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Ideals in pseudo-hoop algebras

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Abstract

Pseudo-hoop algebras are non-commutative generalizations of hoop-algebras, originally introduced by Bosbach. In this paper, we study ideals in pseudo-hoop algebras. We define congruences induced by ideals and construct the quotient structure. We show that there is a one-to-one correspondence between the set of all normal ideals of a pseudo-hoop algebra A with condition (pDN) and the set of all congruences on A. Also, we prove that if A is a good pseudo-hoop algebra with pre-linear condition, then a normal ideal P of A is prime if and only if A/P is a pseudo-hoop chain. Furthermore, we analyse the relationship between ideals and filters of A.

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1 Introduction

Hoop algebras were presented by Bosbach in [4, 5]. Then Büchi and Owens investigated this algebraic structure in an unpublished paper. Pseudo-hoop algebras were presented as non-commutative generalizations of hoop algebras by Georgescu, Leuştean and Preoteasa in [13], following after the notions of pseudo-MV algebras in [12] and pseudo-BL algebras ([10]). Pseudo-hoop algebras are weaker structures. Pseudo-MV algebras and pseudo-BL algebras are particular cases of pseudo-hoop algebras. In recent years, the study of hoop algebras and pseudo-hoop algebras has made great progress. And the main focus has been on filters in [2, 6, 9, 15].

Ideal theory plays a fundamental role in many algebraic structures, such as lattices, rings and pseudo-MV algebras. Georgescu and Iorgulescu in [12] introduced the notion of ideals in pseudo-MV algebras, which was shown effective in studying structure properties of pseudo-MV algebras. In addition, Dvurečenskij in [11] studied states on pseudo-MV algebras by exploiting ideals. In recent years, the notion of ideals has been introduced as a dual notion of filters in some algebraic structures using multiplication operations. Lele and Nganou in [14] presented the notion of ideals

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in BL-algebras and defined quotient algebraic structures by ideals. Using ideals, they proved that an ideal of a BL-algebra is prime if and only if the quotient algebraic structure is a linear MV-algebra. Also, Rachůnek and Šalounová in [16] introduced ideals of general residuated lattices. It was proved that a congruence can be defined by an ideal in some cases, and the corresponding quotient structure is involutive. In [1], Kologani and Borzooei introduced the notions of ideals, implicative (maximal, prime) ideals of hoop algebras and studied the relationships between these ideals.

In (pseudo-) MV-algebras, filters and ideals are dual. However, in pseudo-hoop algebras, we mainly study filters. As pseudo-hoop algebras may not have lattice structures, not all pseudo-hoop algebras are general residuated lattices. Since pseudo-MV algebras are particular cases of general residuated lattices, the notion of ideals in pseudo-hoop algebras can not be similar to that in pseudo-MV algebras. Therefore, we want to introduce the notion of ideals in pseudo-hoop algebras, as a dual notion of filters in [2]. Another inspiration is the notion of ideals in hoop algebras defined in [1]. Since pseudo-hoop algebras are non-commutative generalizations of hoop algebras, we shall generalize the notion of ideals in hoop algebras to the case of pseudo-hoop algebras. Also, by Theorem 6.5 and Theorem 6.6, it is noticeable that ideals and filters behave differently in pseudo-hoop algebras. Therefore, it is meaningful to investigate ideals in pseudo-hoop algebras.

The paper is constructed as follows. In Section 2, we recall some definitions and results on pseudo-hoop algebras which are useful. In Section 3, we define the notions of left, right and both-sided ideals of pseudo-hoop algebras. In Section 4, we analyse congruences induced by ideals and construct the quotient pseudo-hoop algebras via ideals. In addition, we get an isomorphism theorem. In Section 5, we introduce the notion of prime ideals in pseudo-hoop algebras and give some equivalent conditions of prime ideals. In Section 6, we analyse the relationship between ideals and filters. Also, we introduce the notion of $\odot$-prime ideals in pseudo-hoop algebras. The relationship between $\odot$-prime ideals and maximal filters is discussed.

2 Preliminaries

In this section, we recall some definitions and results to be used in this paper.

Definition 2.1. [13] *A pseudo-hoop algebra is an algebra* $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ *of the type* $(2,2,2,0)$ *that for all* $u, v, w \in A$*, it is satisfying in the following conditions:*
$(ph1)$ $u \odot 1 = 1 \odot u = u$;
$(ph2)$ $u \rightarrow u = u \rightsquigarrow u = 1$;
$(ph3)$ $(u \odot v) \rightarrow w = u \rightarrow (v \rightarrow w)$;
$(ph4)$ $(u \odot v) \rightsquigarrow w = v \rightsquigarrow (u \rightsquigarrow w)$;
$(ph5)$ $(u \rightarrow v) \odot u = (v \rightarrow u) \odot v = u \odot (u \rightsquigarrow v) = v \odot (v \rightsquigarrow u)$.

