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Connections between reversible regular hypergroups, t-fuzzy subgroups and t-fuzzy graphs

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Abstract

In this paper, we obtain a reversible regular hypergroup from fuzzy sets by using a t-norm. Some properties of isomorphism of t-fuzzy graphs are considered and we show that a t-fuzzy subgroup can be associated with a t-fuzzy graph. Finally, using the group of automorphisms of fuzzy graph, we explain the relationship between the hypergroup and the t-fuzzy subgroup with the t-fuzzy graph.

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1 Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of hyperstructures was first introduced by Marty [21] at the eighth congress of Scandinavian Mathematicians in 1934. Nowadays, the well-written books for the introduction to hyperstructures, which include hypergroups [9], application of hyperstructures [10], semihypergroups [12], hyperrings [14], polygroups [11] and weak hyperstructures [15, 32]. A more recent book [13] gives an introduction into fuzzy algebraic hyperstructures.

In [28], Rosenfield defined the notion of a fuzzy subgroup. Many authors have worked on fuzzy subgroup theory [22, 26, 27, 33]. Anthony and Sherwood [2] gave a definition of fuzzy subgroup

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based on t-norm. Borzooei et al. [3] considered the fuzzy subgroups with respect to a t-norm and gave some results.

Rosenfield et al. [6, 7, 20, 24, 29] introduced the concept of fuzzy graphs and obtained the fuzzy analogs of several basic graph theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. Fuzzy matrices were introduced for the first time by Thomason [31] and several authors have presented a number of results on the convergence of the power sequence of fuzzy matrices [16, 18]. Ragab et al. [25] presented some properties on determinant and adjoint of square fuzzy matrix. Kim, Roush and Hashimoto [16, 19] studied the canonical form of an idempotent matrix. In 2018, Mordeson et al. [20, 23, 24] introduced the notion of t-fuzzy graph and obtained t-norm fuzzy incidence graphs. Borzooei et al. [8] considered all types of t-fuzzy graphs based on [23] and they obtained adjacency matrix of t-fuzzy graphs.

Ameri, Zahedi and Sen [1, 30] considered a fuzzy subset μ of a group G and the fuzzy left (right) cosets of μ in G and defined the hyperoperation on G. They obtained hypergroup and polygroup when μ is subnormal. Bhattacharya, Bhutani and Y.B Jun in [4, 5, 17] obtained other graph-theoretic results concerning center and eccentricity and showed that with a given fuzzy graph we can associate a fuzzy subgroup in a natural way as an automorphism group.

The organization of this paper is as follows: In Section 2, some preliminary definitions and concepts are given. In Section 3, we obtain a reversible regular hypergroup from a t-fuzzy subgroup. In Section 4, after expressing several proposition on how to generate a t-fuzzy subgroup by a t-fuzzy graph, and finally by Sections 3 and 4, we get a reversible regular hypergroup obtain from t-fuzzy graph.

2 Preliminaries

In this section, we introduce some preliminary notions and definitions which be used in this paper.

First, we present notions of hypergroups [21] which will be used during this paper. More exactly, we define the notion of hypergroup, regular hypergroup and reversible hypergroup.

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a map $\circ : H \times H \longrightarrow P^*(H)$ and the couple $(H, \circ)$ is called a hypergroupoid. For non-empty subsets A and B of H, $A \circ B = \bigcup\limits_{a\in A, b\in B} a \circ b$.

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for x, y, z of H we have $(x\circ y)\circ z = x\circ(y\circ z)$, which means that $\bigcup\limits_{u\in x\circ y} u \circ z = \bigcup\limits_{v\in y\circ z} x \circ v$.

A semihypergroup $(H, \circ)$ is called a hypergroup if for all $x \in H$, we have $x \circ H = H \circ x = H$.

Recall that a regular hypergroup $(H, \circ)$ is a hypergroup which has at least an identity and any element of H has at least an inverse. In other words, there exists $e \in H$ such that for all $x \in H$, we have $x \in x \circ e \cap e \circ x$ and there exists $x^{-1} \in H$ such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$.

