surjection defined by $p_A(x) = [x]$, for every $x \in A$.

Obviously, on L_A the relation $[x] \sqsubseteq [y]$ iff for every $M \in Max(A)$, $x \in M$ implies $y \in M$ is an order relation on A.

Proposition 4.3. *Let A be a bounded BCK-algebra and $x, y \in A$. The following assertions hold:*

(i) $[x] = [x^{**}]$;

(ii) *if $x \leq y$, then $[x] \sqsubseteq [y]$;*

(iii) $[x] \sqsubseteq [y]$ *iff* $[x \curlywedge y] = [x]$;

(iv) $[x] \sqsubseteq [y]$ *iff* $[x \sqcup y] = [y]$.

Proof. (i). Follows from Lemma 2.12.

(ii). Let $M \in Max(L)$ such that $x \in M$. Since $x \leq y$ we deduce that $y \in M$, so, $[x] \sqsubseteq [y]$.

(iii). Suppose that $[x] \sqsubseteq [y]$. Since $x \curlywedge y \leq x^{**}$, by (i) and (ii) we deduce that $[x \curlywedge y] \sqsubseteq [x^{**}] = [x]$. Now, let $M \in Max(L)$ such that $x \in M$. Since $[x] \sqsubseteq [y]$ we deduce that $y \in M$. Using Lemma 2.6, $x \curlywedge y \in M$, so, $[x] \sqsubseteq [x \curlywedge y]$. We conclude that $[x \curlywedge y] = [x]$. Conversely, we suppose that $[x \curlywedge y] = [x]$. Since $x \curlywedge y \leq y^{**}$, using (ii), we have $[x \curlywedge y] \sqsubseteq [y^{**}] = [y]$. Thus, $[x] \sqsubseteq [y]$.

(iv). If $[x] \sqsubseteq [y]$, since $y \leq x \sqcup y$, from (ii) we deduce that $[y] \sqsubseteq [x \sqcup y]$. Now, let $M \in Max(L)$ such that $x \sqcup y \in M$. From Theorem 2.11, $x \in M$ or $y \in M$. If $y \in M$, then $[x \sqcup y] \sqsubseteq [y]$, so, $[x \sqcup y] = [y]$. If $x \in M$, since $[x] \sqsubseteq [y]$, we deduce that $y \in M$, so, $[x \sqcup y] = [y]$. Conversely, suppose that $[x \sqcup y] = [y]$ and let $M \in Max(L)$ such that $x \in M$. Since $x \leq x \sqcup y$ we obtain that $x \sqcup y \in M$, so $y \in M$. Thus, $[x] \sqsubseteq [y]$. □

Theorem 4.4. *$(L_A, \wedge, \vee, [0], [1])$ is a bounded distributive lattice, relative to the above order, in which $[x] \wedge [y] = [x \curlywedge y]$ and $[x] \vee [y] = [x \sqcup y]$, for every $x, y \in A$.*

Proof. Obviously, $[x \curlywedge y] \sqsubseteq [x], [y]$, for every $x, y \in A$. Let $z \in A$ such that $[z] \sqsubseteq [x], [y]$. To prove that $[z] \sqsubseteq [x \curlywedge y]$ we consider $M \in Max(A)$ such that $z \in M$. By definition we deduce that $x, y \in M$, hence, using Lemma 2.6, $x \curlywedge y \in M$. Thus, $[x] \wedge [y] = [x \curlywedge y]$.

Clearly, $[x], [y] \sqsubseteq [x \sqcup y]$. Let $z \in A$ such that $[x], [y] \sqsubseteq [z]$. To prove that $[x \sqcup y] \sqsubseteq [z]$ we consider $M \in Max(A)$ such that $x \sqcup y \in M$. By Theorem 2.11 we deduce that $x \in M$ or $y \in M$. In both cases, $z \in M$, hence $[x] \vee [y] = [x \sqcup y]$.

Since $[0] \wedge [x] = [0 \curlywedge x] = [0]$ and $[x] \wedge [1] = [x \curlywedge 1] = [x^{**}] = [x]$ we deduce that $[0] \sqsubseteq [x] \sqsubseteq [1]$, for every $x \in A$, so $(L_A, \wedge, \vee, [0], [1])$ is a bounded lattice.