We define $u^0 = 1$ *and* $u^n = u^{n-1} \odot u$ *for any* $n \in \mathbb{N}_+$ *on* A*. The relation* $\leq$ *defined by* $u \leq v \Leftrightarrow u \rightarrow v = 1 \Leftrightarrow u \rightsquigarrow v = 1$ *is a partial order on* A*. If* $\odot$ *is commutative or equivalently* $\rightarrow = \rightsquigarrow$*,* A *is called to be a hoop algebra. Also,* A *is called bounded if* $u \geq 0$ *for any* $u \in A$*. In this case, we define* $u^- = u \rightarrow 0$ *and* $u^\sim = u \rightsquigarrow 0$ *on* A*. If* $u^{-\sim} = u^{\sim -}$ *for all* $u \in A$*, then the bounded pseudo-hoop algebra is called good (see [8]). In a bounded pseudo-hoop algebra* A*, if* $u^{-\sim} = u^{\sim -} = u$ *for all* $u \in A$*, then* A *is called satisfying the (pDN) condition (see [8]). A good pseudo-hoop algebra* A *is called normal if it satisfies* $(u \odot v)^{-\sim} = u^{-\sim} \odot v^{-\sim}$ *for all* $u, v \in A$*.*

We summarize some properties of pseudo-hoop algebras that we will use later. For more details, see [8] and [13].

Proposition 2.2. [13] *Let* $(A, \odot, \to, \rightsquigarrow, 1)$ *be a pseudo-hoop algebra. Then for all* $u, v, w \in A$*, the following conditions hold:*
(1) $u \odot v \le w$ *iff* $u \le v \to w$ *iff* $v \le u \rightsquigarrow w$*;*
(2) $(A, \odot, 1)$ *is a monoid;*
(3) *if* $u \le v$*, then* $u \odot w \le v \odot w$ *and* $w \odot u \le w \odot v$*;*
(4) $u \wedge v = (u \to v) \odot u = (v \to u) \odot v = u \odot (u \rightsquigarrow v) = v \odot (v \rightsquigarrow u)$*;*
(5) *if* $u \le v$*, then* $v \to w \le u \to w$ *and* $v \rightsquigarrow w \le u \rightsquigarrow w$*;*
(6) *if* $u \le v$*, then* $w \to u \le w \to v$ *and* $w \rightsquigarrow u \le w \rightsquigarrow v$*;*
(7) $(v \to w) \odot (u \to v) \le u \to w$, $(u \rightsquigarrow v) \odot (v \rightsquigarrow w) \le u \rightsquigarrow w$.

Proposition 2.3. [8] *Let* A *be a bounded pseudo-hoop algebra. Then for all* $u, v, w \in A$ *the following statements hold:*
(1) $u \odot 0 = 0 \odot u = 0$*;*
(2) $u^- \odot u = 0$, $u \odot u^\sim = 0$*;*
(3) $u \odot v = 0$ *iff* $u \le v^-$ *iff* $v \le u^\sim$*;*
(4) $u \le u^{-\sim}$, $u \le u^{\sim -}$*;*
(5) $u^{-\sim -} = u^-$, $u^{\sim - \sim} = u^\sim$*;*
(6) *if* A *is good, then* $(u \to v)^{-\sim} = u^{-\sim} \to v^{-\sim}$ *and* $(u \rightsquigarrow v)^{-\sim} = u^{-\sim} \rightsquigarrow v^{-\sim}$*;*
(7) *if* A *is good, then* $u \to v^- = u^{-\sim} \to v^-$ *and* $u \rightsquigarrow v^\sim = u^{-\sim} \rightsquigarrow v^\sim$*.*

A pseudo-hoop algebra A is said to satisfy the pre-linear condition if we have $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ for any $x, y \in A$. By [7, Proposition 3.4], $(A, \odot, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-hoop algebra with pre-linear condition if and only if $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-BL algebra.

A filter F of a pseudo-hoop algebra A is a nonempty subset of A which satisfies (F1): $u, v \in F$ implies $u \odot v \in F$ and (F2): for any $u, v \in A$, if $u \le v$ and $u \in F$, then $v \in F$ (see [13]). In a pseudo-hoop algebra A, filters are coincided with deductive systems. A filter F of A satisfying $F \ne A$ is called proper. If F is a proper filter of A and there is no proper filter containing F, F is called maximal. A filter F of A is normal if $u \to v \in F$ iff $u \rightsquigarrow v \in F$ for any $u, v \in A$. Let X be a subset of A. We use $(X]$ to denote the filter of A generated by X.