Also, a reversible hypergroup $(H, \circ)$ is a hypergroup and for all $x, y, z \in H$, $x \in y \circ z$ implies that $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Definition 2.1. *A t-norm is a function $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which satisfies the following properties:*

1. $T(a, b) = T(b, a)$;
2. $T(a, b) \leq T(a, c)$ *if* $b \leq c$;
3. $T(a, T(b, c)) = T(T(a, b), c)$;

4. $T(a,1) = a$,

for all $a, b, c, d \in [0,1]$.

Here are some examples of t-norms:

$$T_D(x,y) = \begin{cases} x \wedge y & ; \quad x \vee y = 1, \\ 0 & ; \quad \text{otherwise.} \end{cases} \qquad \text{(Drastic product)}$$
$$T_{Luk}(x,y) = 0 \vee (x+y-1). \qquad \text{(Lukasiewicz)}$$
$$T_P(x,y) = xy. \qquad \text{(Product)}$$
$$T_M(x,y) = x \wedge y. \qquad \text{(Minimum)}$$
$$T_{nM}(x,y) = \begin{cases} 0 & ; \quad x+y \leq 1, \\ x \wedge y & ; \quad \text{otherwise} \end{cases} \qquad \text{(Nilpotent minimum).}$$
$$T_{H_0}(x,y) = \begin{cases} 0 & ; \quad x = y = 0, \\ \frac{xy}{x+y-xy} & ; \quad \text{otherwise} \end{cases} \qquad \text{(Hamacher product).}$$

where $\wedge = \min$ and $\vee = \max$. Every t-norm T satisfies the inequality

$$T_D(x,y) \leq T(x,y) \leq T_M(x,y).$$

Let G be a non-empty set. Any map $\mu : G \to [0,1]$ is called a fuzzy set on G.
In this paper $FS(G)$ denotes the set of all fuzzy subsets of G.

Definition 2.2. [23] *Let G be a non-empty finite set. A t-fuzzy graph with underlying set G is a quadruple (G, σ, μ, T), where $\sigma : S \longrightarrow [0,1]$ is a fuzzy subset of G, $\mu : G \times G \longrightarrow [0,1]$ is a symmetric fuzzy relation on the fuzzy subset σ and T is a t-norm such that for all $x, y \in G$, $\mu(x,y) \leq T(\sigma(x), \sigma(y))$.*

If for all $a, b \in [0,1]$, $T(a,b) = a \wedge b$, then a t-fuzzy graph is called fuzzy graph and set $(G, \sigma, \mu) := (G, \sigma, \mu, \wedge)$.

In the Definition 2.2, we write $\mu(x,y)$ instead of $\mu(\{x,y\})$. We say that μ is a symmetric fuzzy relation, if $\mu(x,y) = \mu(y,x)$.

Definition 2.3. [2] *Let G be a group, $\mu : G \longrightarrow [0,1]$ be a fuzzy subset of G and T be a t-norm. A t-fuzzy subgroup is a triple (G, μ, T) such that satisfies the following:*

1. $\mu(x,y) \geq T(\mu(x), \mu(y))$,
2. $\mu(x^{-1}) = \mu(x)$,

for all $x, y \in G$.

If for all $a, b \in [0,1]$, $T(a,b) = a \wedge b$, then a t-fuzzy subgroup is called fuzzy subgroup and set $(G, \mu) := (G, \mu, \wedge)$.

3 Hypergroups obtain from t-fuzzy subsets

The first connection between fuzzy sets and algebraic structures has been considered by Rosenfeld in 1971 [28] when he defined the notion of fuzzy subgroup of a group $(G, \cdot)$, with applications in the economical mathematics. Later, Ameri, Zahedi and Sen [1, 30] have extended this association for constructing a new hyperstructure $(G, \circ_\mu)$, which under suitable conditions is a hypergroup, or moreover, a join space.

In the following, we present some results obtained in these directions by using t-norm.

In this section G is a finite group and T is a t-norm.