To prove the distributivity of L_A, let $x, y, z \in A$. We show that $[x] \wedge ([y] \vee [z]) = ([x] \wedge [y]) \vee ([x] \wedge [z])$. This is equivalent to show that $[x \curlywedge (y \sqcup z)] = [(x \curlywedge y) \sqcup (x \curlywedge z)]$. First, let $M \in Max(A)$ such that $x \curlywedge (y \sqcup z) \in M$. Thus, $M \in V_{Max}(x \curlywedge (y \sqcup z)) = V_{Max}(x) \cap V_{Max}(y \sqcup z)$. Hence $x \in M$ and ($y \in M$ or $z \in M$). If $x, y \in M$, then $x \curlywedge y \in M$, so, $(x \curlywedge y) \sqcup (x \curlywedge z) \in M$. Similarly if $x, z \in M$. We conclude that $[x \curlywedge (y \sqcup z)] \sqsubseteq [(x \curlywedge y) \sqcup (x \curlywedge z)]$. Conversely, let $M \in Max(A)$ such that $(x \curlywedge y) \sqcup (x \curlywedge z) \in M$. Thus, $x \curlywedge y \in M$ or $x \curlywedge z \in M$. Since $x \curlywedge y, x \curlywedge z \leq x \curlywedge (y \sqcup z)$ we deduce that $x \curlywedge (y \sqcup z) \in M$. Thus, $[(x \curlywedge y) \sqcup (x \curlywedge z)] \sqsubseteq [x \curlywedge (y \sqcup z)]$. We conclude that L_A is a distributive lattice. □

Definition 4.5. For a bounded BCK-algebra A, *the bounded distributive lattice* L_A is called the *Belluce lattice associated with* A.

Proposition 4.6. *Let A be a bounded BCK-algebra and $x, y \in A$. Then the following assertions hold:*

(*i*) $[x] \sqsubseteq [y]$ *iff* $D_{Max}(y) \subseteq D_{Max}(x)$;

(*ii*) $[x] = [y]$ *iff* $\langle x \rangle = \langle y \rangle$;

(*iii*) $[x] = [0]$ *iff* $x \rightarrow_n 0 = 1$ *for some* $n \geq 1$;

(*iv*) $[x] = [1]$ *iff* $x = 1$.

Proof. (*i*). We have $[x] \sqsubseteq [y]$ iff $[x \curlywedge y] = [x]$ iff $D_{Max}(x) = D_{Max}(x \curlywedge y) = D_{Max}(x) \cup D_{Max}(y)$ iff $D_{Max}(y) \subseteq D_{Max}(x)$.

(*ii*). Follows from Remark 4.1.

(*iii*). By (*ii*), $[x] = [0]$ iff $\langle x \rangle = \langle 0 \rangle = A$ iff $x \rightarrow_n 0 = 1$, for some $n \geq 1$.

(*iv*). By (*ii*), $[x] = [1]$ iff $\langle x \rangle = \langle 1 \rangle$ iff $\langle x \rangle = \{1\}$ iff $x = 1$. □

We recall that if A and B are two BCK-algebras, then $f : A \rightarrow B$ is a *morphism* of BCK-algebras if $f(x \rightarrow y) = f(x) \rightarrow f(y)$, for every $x, y \in A$. If A and B are bounded BCK-algebras, we ask that $f(0) = 0$, see [12].

We denote by $\overline{\mathcal{BCK}}$ the category of bounded BCK-algebras and by $\mathbf{Ld(0,1)}$ the category of bounded distributive lattices.

Remark 4.7. *If $f : A \rightarrow B$ is a morphism in $\overline{\mathcal{BCK}}$, then for every $x, y \in A$, $f(x^*) = (f(x))^*$, $f(x \curlywedge y) = f(x) \curlywedge f(y)$ and $f(x \sqcup y) = f(x) \sqcup f(y)$.*

Proposition 4.8. *Let $f : A \rightarrow B$ be a morphism in $\overline{\mathcal{BCK}}$.*

(*i*) *If $D \in Ds(B)$, then $f^{-1}(D) \in Ds(A)$ and if D is proper, then $f^{-1}(D)$ is also proper;*

(*ii*) *If $M \in Max(B)$, then $f^{-1}(M) \in Max(A)$;*

(*iii*) *If $x, y \in A$ such that $D_{Max}(x) = D_{Max}(y)$, then $D_{Max}(f(x)) = D_{Max}(f(y))$.*