Proposition 2.4. [13] *Let* A *be a pseudo-hoop algebra,* W *a normal filter of* A *and* $u \in A$*. Then*

$$\begin{aligned}(W \cup \{u\}] &= \{a \in A | w \odot u^n \le a,\ for\ some\ n \in \mathbb{N},\ w \in W\} \\ &= \{a \in A | u^n \odot w \le a,\ for\ some\ n \in \mathbb{N},\ w \in W\}.\end{aligned}$$

Let A_1 and A_2 be pseudo-hoop algebras. In [9], a map $f : A_1 \to A_2$ is called a pseudo-hoop homomorphism if f preserves the operations $\odot$, $\to$ and $\rightsquigarrow$. The pseudo-hoop homomorphism $f : A_1 \to A_2$ is called a bounded pseudo-hoop homomorphism if A_1, A_2 are bounded and $f(0) = 0$.

3 Ideals

In this section, we shall introduce two kinds of binary operations (left and right additions) and the notion of ideals in pseudo-hoop algebras. We give some equivalent characterizations of ideals of good pseudo-hoop algebras.

Definition 3.1. *Let* $(A, \odot, \to, \rightsquigarrow, 1)$ *be a bounded pseudo-hoop algebra. We define left addition* $\oslash$ *and right addition* $\otimes$ *as follows: for any* $x, y \in A$,

$$x \oslash y = y^- \rightsquigarrow x \quad and \quad x \otimes y = x^\sim \to y.$$

Example 3.2. [15] *Let $A = \{0, a, b, c, d, 1\}$. Define the operations $\to$, $\rightsquigarrow$ and $\odot$ on A as follows:*

$\to=\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

Then $(A, \odot, \to, \rightsquigarrow, 1)$ is a bounded hoop algebra. It is easy to see that $b \oslash c = c^- \rightsquigarrow b = a \rightsquigarrow b = b$ and $c \otimes a = c^\sim \to a = a \to a = 1$.

Proposition 3.3. *Let A be a pseudo-hoop algebra. For all $x, y, m, n \in A$, if $x \leq y$ and $m \leq n$, then $x \oslash m \leq y \oslash n$ and $x \otimes m \leq y \otimes n$.*

Proof. If $x \leq y$ and $m \leq n$, then $y^\sim \leq x^\sim$, $n^- \leq m^-$. By Proposition 2.2(5) and (6), we have $x \oslash m = m^- \rightsquigarrow x \leq n^- \rightsquigarrow x \leq n^- \rightsquigarrow y = y \oslash n$. Similarly, we have $x \otimes m \leq y \otimes n$. □

Proposition 3.4. *Let A be a pseudo-hoop algebra. If A is normal, then left addition $\oslash$ and right addition $\otimes$ are associative.*

Proof. For all $x, y, z \in A$, we obtain

$$\begin{aligned} x \otimes (y \otimes z) &= x^\sim \to (y^\sim \to z) \quad (ph3) \\ &= (x^\sim \odot y^\sim) \to z \quad (Proposition\ 2.3(5)) \\ &= (x^{\sim-\sim} \odot y^{\sim-\sim}) \to z \quad (A\ is\ normal) \\ &= (x^\sim \odot y^\sim)^{-\sim} \to z \quad (ph3) \\ &= (x^\sim \to y^{\sim-})^\sim \to z \quad (A\ is\ good\ and\ Proposition\ 2.3(5)) \\ &= (x^{\sim-\sim} \to y^{-\sim})^\sim \to z \quad (A\ is\ good\ and\ Proposition\ 2.3(6)) \\ &= (x^\sim \to y)^{\sim-\sim} \to z \quad (Proposition\ 2.3(5)) \\ &= (x^\sim \to y)^\sim \to z \\ &= (x \otimes y) \otimes z. \end{aligned}$$

Similarly, we can prove $(x \oslash y) \oslash z = x \oslash (y \oslash z)$. □

Definition 3.5. *Let I be a nonempty subset of a bounded pseudo-hoop algebra A. Then I is called a left ideal of A if it satisfies:*
$(LI1)$ $x, y \in I$ implies $x \oslash y \in I$;
$(I2)$ for any $x, y \in A$, $x \leq y$ and $y \in I$ imply $x \in I$.

Similarly, I is called a right ideal of A if it satisfies:
$(RI1)$ $x, y \in I$ implies $x \otimes y \in I$;
$(I2)$ for any $x, y \in A$, $x \leq y$ and $y \in I$ imply $x \in I$.

If I is both a left ideal and a right ideal of A, we call I to be an ideal of A.

For any ideal I of A, we have $0 \in I$. For all $x \in A$, we have $x \in I$ iff $x^{-\sim} \in I$ iff $x^{\sim-} \in I$. An ideal I of A is called proper if $I \neq A$. An ideal I of A is called normal if $x^- \odot y \in I$ iff $y \odot x^\sim \in I$ for all $x, y \in A$. The intersection of any family of ideals of a bounded pseudo-hoop algebra A is also an ideal of A. For any subset $H \subseteq A$, the smallest ideal of A containing H is said to be the ideal generated by H, and it is denoted by $\langle H \rangle$.