Let $a, b \in G$. Ameri and Zahedi [1] defined the following notions:

Let $\alpha_\mu = \bigvee_{x \in [0,1]} \mu(x)$.

Definition 3.1. [1] *Let $\mu \in FS(G)$ and $x \in G$. Then the left fuzzy coset $x\mu \in FS(G)$ of μ is defined by*

$$x\mu(g) = \mu(x^{-1}g), \ \forall g \in G.$$

Similarly, the right fuzzy coset $\mu x \in FS(G)$ is defined by

$$\mu x(g) = \mu(gx^{-1}), \ \forall g \in G.$$

By the above notations we put:

$$\mu^a = \{x \in G : \mu x = \mu a\} \text{ and } P_\mu = \{\mu^a \mid a \in G\}.$$

By $a\mu^e$ and $\mu^a \mu^b$ we mean the following sets

$$a\mu^e = \{ax : x \in \mu^e\} \text{ and } \mu^a \mu^b = \left\{xy : x \in \mu^a, y \in \mu^b\right\}.$$

Ameri and Zahedi [1] defined the hyperoperation $\circ_\mu$ as follows.

$$\begin{gathered} \circ_\mu : G \times G \longrightarrow P^*(G) \\ (a,b) \mapsto \mu^a \mu^b \end{gathered}$$

Clearly, $\circ_\mu$ is a hyperoperation on G. The hyperoperation $\circ_\mu$ is called the hyperoperation induced by μ.

Ameri and Zahedi [1] showed that if μ is subnormal, then $(G, \circ_\mu)$ is both a hypergroup and a polygroup. In particular, they proved that if the commutator subgroup of G is a subgroup of μ^e then $(G, \circ_\mu)$ is a join space.

Now, we generalize the above notions by t-norm.

Definition 3.2. *Let $\mu \in FS(G)$ and $x \in G$. Then the left t-fuzzy coset $(x\mu)_T \in FS(G)$ of μ is defined by*

$$(x\mu)_T(g) = T(\alpha_\mu, \mu(x^{-1}g)), \ \forall g \in G.$$

Similarly, the right t-fuzzy coset $(\mu x)_T \in FS(G)$ is defined by

$$(\mu x)_T(g) = T(\alpha_\mu, \mu(gx^{-1})), \ \forall g \in G.$$

Theorem 3.3. *Let $\mu \in FS(G)$. For every $x \in G$ set $\mu_T(x) = T(\alpha_\mu, \mu(x))$. Then*

(1) $\mu_T \in FS(G)$.

(2) The left t-fuzzy coset $(x\mu)_T \in FS(G)$ is equal to the left fuzzy coset $x\mu_T \in FS(G)$.

(3) The right t-fuzzy coset $(\mu x)_T \in FS(G)$ is equal to the right fuzzy coset $\mu_T x \in FS(G)$.

Proof. It obtains from Definitions 3.1 and 3.2. □

Definition 3.4. [22] *$\mu \in FS(G)$ is said to be*

(1) *symmetric if* $\mu(x) = \mu(x^{-1}),\ \forall x \in G$,

(2) *invariant if* $\mu(xy) = \mu(yx), \forall x, y \in G$,

(3) *subnormal if it is both symmetric and invariant.*

Now, let $\mu \in FS(G)$ and $a, b \in G$.

$$\mu_T^a = \{x \in G : (\mu x)_T = (\mu a)_T\} \text{ and } P_\mu^T = \{\mu_T^a \mid a \in G\}.$$

By $a\mu_T^e$ and $\mu_T^a\mu_T^b$ we mean the following sets

$$a\mu_T^e = \{ax : x \in \mu_T^e\} \text{ and } \mu_T^a\mu_T^b = \left\{xy : x \in \mu_T^a, y \in \mu_T^b\right\}.$$

We define the hyperoperation $\circ_\mu^T$ as follows.

$$\begin{gathered}\circ_\mu^T : G \times G \longrightarrow P^*(G)\\ (a, b) \mapsto \mu_T^a\mu_T^b\end{gathered}$$

Clearly, $\circ_\mu^T$ is a hyperoperation on G. The hyperoperation $\circ_\mu^T$ is called the hyperoperation induced by fuzzy subset μ and t-norm T.