Proof. (*i*). For $D \in Ds(B)$, since $f(1) = 1$ we deduce that $1 \in f^{-1}(D)$. Let $x, y \in A$ such that $x, x \rightarrow y \in f^{-1}(D)$. Then $f(x), f(x \rightarrow y) = f(x) \rightarrow f(y) \in D$. Since $D \in Ds(B)$ we deduce that $f(y) \in D$, hence $y \in f^{-1}(D)$, that is, $f^{-1}(D) \in Ds(A)$. If D is proper, then $D \neq B$, so $0 \notin D$. If

$f^{-1}(D) = A$, then $0 \in f^{-1}(D)$, hence $0 = f(0) \in D$, a contradiction. We deduce that $f^{-1}(D)$ is a proper filter of A.

(ii). For $M \in Max(B)$, using (i), $f^{-1}(M) \neq A$. To prove that $f^{-1}(M) \in Max(A)$, let $x \in A$ such that $x \notin f^{-1}(M)$. By Theorem 2.10, there exists $n \geq 1$ such that $f(x \rightarrow_n 0) = f(x) \rightarrow_n 0 \in M$. Thus $x \rightarrow_n 0 \in f^{-1}(M)$, so, $f^{-1}(M) \in Max(A)$.

(iii). For $M \in Max(B)$, using (ii), $f^{-1}(M) \in Max(A)$. We have $M \in D_{Max}(f(x))$ iff $f(x) \notin M$ iff $x \notin f^{-1}(M)$ iff $f^{-1}(M) \in D_{Max}(x)$ iff $f^{-1}(M) \in D_{Max}(y)$ iff $y \notin f^{-1}(M)$ iff $f(y) \notin M$ iff $M \in D_{Max}(f(y))$. We deduce that $D_{Max}(f(x)) = D_{Max}(f(y))$. □

Theorem 4.9. *Let $f : A \rightarrow B$ be a morphism in $\overline{\mathcal{BCK}}$. Then $\mathcal{R}(f) : L_A \rightarrow L_B$ defined by $\mathcal{R}(f)([x]) = [f(x)]$, for every $x \in A$, is a morphism in $\mathbf{Ld(0,1)}$ with the property that $p_B \circ f = \mathcal{R}(f) \circ p_A$.*

Proof. By Proposition 4.8 (iii), we deduce that $R(f)$ is well-defined. Clearly, $\mathcal{R}(f)([0]) = [f(0)] = [0]$ and $\mathcal{R}(f)([1]) = [f(1)] = [1]$. Let $x, y \in A$. We have

$$\mathcal{R}(f)([x] \wedge [y]) = \mathcal{R}(f)([x \curlywedge y]) = [f(x \curlywedge y)] = [f(x) \curlywedge f(y)] = [f(x)] \wedge [f(y)] = \mathcal{R}(f)([x]) \wedge \mathcal{R}(f)([y]),$$

and

$$\mathcal{R}(f)([x] \vee [y]) = \mathcal{R}(f)([x \sqcup y]) = [f(x \sqcup y)] = [f(x) \sqcup f(y)] = [f(x)] \vee [f(y)] = \mathcal{R}(f)([x]) \vee \mathcal{R}(f)([y]).$$

We deduce that $\mathcal{R}(f)$ is a morphism in $\mathbf{Ld(0,1)}$.

Since $p_A(x) = [x]$ and $p_B(f(x)) = [f(x)]$ we deduce that $\mathcal{R}(f)(p_A(x)) = p_B(f(x))$, so $(\mathcal{R}(f) \circ p_A)(x) = (p_B \circ f)(x)$, for every $x \in A$. Thus, $p_B \circ f = \mathcal{R}(f) \circ p_A$. □

For every $A \in Ob(\overline{\mathcal{BCK}})$ we denote $\mathcal{R}(A) = L_A$. In this way, we define a functor $\mathcal{R} : \overline{\mathcal{BCK}} \rightarrow \mathbf{Ld(0,1)}$ and we called $\mathcal{R}$ the *reticulation functor*.