Example 3.6. *Let A be a pseudo-hoop algebra as in Example 3.2. Then $I_1 = \{0\}$, $I_2 = \{0, c\}$, $I_3 = \{0, a, d\}$ and $I_4 = A$ are all ideals of A.*

Example 3.7. [13] *Let u be an element of an arbitrary ℓ-group $G = (G, +, -, 0, \vee, \wedge)$ and $u \geq 0$. Define the operations $\to$, $\rightsquigarrow$ and $\odot$ on $G[u] = [0, u]$ as follows:*

$$x \odot y = (x - u + y) \vee 0, \;\; x \to y = (y - x + u) \wedge u, \;\; \text{and } x \rightsquigarrow y = (u - x + y) \wedge u.$$

By [13, Example 5.1], $G[u]$ is a bounded pseudo-hoop algebra. Let W be a normal convex ℓ-subgroup of G and $F = \{x \in G[u] : u - x \in W\}$. We define $I_0 = \{x \in G[u] : x^- \in F\}$ and $I'_0 = \{x \in G[u] : x^\sim \in F\}$. Then I_0 and I'_0 are ideals of $G[u]$.

We shall show that I_0 is an ideal of $G[u]$. Let $x, y \in G[u]$. Then $x \to 0 = (0-x+u) \wedge u = -x+u$, $x \rightsquigarrow 0 = (u - x + 0) \wedge u = u - x$,

$$x \oslash y = x^\sim \to y = (y - (u - x) + u) \wedge u = (y + x - u + u) \wedge u = (y + x) \wedge u,$$

and $x \oslash y = y^- \rightsquigarrow x = (u - (-y + u) + x) \wedge u = (y + x) \wedge u$. Also, we have $x \oslash y = x \oslash y$.

By [13, Proposition 5.2], F is a normal filter of $G[u]$. Suppose $x, y \in G[u]$ such that $x \leq y$ and $y \in I_0$. Then $y^- \leq x^-$ and $y^- \in F$. Using (F2), we obtain $x^- \in F$, i.e. $x \in I_0$. Suppose $x, y \in I_0$, i.e. $x^-, y^- \in F$. We have $x^- \odot y^- \in F$, by (F1). Since

$$x^- \odot y^- = (x^- - u + y^-) \vee 0 = [(-x + u) - u + (-y + u)] \vee 0 = (-x - y + u) \vee 0,$$

and

$$(x \oslash y)^- = ((y + x) \wedge u)^- = -((y + x) \wedge u) + u = (-x - y + u) \vee (-u + u) = (-x - y + u) \vee 0,$$

we obtain $(x \oslash y)^- = (x \oslash y)^- = x^- \odot y^- \in F$. Hence, $x \oslash y, x \oslash y \in I_0$. Thus, I_0 is an ideal of $G[u]$.

Similarly, we can show that I'_0 is an ideal of $G[u]$.

Theorem 3.8. *Let I be a nonempty subset of a good pseudo-hoop algebra A containing 0. The following conditions are equivalent:*
(1) *I is an ideal of A;*
(2) *for any $x, y \in A$, $x^- \odot y \in I$ and $x \in I$ imply $y \in I$;*
(3) *for any $x, y \in A$, $y \odot x^\sim \in I$ and $x \in I$ imply $y \in I$.*

Proof. (1) $\Rightarrow$ (2) Suppose I is an ideal of A. If $x, y \in A$ such that $x, x^- \odot y \in I$, then $(x^- \odot y) \oslash x \in I$. Since $x^- \odot y \leq x^- \odot y$, we obtain $y \leq x^- \rightsquigarrow (x^- \odot y) = (x^- \odot y) \oslash x$ by Proposition 2.2(1). Using (I2), we have $y \in I$.

(2) $\Rightarrow$ (1) Let $x, y \in A$ such that $y \in I$ and $x \leq y$. Then $y^- \leq x^-$. Thus, $y^- \odot x \leq x^- \odot x = 0$. So $y^- \odot x = 0 \in I$. By (2), we obtain $x \in I$. Therefore, condition (I2) holds. Let $x, y \in I$. Since $y^- \odot (x \oslash y) = y^- \odot (y^- \rightsquigarrow x) \leq x \in I$, we have $y^- \odot (x \oslash y) \in I$. Therefore, $x \oslash y \in I$. In addition, we have $x \in I$ and $x^- \odot x^{-\sim} = 0 \in I$. It follows that $x^{-\sim} \in I$. Since $x^{\sim -} = x^{-\sim}$, we have $y^- \odot (x \oslash y) = y^- \odot (x^\sim \to y) \leq x^{\sim -} \in I$ by Proposition 2.2(7). Using (I2), we obtain $y^- \odot (x \oslash y) \in I$. Thus, $x \oslash y \in I$. Therefore, I is an ideal of A.