If $\mu \in FS(G)$ is invariant, then it is easy to see that $(x\mu)_T = (\mu x)_T$, for all $x \in G$.

Lemma 3.5. *Let $\mu \in FS(G)$ be subnormal. Then*

(1) *For every $x, y \in G$, $(x\mu)_T = (y\mu)_T$ if and only if $xy^{-1} \in \mu_T^e$,*

(2) *μ_T^e is a normal subgroup of G,*

(3) *For every $x \in G$, $\mu_T^x = x\mu_T^e$,*

(4) *For every $x, y \in G$, $\mu_T^x\mu_T^y = \mu_T^{xy}$.*

Proof. It is by the similar way by proof of Lemma 3.6 [22]. □

Theorem 3.6. *If $\mu \in FS(G)$ is subnormal, then $(G, \circ_\mu^T)$ is a hypregroup.*

Proof. First, we show that $\circ_\mu^T$ is associative. Let $a, b, c \in G$. Then by subnormality of μ, we have

$$\begin{aligned}(a \circ_\mu^T b) \circ_\mu^T c &= \bigvee_{x \in \mu_T^a\mu_T^b} \mu_T^x\mu_T^c = \bigvee_{x \in \mu_T^{ab}} \mu_T^{xc} = \mu_T^{(ab)c} = \mu_T^{a(bc)} = \bigvee_{y \in \mu_T^{bc}} \mu_T^{ay}\\ &= \bigvee_{y \in \mu_T^b\mu_T^c} \mu_T^a\mu_T^y = a \circ_\mu^T (b \circ_\mu^T c).\end{aligned}$$

Then $\circ_\mu^T$ is associative.

Now, we prove that the reproductive law holds. For every $a \in G$ we have

$$a \circ_\mu^T G = \bigcup_{x \in G} a \circ_\mu^T x = \bigcup_{y \in G} a \circ_\mu^T (a^{-1}y) = \bigvee_{y \in G} \mu_T^a\mu_T^{a^{-1}y} = \bigvee_{y \in G} \mu_T^y = G.$$

By the similar way, we obtain $G \circ_\mu^T a = G$. Therefore, $(G, \circ_\mu^T)$ is a hypregroup. □

Theorem 3.7. *If $\mu \in FS(G)$ is subnormal, then $(G, \circ_\mu^T)$ is a reversible regular hypergroup.*

Proof. Let $a \in G$. Then $a \in \mu_T^a = \mu_T^{ea} = \mu_T^e \mu_T^a = e \circ_\mu^T a$ and by the similar way we obtain $a \in a \circ_\mu^T e$. Then $a \in a \circ_\mu^T e \cap e \circ_\mu^T a$. Similarly, for every $a \in G$, we have $a \in a \circ_\mu^T e \cap e \circ_\mu^T a$. Then $(G, \circ_\mu^T)$ is a regular hypergroup.

Now, we show that $a \in b \circ_\mu^T c$ implies that $c \in b^{-1} \circ_\mu^T a$ and $b \in a \circ_\mu^T c^{-1}$. Let $a \in b \circ_\mu^T c$. Then $a \in \mu_T^b \mu_T^c$ and so $\mu_T^a = \mu_T^b \mu_T^c$. Then $\mu_T^a \mu_T^{c^{-1}} = \mu_T^b$ and $\mu_T^{b^{-1}} \mu_T^a = \mu_T^c$. Hence, $c \in b^{-1} \circ_\mu^T a$ and $b \in a \circ_\mu^T c^{-1}$. Therefore, $(G, \circ_\mu^T)$ is a reversible hypergroup. □

4 Isomorphisms of t-fuzzy graphs

Definition 4.1. *Let (G, σ, μ, T) and (G', σ', μ', T) be two t-fuzzy graphs. A homomorphism $h : G \longrightarrow G'$ of two t-fuzzy graphs is meant a map $h : G \longrightarrow G'$ that satisfies*

1. $\sigma(x) \leq \sigma'(h(x))$,
2. $\mu(x, y) \leq \mu'(h(x), h(y))$,

for all $x, y \in G$.