Lemma 4.10. *Let $f : A \rightarrow B$ be an injective morphism in $\overline{\mathcal{BCK}}$ and $x, y \in A$ such that $\langle f(x) \rangle = \langle f(y) \rangle$. Then $\langle x \rangle = \langle y \rangle$.*

Proof. Let $z \in \langle x \rangle$. Then $x \rightarrow_n z = 1$ for some $n \geq 1$ and $f(x) \rightarrow_n f(z) = f(1) = 1$. Thus, $f(z) \in \langle f(x) \rangle = \langle f(y) \rangle$, so there exists $m \geq 1$ such that $f(y) \rightarrow_m f(z) = 1$. Hence, $f(y \rightarrow_m z) = f(1)$. Since f is injective we deduce that $y \rightarrow_m z = 1$, so, $z \in \langle y \rangle$. Hence $\langle x \rangle \subseteq \langle y \rangle$. Similarly, $\langle y \rangle \subseteq \langle x \rangle$, so $\langle x \rangle = \langle y \rangle$. □

Theorem 4.11. *The reticulation functor $\mathcal{R}$ preserves injective and surjective morphisms.*

Proof. Let $f : A \rightarrow B$ be an injective morphism in $\overline{\mathcal{BCK}}$ and $x, y \in A$ such that $\mathcal{R}(f)([x]) = \mathcal{R}(f)([y])$. Then $[f(x)] = [f(y)]$ and using Proposition 4.6(ii), we obtain $\langle f(x) \rangle = \langle f(y) \rangle$. Since f is injective, by Lemma 4.10, $\langle x \rangle = \langle y \rangle$, hence $[x] = [y]$. We deduce that $\mathcal{R}(f)$ is injective.

Now, let $f : A \rightarrow B$ be a surjective morphism in $\overline{\mathcal{BCK}}$ and we consider $y \in B$. Then there exists $x \in A$ such that $y = f(x)$. We obtain $\mathcal{R}(f)([x]) = [f(x)] = [y]$, that is, $\mathcal{R}(f)$ is surjective. □

We recall that for a set T we denote $\mathcal{P}(T) = \{X : X \subseteq T\}$.

Using this notation, for a bounded BCK-algebra A, we consider the map $p_A^* : \mathcal{P}(L_A) \rightarrow \mathcal{P}(A)$, $p_A^*(S) = p_A^{-1}(S) = \{x \in A : p_A(x) = [x] \in S\}$, for every $S \subseteq L_A$.

Remark 4.12. *Since p_A is a surjective map, we get p_A^* is one-to-one and $p_A(p_A^*(S)) = S$, for every $S \subseteq L_A$.*

Theorem 4.13. *Let A be a bounded BCK-algebra.*

(i) *If $F \in F(L_A)$, then $p_A^*(F) \in Ds(A)$ and if F is proper, then $p_A^*(F)$ is also proper;*

(ii) *If $M \in Max(A)$, then $p_A(M) \in Max(L_A)$.*

Proof. (i). Obviously, $1 \in p_A^*(F)$ since $p_A(1) = [1] \in F$. Let $x, y \in A$ such that $x, x \to y \in p_A^*(F)$. Then $[x], [x \to y] \in F$, hence $[x] \wedge [x \to y] = [x \curlywedge (x \to y)] \in F$. Using (c_{12}), $x \curlywedge (x \to y) \leq y^{**}$, so, by Proposition 4.3, $[x \curlywedge (x \to y)] \sqsubseteq [y^{**}] = [y]$. We deduce that $[y] \in F$, so $y \in p_A^*(F)$ and $p_A^*(F) \in Ds(A)$. If F is proper, then $F \neq L_A$. Since p_A^* is one-to-one we deduce that $p_A^*(F) \neq p_A^*(L_A) = A$, so $p_A^*(F)$ is proper.