This proves that (1) $\Leftrightarrow$ (2). Similarly, we can prove that (1) $\Leftrightarrow$ (3). □

Remark 3.9. *Let I be a nonempty subset of a bounded pseudo-hoop algebra A containing 0, where A does not have to be good. By the previous proof, if I is an ideal of A, then conditions (2) and (3) hold. Also, I is a left (right) ideal of A if and only if condition (2) ((3)) holds.*

Theorem 3.10. *Let I be a nonempty subset of a good pseudo-hoop algebra A containing 0. The following conditions are equivalent:*
(1) *I is an ideal of A;*
(2) *for $x, y \in A$, $(x^- \to y^-)^\sim \in I$ and $x \in I$ imply $y \in I$;*
(3) *for $x, y \in A$, $(x^\sim \rightsquigarrow y^\sim)^- \in I$ and $x \in I$ imply $y \in I$.*

Proof. (1) $\Rightarrow$ (2) Suppose I is an ideal of A. Let $x, y \in A$ such that $(x^- \to y^-)^\sim \in I$ and $x \in I$. Then $x^- \odot y^{-\sim} \leq (x^- \odot y^{-\sim})^{-\sim} = (x^- \to y^{-\sim-})^\sim = (x^- \to y^-)^\sim \in I$. Using (I2), we obtain $x^- \odot y^{-\sim} \in I$. Thus $y^{-\sim} \in I$ by Theorem 3.8. Since $y \leq y^{-\sim}$, we obtain $y \in I$.

(2) $\Rightarrow$ (1) Suppose that the condition (2) holds. Let $x \in I$. Then $(x^- \to x^{-\sim-})^\sim = (x^- \to x^-)^\sim = 0 \in I$. It follows that $x^{-\sim} \in I$ by (2). Hence, we show that $x \in I$ implies $x^{-\sim} \in I$. Let $x^- \odot y$, $x \in I$. Then $(x^- \odot y)^{-\sim} \in I$, and so $(x^- \to y^-)^\sim \in I$. Thus, $y \in I$ by (2). Therefore, I is an ideal of A by Theorem 3.8.

This proves that (1) $\Leftrightarrow$ (2). Similarly, we can prove that (1) $\Leftrightarrow$ (3). $\square$

Proposition 3.11. *Let H be a subset of a bounded pseudo-hoop algebra A.*
(1) *If H is empty, then $\langle H \rangle = \{0\}$.*
(2) *If H is not empty and A is normal, then*

$$\begin{aligned}\langle H \rangle &= \{h \in A : h \leq x_1 \oslash x_2 \oslash x_3 \oslash \ldots \oslash x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\} \\ &= \{h \in A : h \leq x_1 \otimes x_2 \otimes x_3 \otimes \ldots \otimes x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}.\end{aligned}$$

Proof. (1) It is obvious.

(2) If A is normal, $\oslash$ and $\otimes$ are associative. Let

$$B = \{h \in A : h \leq x_1 \oslash x_2 \oslash x_3 \oslash \ldots \oslash x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}.$$

Let $a, b \in A$ such that $a \in B$ and $a^- \odot b \in B$. We obtain $a \leq x_1 \oslash x_2 \oslash x_3 \oslash \ldots \oslash x_n$ and $a^- \odot b \leq y_1 \oslash y_2 \oslash y_3 \oslash \ldots \oslash y_m$, for some $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in H$. Since

$$b \leq a^- \rightsquigarrow (a^- \odot b) = (a^- \odot b) \oslash a \leq y_1 \oslash y_2 \oslash y_3 \oslash \ldots \oslash y_m \oslash x_1 \oslash x_2 \oslash x_3 \oslash \ldots \oslash x_n,$$

we have $b \in B$. By the notion of normal pseudo-hoop algebras, we know that A is good. Thus, B is an ideal of A by Theorem 3.8.

Suppose D is an ideal of A containing H. For any $b \in B$, we have $b \leq x_1 \oslash x_2 \oslash x_3 \oslash \ldots \oslash x_n$ for some $x_1, x_2, \ldots, x_n \in H$. Since $H \subseteq D$, we obtain $x_1 \oslash x_2 \oslash x_3 \oslash \ldots \oslash x_n \in D$. Then $b \in D$. Thus, $B \subseteq D$. Therefore, $B = \langle H \rangle$.

Similarly, $\langle H \rangle = \{h \in A : h \leq x_1 \otimes x_2 \otimes x_3 \otimes \ldots \otimes x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}$. $\square$

4 Ideals and congruences

In this section, we define congruences on pseudo-hoop algebras induced by ideals. We construct the quotient pseudo-hoop algebras via ideals and prove that there is a one-to-one correspondence between the set of all normal ideals of a pseudo-hoop algebra A with condition (pDN) and the set of all congruences relation on A. Also, we obtain an isomorphism theorem.