Example 4.2. *Let $(G = \{a, b, c\}, \sigma, \mu, T_{Luk})$ and $(G' = \{a', b', c'\}, \sigma', \mu', T_{Luk})$ be two t-fuzzy graphs as follows:*

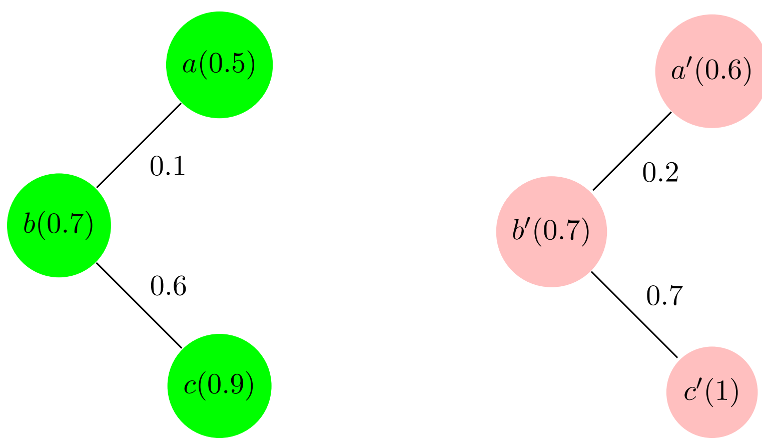

Figure 1: Two t-fuzzy graphs

The map $h : G \longrightarrow G'$ by $h(x) = x'$, for all $x \in G$, is a homomorphism of two t-fuzzy graphs.

Definition 4.3. *Let (G, σ, μ, T) and (G', σ', μ', T) be two fuzzy graphs. An isomorphism $h : G \longrightarrow G'$ of fuzzy graphs is meant a bijective map $h : G \longrightarrow G'$ that satisfies*

1. $\sigma(x) = \sigma'(h(x))$,
2. $\mu(x, y) = \mu'(h(x), h(y))$,

for all $x, y \in G$.

Remark 4.4. *In Example 4.2 the bijective map h is a homomorphism but it is not an isomorphism of fuzzy graphs.*

Theorem 4.5. *Let (G, σ, μ) and (G', σ', μ') be two finite fuzzy graphs, G and G' be two groups. Suppose $h : G \longrightarrow G'$ is a group isomorphism for which satisfies the following conditions:*

1. $\sigma(x) = \sigma'(h(x))$,
2. $\mu(x, y) = \mu'(h(x), h(y))$,

for all $x, y \in G$. Then

(i) If σ is a t-fuzzy subgroup of G respect to T, then σ' is a t-fuzzy subgroup of G'.

(ii) If μ is a t-fuzzy subgroup of $G \times G$ respect to T, then μ' is a t-fuzzy subgroup of $G' \times G'$.

Proof. (i) Let $x', y' \in G'$, then there are $x, y \in G$ such that $h(x) = x'$ and $h(y) = y'$. Thus,

$$\begin{aligned} \sigma'(x'y') &= \sigma'(h(x)h(y)) = \sigma'(h(xy)) = \sigma(xy) \\ &\geq T(\sigma(x), \sigma(y)) \\ &= T(\sigma'(h(x)), \sigma'(h(y))), \end{aligned}$$

and

$$\sigma'(x'^{-1}) = \sigma'(h(x)^{-1}) = \sigma'(h(x^{-1})) = \sigma(x^{-1}) = \sigma(x) = \sigma'(h(x)) = \sigma'(x').$$

(ii) The proof is similar to the case (i). □

In the Theorem 4.5, $h : G \longrightarrow G'$ should be a group isomorphism. By the following examples, for two isomorphic t-fuzzy graphs (G, σ, μ, T) and (G', σ', μ', T) one can show that:

(i) If σ is a t-fuzzy subgroup of G, maybe σ' is not a t-fuzzy subgroup of G'.