(ii). Since $M \in Max(A)$ we have $M \neq A$, so, there exists $x \in A \backslash M$. If $p_A(M) = L_A$, then $p_A^*(p_A(M)) = p_A^*(L_A) = A$. Thus $x \in p_A^*(p_A(M))$, hence $p_A(x) = [x] \in p_A(M)$, so there exists $y \in M$ such that $[x] = [y]$. Since $x \equiv y$ and $y \in M$ we deduce that $x \in M$, a contradiction. Thus $M \neq A$ implies $p_A(M) \neq L_A$. To prove $p_A(M) \in F(L_A)$, obviously $[1] = p_A(1) \in p_A(M)$ and let $\alpha, \beta \in p_A(M)$, that is, $\alpha = [x], \beta = [y]$ with $x, y \in M$. We have $\alpha \wedge \beta = [x] \wedge [y] = [x \curlywedge y]$. Using Lemma 2.6, $x \curlywedge y \in M$, so $\alpha \wedge \beta \in p_A(M)$. Now, let $\alpha \in p_A(M)$ and $\beta \in L_A$ such that $\alpha \sqsubseteq \beta$. Then $\alpha = [x]$, $x \in M$ and $\beta = [y]$, $y \in A$. Since $\alpha \sqsubseteq \beta$, we have $\alpha = \alpha \wedge \beta = [x] \wedge [y] = [x \curlywedge y]$, hence $x \equiv (x \curlywedge y)$. But $x \in M$ so $x \curlywedge y \in M$ and $M \in V_{Max}(x \curlywedge y) = V_{Max}(x) \cap V_{Max}(y)$. Thus, $M \in V_{Max}(y)$, so, $y \in M$. Hence $\beta = [y] \in p_A(M)$ and $p_A(M) \in F(L_A)$. To prove that $p_A(M) \in Max(L_A)$, let $F \in F(L_A)$ such that $p_A(M) \subseteq F$. Then $p_A^*(p_A(M)) \subseteq p_A^*(F)$. Since $M \subseteq p_A^*(p_A(M))$ we have $M \subseteq p_A^*(F)$. Since $p_A^*(F) \in Ds(A)$ and $M \in Max(A)$ we obtain $M = p_A^*(F)$ or $p_A^*(F) = A$. If $p_A^*(F) = A$, then $p_A(p_A^*(F)) = p_A(A) = L_A$, hence by Remark 4.12, $F = L_A$. If $M = p_A^*(F)$, then $p_A(M) = p_A(p_A^*(F)) = F$. So, $p_A(M) \in Max(L_A)$. □

5 The reticulation of a bounded BCK-algebra

Definition 5.1. *A reticulation of a bounded BCK-algebra A is a pair (L, λ), where $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\lambda : A \to L$ is a surjective map that satisfies the following conditions for every $x, y \in A$:*

(r_1) *$\lambda(0) = 0$, $\lambda(1) = 1$, $\lambda(x \curlywedge y) = \lambda(x) \wedge \lambda(y)$ and $\lambda(x \sqcup y) = \lambda(x) \vee \lambda(y)$;*

(r_2) *$\lambda(x) = \lambda(y)$ iff $\langle x \rangle = \langle y \rangle$.*

Theorem 5.2. *Let A be a bounded BCK-algebra. If (L_1, λ_1) and (L_2, λ_2) are two reticulations of A, then there exists a unique isomorphism of bounded lattices $f : L_1 \to L_2$ such that $f \circ \lambda_1 = \lambda_2$.*

Proof. Let $z \in L_1$ and $x \in A$ such that $z = \lambda_1(x)$. We define $f(z) = \lambda_2(x)$. Obviously, $f \circ \lambda_1 = \lambda_2$. If $x_1, x_2 \in A$ such that $z_1 = \lambda_1(x_1)$ and $z_2 = \lambda_1(x_2)$, using (r_2) we have $\lambda_1(x_1) = \lambda_1(x_2)$ iff $\langle x_1 \rangle = \langle x_2 \rangle$ iff $\lambda_2(x_1) = \lambda_2(x_2)$.

These implications prove that f is well-defined and injective. The surjectivity of λ_2 implies that f is surjective. We conclude that f is bijective. Also, we have $f(0) = f(\lambda_1(0)) = \lambda_2(0) = 0$ and $f(1) = f(\lambda_1(1)) = \lambda_2(1) = 1$.

Let $x, y \in L_1$. Since λ_1 is surjective, there are $a, b \in A$ such that $x = \lambda_1(a)$ and $y = \lambda_1(b)$. Applying (r_1) we obtain the following equalities:

$$
\begin{aligned}
f(x \wedge y) &= f(\lambda_1(a) \wedge \lambda_1(b)) = f(\lambda_1(a \curlywedge b)) = \lambda_2(a \curlywedge b) \\
&= \lambda_2(a) \wedge \lambda_2(b) = f(\lambda_1(a)) \wedge f(\lambda_1(b)) = f(x) \wedge f(y),
\end{aligned}
$$

and analogous $f(x \vee y) = f(x) \vee f(y)$. We conclude that f is an isomorphism in $\mathbf{Ld(0,1)}$ such that $f \circ \lambda_1 = \lambda_2$.