Definition 4.1. *Let $(A, \odot, \to, \rightsquigarrow)$ be a pseudo-hoop algebra and $\sim$ an equivalence relation on A.*

The equivalence relation $\sim$ is called a left congruence relation if $x \sim y$ implies $(a \odot x) \sim (a \odot y)$, $(a \to x) \sim (a \to y)$ and $(a \rightsquigarrow x) \sim (a \rightsquigarrow y)$ for any $x, y, a \in A$.

The equivalence relation $\sim$ is called a right congruence relation if $x \sim y$ implies $(x \odot a) \sim (y \odot a)$, $(x \to a) \sim (y \to a)$ and $(x \rightsquigarrow a) \sim (y \rightsquigarrow a)$ for any $x, y, a \in A$.

The equivalence relation $\sim$ is called a congruence relation if $x_1 \sim y_1$ and $x_2 \sim y_2$ imply $(x_1 \odot x_2) \sim (y_1 \odot y_2)$, $(x_1 \to x_2) \sim (y_1 \to y_2)$ and $(x_1 \rightsquigarrow x_2) \sim (y_1 \rightsquigarrow y_2)$.

Example 4.2. *Let A be a hoop algebra of Example 3.2. It is easy to check that*

$$\begin{aligned}\rho = \{&(0,0),\ (0,a),\ (0,d),\ (a,0),\ (a,a),\ (a,d),\ (d,0),\ (d,a),\ (d,d),\\ &(b,b),\ (b,c),\ (b,1),\ (c,b),\ (c,c),\ (c,1),\ (1,b),\ (1,c),\ (1,1)\},\end{aligned}$$

is a congruence relation on A.

Proposition 4.3. *A relation on a pseudo-hoop algebra $(A, \odot, \to, \rightsquigarrow)$ is a congruence relation if and only if it is both a left and a right congruence relation.*

Proof. The proof is obvious. □

If I is an ideal of a bounded pseudo-hoop algebra A, then define $\sim_I$ on A as follows:

$$\forall\, x, y \in A,\ x \sim_I y \text{ iff } x^- \odot y \in I,\, y^- \odot x \in I,\, x \odot y^\sim \in I,\, y \odot x^\sim \in I.$$

Proposition 4.4. *Let A be a bounded pseudo-hoop algebra and I an ideal of A. Then $\sim_I$ is an equivalence relation on A.*

Proof. It is clear that $\sim_I$ is symmetric. And we know that $\sim_I$ is reflexive by Proposition 2.3(2). We only need to show that $\sim_I$ is transitive. If $x \sim_I y$ and $y \sim_I z$, then

$$(z^- \odot y)^- \odot (z^- \odot x) = ((z^- \to y^-) \odot z^-) \odot x \le y^- \odot x \in I.$$

So $(z^- \odot y)^- \odot (z^- \odot x) \in I$. Since $z^- \odot y \in I$, we get $z^- \odot x \in I$ by Theorem 3.8 and Remark 3.9. Similarly, $x^- \odot z \in I$.

Since $(x \odot z^\sim) \odot (y \odot z^\sim)^\sim = x \odot (z^\sim \odot (z^\sim \rightsquigarrow y^\sim)) \le x \odot y^\sim \in I$, we get $x \odot z^\sim \in I$. Similarly, $z \odot x^\sim \in I$. Therefore, $x \sim_I z$. □

Theorem 4.5. *Let A be a good pseudo-hoop algebra and I a normal ideal of A. Then $\sim_I$ is a congruence relation on A.*

Proof. Let $x, y \in A$. By Propositions 4.3 and 4.4, we only need to show that $x \sim_I y$ implies $(x \odot a) \sim_I (y \odot a)$, $(a \odot x) \sim_I (a \odot y)$, $(x \to a) \sim_I (y \to a)$, $(a \to x) \sim_I (a \to y)$, $(x \rightsquigarrow a) \sim_I (y \rightsquigarrow a)$ and $(a \rightsquigarrow x) \sim_I (a \rightsquigarrow y)$ for any $a \in A$.

Suppose $x \sim_I y$. Then $x^- \odot y \in I$, $y^- \odot x \in I$, $x \odot y^\sim \in I$ and $y \odot x^\sim \in I$. Since

$$(x \odot a) \odot (y \odot a)^\sim = x \odot (a \odot (a \rightsquigarrow y^\sim)) \le x \odot y^\sim \in I,$$

we obtain $(x \odot a) \odot (y \odot a)^\sim \in I$. Since I is normal, we have $(y \odot a)^- \odot (x \odot a) \in I$. Similarly, we have $(y \odot a) \odot (x \odot a)^\sim \in I$ and $(x \odot a)^- \odot (y \odot a) \in I$. So $(x \odot a) \sim_I (y \odot a)$.