(ii) If μ is a t-fuzzy subgroup of $G \times G$, maybe μ' is not a t-fuzzy subgroup of $G' \times G'$.

Example 4.6. *Consider two fuzzy graphs $(G = \mathbb{Z}_3, \sigma, \mu, \wedge)$ and $(G' = \mathbb{Z}_3, \sigma', \mu', \wedge)$. Let $h : G \longrightarrow G'$ be defined as follows.*

$$h(x) = \begin{cases} 2 & x = 0 \\ 1 & x = 1 \\ 0 & x = 2 \end{cases},$$

and

$$\sigma(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{2} & x = 1, 2 \end{cases}, \qquad \mu(x, y) = \begin{cases} \frac{1}{2} & (x, y) = (0, 1) \text{ or } (0, 2) \\ \frac{1}{3} & (x, y) = (1, 2) \\ 0 & x = y \end{cases}$$

$$\sigma'(x) = \begin{cases} \frac{1}{2} & x = 0, 1 \\ 1 & x = 2 \end{cases}, \qquad \mu'(x, y) = \begin{cases} \frac{1}{3} & (x, y) = (0, 1) \\ \frac{1}{2} & (x, y) = (1, 2) \text{ or } (0, 2) \\ 0 & x = y. \end{cases}$$

We have $h : G \longrightarrow G'$ is an isomorphism of two fuzzy graphs. Then these two fuzzy graphs are isomorphic and σ is a t-fuzzy subgroup of G but σ' is not a t-fuzzy subgroup of G'.

Example 4.7. *Consider two fuzzy graphs* $(G = \mathbb{Z}_{12}, \sigma, \mu, \wedge)$ *and* $(G' = \mathbb{Z}_{12}, \sigma', \mu', \wedge)$. *Let* $h : G \longrightarrow G'$ *be defined by*

$$h(x) = \begin{cases} 11 & x = 4 \\ 4 & x = 11 \\ x & elsewhere \end{cases},$$

and set $\sigma(i) = \frac{1}{i+2}$, $\sigma'(i) = \frac{1}{h^{-1}(i)+2}$, *where* $i \in \mathbb{Z}_{12}$. *Also, suppose that*

$$\mu(x,y) = \begin{cases} \frac{1}{13} & x = y = 0 \\ \frac{1}{13} & \{x,y\} = \{4,8\} \\ \frac{1}{15} & elsewhere \end{cases}, \qquad \mu'(x,y) = \begin{cases} \frac{1}{13} & x = y = 0 \\ \frac{1}{13} & \{x,y\} = \{11,8\} \\ \frac{1}{15} & elsewhere \end{cases}.$$

Then f *is an isomorphism. Hence, two fuzzy graphs* $(G = \mathbb{Z}_{12}, \sigma, \mu, \wedge)$ *and* $(G' = \mathbb{Z}_{12}, \sigma', \mu', \wedge)$ *are isomorphic. Clearly,* μ *is a* t*-fuzzy subgroup of* $G \times G$, *but* μ' *is not a* t*-fuzzy subgroup of* $G' \times G'$. *In fact,*

$$\mu'((11,8) + (11,8)) = \mu'(10,4) = \frac{1}{15} \ngeq \mu'(11,8) \wedge \mu'(11,8) = \frac{1}{13}.$$

5 Hypergroups and t-Fuzzy Subgroups Obtained From t-Fuzzy Graphs

In this section, we show that how to assign a t-fuzzy graph to the appropriate t-fuzzy subgroups and hypergroups.

Lemma 5.1. *Let* (G, σ, T) *be a* t*-fuzzy subgroup and* $\mu_\sigma : G \times G \longrightarrow [0,1]$ *defined by* $\mu_\sigma(x,y) = T(\sigma(x), \sigma(y))$ *for all* $x, y \in G$, *then* $(G, \sigma, \mu_\sigma, T)$ *is a* t*-fuzzy graph.*

Definition 5.2. *Let* (G, σ, μ, T) *be a* t*-fuzzy graph. A bijective map* $h : G \longrightarrow G$ *is an automorphism of* G *if*

1. $\mu(x,y) = \mu(h(x), h(y))$,
2. $\sigma(x) = \sigma(h(x))$,

for all $x, y \in G$.