If we have two isomorphisms of bounded lattices $f, g : L_1 \to L_2$ such that $f \circ \lambda_1 = g \circ \lambda_1 = \lambda_2$, then for $y \in L_1$ there exists $x \in A$ such that $y = \lambda_1(x)$. We have $f(y) = f(\lambda_1(x)) = \lambda_2(x)$ and $g(y) = g(\lambda_1(x)) = \lambda_2(x) = f(y)$, hence $f(y) = g(y)$ for every $y \in L_1$. We conclude that $f = g$. $\square$

From Theorem 4.4 and Proposition 4.6 we deduce that:

Corollary 5.3. *If A is a bounded BCK-algebra, then the pair (L_A, p_A) is a reticulation of A.*

We recall that in Section 4, for $x \in A$ we defined $V_{Max}(x) = \{M \in Max(A) : x \in M\}$.

Now, we consider $S_A = \{V_{Max}(x) : x \in A\} \subseteq \mathcal{P}(Max(A))$.

Following Proposition 3.8 and Remark 4.1 we deduce that S_A is a distributive sublattice of the lattice $(\mathcal{P}(Max(A)), \subseteq)$ and

Corollary 5.4. *If A is a bounded BCK-algebra, then the pair (S_A, V_{Max}) is a reticulation of A.*

From Theorem 5.2, Corollaries 5.3 and 5.4 we obtain:

Corollary 5.5. *The lattices L_A and S_A are isomorphic.*

6 Conclusion

We have introduced the concept of Belluce lattice L_A associated with a bounded BCK algebra A, that enables us to transfer many properties between L_A and A. Moreover, we gave a description of the reticulation for a bounded BCK algebra and we proved the uniqueness of this reticulation.

For future work, we could generalize these results to the non-commutative case.

References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Series in Mathematics 361, Addison-Wesley (Reading, M A), 1969.

[2] R. Balbes, P. Dwinger, *Distributive lattices*, University of Missouri Press, XIII, 1974.

[3] L.P. Belluce, *Semisimple algebras of infinite-valued logic and bold fuzzy set theory*, Canadian Journal of Mathematics, 38(6) (1986), 1356–1379.

[4] L.P. Belluce, *Spectral spaces and non-commutative rings*, Communications in Algebra, 19 (1991), 1855–1865.

[5] D. Buşneag, D. Piciu, *The Belluce lattice associated with a bounded Hilbert algebra*, Soft Computing, 19 (2015), 3031–3042.

[6] S.A. Celani, *Deductive systems of BCK-algebras,* Acta Univ. Palackianae Olomoucensis, Facultas Rervoum Naturalium Mathematisa, 43(1) (2004), 27–32.

[7] L. Ciungu, *Non-commutative multiple-valued logic algebras*, Springer International Publishing, 2014.

[8] G. Georgescu, *The reticulation of a quantale*, Revue Roumaine des Mathematiques Pures et Appliquees, 7-8 (1995), 619–631.

[9] M. Ghita, *Contributions to the study of the category of Hilbert algebras*, Ph. D. Thesis, University of Craiova, 2011.

[10] K. Iseki, *On some ideals in BCK-algebras*, Mathematics Seminar Notes, 3 (1975), 65–70.

[11] K. Isèki, S. Tanaka, *An introduction to the theory of BCK-algebras*, Mathematica Japonica, 23(1) (1978), 1–26.

[12] J. Kühr, *Pseudo BCK-algebras and related structures*, Univerzita Palackého v Olomouci, 2007.

[13] L. Leustean, *The prime and maximal spectra and the reticulation of BL-algebras*, Central European Journal of Mathematics, 3 (2003), 382–397.

[14] C. Mureşan, *The reticulation of a residuated lattice*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, 51(1) (2008), 47–65.

[15] C. Mureşan, *Characterization of the reticulation of a residuated lattice*, Journal of Multiple-Valued Logic and Soft Computing, 16 (2010), 427–447.

[16] A.B. Saeid, C. Flaut, S. Hoskova-Mayerova, M. Afshar, *Some connections between BCK-algebras and n-ary block codes*, Soft Computing, 22(1) (2018), 41–46.

[17] H. Simmons, *Reticulated rings*, Journal of Algebra, 66 (1980), 169–192.