Similarly, $x \sim_I y$ implies $(a \odot x) \sim_I (a \odot y)$ for any $a \in A$. Moreover, by

$$(x^- \odot y)^- \odot (x^- \odot y^{-\sim}) = ((x^- \to y^-) \odot x^-) \odot y^{-\sim} \le y^- \odot y^{-\sim} = 0 \in I,$$

we obtain $(x^- \odot y)^- \odot (x^- \odot y^{-\sim}) \in I$. Thus, $x^- \odot y^{-\sim} \in I$ by Theorem 3.8. Similarly, we have $y^- \odot x^{-\sim} \in I$. Since I is normal, we obtain $y^{--} \odot x^- \in I$ and $x^{--} \odot y^- \in I$. Hence, $x^- \sim_I y^-$. Similarly, $x \sim_I y$ implies $x^\sim \sim_I y^\sim$.

If $x \sim_I y$, for any $a \in A$, then $(x \odot a^{\sim})^{-} \sim_I (y \odot a^{\sim})^{-}$, and so $(x \to a^{\sim -}) \sim_I (y \to a^{\sim -})$. Since A is good, we obtain $(x \to a^{-\sim}) \sim_I (y \to a^{-\sim})$. For any $b \in A$, we have $b^{\sim -} \odot b^{\sim} = 0 \in I$ and $b \odot b^{\sim -\sim} = b \odot b^{\sim} = 0 \in I$. Since I is normal, we have $b \sim_I b^{\sim -}$. Also, from A is good, we obtain $b^{-\sim} = b^{\sim -} \sim_I b$. Thus, $x^{-\sim} \sim_I x \sim_I y \sim_I y^{-\sim}$. Then $(x^{-\sim} \to a^{-\sim}) \sim_I (y^{-\sim} \to a^{-\sim})$ for any $a \in A$. By Proposition 2.3(6), we have $(x \to a)^{-\sim} \sim_I (y \to a)^{-\sim}$. Hence, $(x \to a) \sim_I (y \to a)$. Similarly, we can show $(x \rightsquigarrow a) \sim_I (y \rightsquigarrow a)$ for any $a \in A$.

If $x \sim_I y$, for any $a \in A$, then $(a \odot x^{\sim})^{-} \sim_I (a \odot y^{\sim})^{-}$, and so $(a \to x^{\sim -}) \sim_I (a \to y^{\sim -})$. Since $a \sim_I a^{\sim -}$, we obtain $(a \to x^{\sim -}) \sim_I (a^{\sim -} \to x^{\sim -})$ and $(a \to y^{\sim -}) \sim_I (a^{\sim -} \to y^{\sim -})$ by the above proof. Thus, $(a^{\sim -} \to x^{\sim -}) \sim_I (a^{\sim -} \to y^{\sim -})$ by transitivity. Hence $(a \to x)^{-\sim} \sim_I (a \to y)^{-\sim}$ by Proposition 2.3(6). Therefore, $(a \to x) \sim_I (a \to y)$. Analogously, we have $(a \rightsquigarrow x) \sim_I (a \rightsquigarrow y)$. $\square$

Let A be a good pseudo-hoop algebra and I a normal ideal of A. We define $A/I = \{[a] : a \in A\}$ where $[a] = \{x \in A : x \sim_I a\}$. For any $x, y \in A$, we define the operations $\odot$, $\to$ and $\rightsquigarrow$ on A/I by:

$$[x] \odot [y] = [x \odot y],\ [x] \to [y] = [x \to y] \text{ and } [x] \rightsquigarrow [y] = [x \rightsquigarrow y].$$

It is easy to know that $(A/I, \odot, \to, \rightsquigarrow, [1])$ is a bounded pseudo-hoop algebra with condition (pDN).

Proposition 4.6. *Let A be a good pseudo-hoop algebra.*
(1) *If $\sim$ is a congruence relation on A, then $B = \{x \in A : x \sim 0\}$ is a normal ideal of A. Also, $\sim_B$ is a congruence relation on A. If A satisfies the condition (pDN), then $\sim_B$ coincides with $\sim$.*
(2) *If I is a normal ideal of A, then $\sim_I$ is a congruence relation on A. Also, $[0] = \{x \in A : x \sim_I 0\}$ is a normal ideal of A and coincides with I.*
(3) *If A satisfies the condition (pDN), then there is a one-to-one correspondence between the set of congruence relations on A and the set of normal ideals of A.*

Proof. (1) By reflexivity, we have $0 \in B$. So $B \neq \emptyset$. Let $x, y \in B$. Then $(y^{-} \rightsquigarrow x) \sim (0^{-} \rightsquigarrow x)$, i.e. $(x \oslash y) \sim x$. Since $x \sim 0$, we obtain $x \oslash y \in B$. Similarly, $x \otimes y \in B$. Suppose $x, y \in A$ such that $x \leq y$ and $y \in B$. Then $(x \odot y^{\sim}) \sim (x \odot 0^{\sim}) = x$. Since $x \leq y \leq y^{\sim -}$, we have $x \odot y^{\sim} = 0$ by Proposition 2.3(3). Thus, $x \sim 0$. Hence, B is an ideal of A.