Proposition 5.3. *Let* $Aut(G)$ *be the set of all automorphisms of a fuzzy graph* (G, σ, μ, T). *Then* $Aut(G)$ *is a group under the set theoretic product of maps as the binary operation.*

Proof. Proof is straightforward. □

Let $c \in [0,1]$ and T be a t-norm. For all $n \geq 3$ set

$$c^2 := T(c,c), \qquad c^n = T(c^{n-1}, c).$$

Then

Theorem 5.4. *Let* (G, σ, μ, T) *be a* t*-fuzzy graph. For all* $f \in Aut(G)$ *define a map* $\tau_c : Aut(G) \longrightarrow [0,1]$ *as follows,*

$$\tau_c(f) = \begin{cases} c^2 & f \text{ is an even permutation} \\ 0 & f \text{ is an odd permutation,} \end{cases} \tag{1}$$

for all $f \in Aut(G)$. *Then* τ_c *is a* t*-fuzzy subgroup on* $Aut(G)$.

Proof. We prove this theorem in two following cases:

Case 1) Suppose that $f, g \in Aut(G)$ both are even or odd permutations. Thus, $f \circ g$ becomes an even permutation. If f, g both are even permutations:

$$\begin{aligned}\tau_c(f \circ g) &= c^2 \\ &\geq \min\{c^2, c^2\} \\ &= \min\{\tau_c(f), \tau_c(g)\} \\ &\geq T(\tau_c(f), \tau_c(g)).\end{aligned}$$

If f, g both are odd permutations:

$$\begin{aligned}\tau_c(f \circ g) &= c^2 \\ &\geq \min\{0, 0\} \\ &= \min\{\tau_c(f), \tau_c(g)\} \\ &\geq T(\tau_c(f), \tau_c(g)).\end{aligned}$$

Case 2) Suppose that $f, g \in Aut(G)$ such that one of them is even permutation and the other one is odd permutation Thus, $f \circ g$ becomes an odd permutation. Hence

$$\begin{aligned}0 &= \tau_c(f \circ g) \\ &= min\{0, c^2\} \\ &= min\{\tau_c(f), \tau_c(g)\} \\ &\geq T(\tau_c(f), \tau_c(g)).\end{aligned}$$

Also, for all $f \in Aut(G)$, f and f^{-1} both are even permutations or both are odd permutations. This follows that $\tau_c(f^{-1}) = \tau_c(f)$. Therefore, τ_c is a t-fuzzy subgroup on $Aut(G)$. □

Example 5.5. *Let $c = 0.1$ and $(G = \{0, 1, 2, 3, 4, 5, 6, 7\}, \sigma, \mu, T)$ be a t-fuzzy graph as follows:*

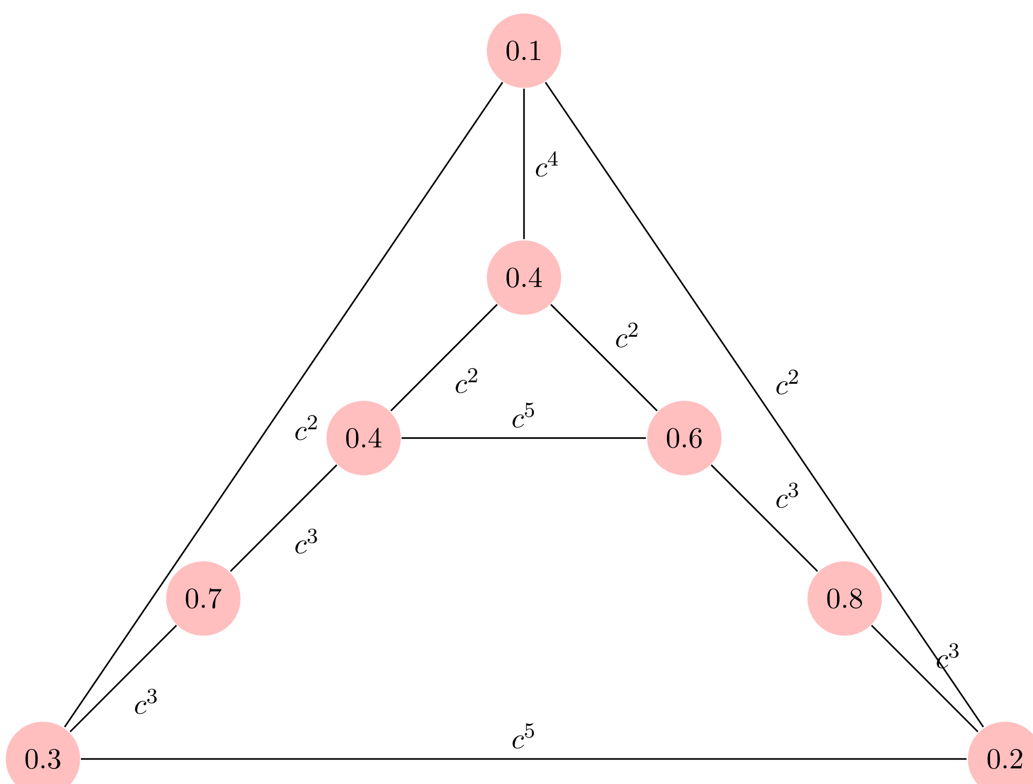

Figure 2: t-fuzzy graph (G,σ,μ,T)

We have

(i) $Aut(G)=\{id,(53)(62)(41),(87)(56)(32),(52)(63)(41)(87)\}$ *and*

(ii)

$$\tau_C(f)=\begin{cases} c^2 & f=(5\ 2)(6\ 3)(4\ 1)(8\ 7)\\ 0 & f\in\{(5\ 3)(6\ 2)(4\ 1),\ (8\ 7)(5\ 6)(3\ 2)\},\end{cases} \tag{2}$$

It is not difficult to see that τ_c is a t-fuzzy subgroup on $Aut(G)$.

By the definition τ_c in the Theorem 5.4 we have

Lemma 5.6. *τ_c is subnormal.*

Proof. By Theorem 5.4 τ_c is a t-fuzzy subgroup on $Aut(G)$. Hence for all $f\in Aut(G)$, $\tau_c(f^{-1})=\tau_c(f)$. Then τ_c is symmetric. If $f,g\in Aut(G)$, then both $f\circ g$ and $g\circ f$ are odd or even. So $\tau_c(f\circ g)=\tau_c(g\circ f)$ and this means that τ_c is invariant. Therefore, τ_c is subnormal. □

Corollary 5.7. *Let (G,σ,μ,T) be a t-fuzzy graph and $Aut(G)$ be the set of all automorphism on G. Define a map $\tau_c: Aut(G)\longrightarrow[0,1]$ as follows*

$$\tau_c(f)=\begin{cases} c^2 & f \text{ is an even permutation}\\ 0 & f \text{ is an odd permutation,}\end{cases} \tag{3}$$

for all $f\in Aut(G)$. Then $(Aut(G),\circ^T_{\tau_c})$ is reversible regular hypergroup.

Proof. It obtains from Theorems 3.6, 3.7 and Lemma 5.6. □

6 Conclusion

We generalized the concept of hyperoperation derived from fuzzy subgroup by replacing minimum in the basic definitions with an arbitrary t-norm. The reason for this is that some applications are better modeled with a t-norm other than minimum. Also, we constructed a fuzzy subgroup by the concept of automorphism of t-fuzzy graph and presented some new results in this respect. Finally, we obtained a reversible regular hypergroup by concept of the automorphism group of t-fuzzy graph.

For future work, it will be interesting to introduce a t-norm over an interval valued fuzzy graph and an intuitionistic fuzzy graph and study intuitionistic fuzzy subgroup by t-norm in a natural way.

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