Suppose $x, y \in A$ such that $x^{-} \odot y \in B$. Then $y \rightsquigarrow x^{-\sim} = (x^{-} \odot y)^{\sim} \sim 1$. Thus $(y \odot (y \rightsquigarrow x^{-\sim})) \sim (y \odot 1)$, and so $(y \wedge x^{-\sim}) \sim y$. Therefore, $(y \odot x^{\sim}) \sim ((y \wedge x^{-\sim}) \odot x^{\sim})$. Since A is good, we obtain

$$(y \wedge x^{-\sim}) \odot x^{\sim} = (x^{-\sim} \to y) \odot x^{-\sim} \odot x^{\sim} = (x^{-\sim} \to y) \odot (x^{\sim -} \odot x^{\sim}) = 0.$$

Then $y \odot x^{\sim} \in B$. Similarly, $y \odot x^{\sim} \in B$ implies $x^{-} \odot y \in B$. Therefore, B is normal.

By Theorem 4.5, $\sim_B$ is a congruence on A. Suppose A satisfies condition (pDN). If $x \sim y$, we have $(x^{-} \odot y) \sim (y^{-} \odot y) = 0$, $(y^{-} \odot x) \sim (x^{-} \odot x) = 0$, $(y \odot x^{\sim}) \sim (y \odot y^{\sim}) = 0$ and $(x \odot y^{\sim}) \sim (x \odot x^{\sim}) = 0$. So $x \sim_B y$. Conversely, if $x \sim_B y$, then $(y \odot x^{\sim}) \sim 0$. Thus $((y \odot x^{\sim})^{-} \odot y) \sim (0^{-} \odot y)$, and so $(y \wedge x^{\sim -}) \sim y$. Using condition (pDN), we have $(y \wedge x) \sim y$. Similarly, $(y \wedge x) \sim x$. Hence, $x \sim y$. Therefore $\sim_B$ coincides with $\sim$.

(2) By Theorem 4.5, $\sim_I$ is a congruence relation on A. Then $[0]$ is a normal ideal of A by (1). So we only need to show that $[0]$ coincides with I. For any $x \in I$, we have $x^{-} \odot 0 = 0 \in I$, $0^{-} \odot x = x \in I$, $x \odot 0^{\sim} = x \in I$ and $0 \odot x^{\sim} = 0 \in I$. So $x \sim_I 0$, i.e. $x \in [0]$. Therefore, $I \subseteq [0]$. Conversely, if $x \in [0]$, then $x \odot 0^{\sim} \in I$. Thus $x = x \odot 0^{\sim} \in I$. Hence, $I = [0]$.

(3) It is obvious by (1) and (2). $\square$

Proposition 4.7. *Let X, Y be two bounded pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. We have the following results:*

(1) *If I is an (normal) ideal of Y, then $f^{-1}(I)$ is an (normal) ideal of X.*
(2) *If $f : X \to Y$ is a bounded pseudo-hoop isomorphism and J is an (normal) ideal of X, then $f(J)$ is an (normal) ideal of Y.*

Proof. (1) Let I be an ideal of Y. Since $0 \in f^{-1}(I)$, we have $f^{-1}(I) \neq \emptyset$. Let $x, y \in X$ such that $x \leq y$ and $y \in f^{-1}(I)$. Then $f(y) \in I$ and $f(x) \to f(y) = f(x \to y) = f(1) = 1$, i.e. $f(x) \leq f(y)$. Using (I2), we have $f(x) \in I$, i.e. $x \in f^{-1}(I)$. Suppose $x, y \in f^{-1}(I)$. Since $f(x \oslash y) = f(x) \oslash f(y)$ and $f(x), f(y) \in I$, we obtain $f(x \oslash y) \in I$, i.e. $x \oslash y \in f^{-1}(I)$. Similarly, $x \otimes y \in f^{-1}(I)$. Hence, $f^{-1}(I)$ is an ideal of X.

Let I be a normal ideal of Y. Then $x^- \odot y \in f^{-1}(I)$ iff $f(x)^- \odot f(y) \in I$ iff $f(y) \odot f(x)^\sim \in I$ iff $y \odot x^\sim \in f^{-1}(I)$ for any $x, y \in X$. Therefore, $f^{-1}(I)$ is a normal ideal of X.

(2) Let J be an ideal of X. Suppose $x, y \in Y$ such that $x \leq y$ and $y \in f(J)$. Then there is $v \in J$ such that $f(v) = y$. Since f is surjective, there is $u \in X$ such that $f(u) = x$. Since $f(u \to v) = x \to y = f(1)$ and f is injective, we have $u \to v = 1$, i.e. $u \leq v \in J$. Thus, $u \in J$. So $x \in f(J)$. Let $x, y \in f(J)$. Then there exist $u, v \in J$ such that $f(u) = x$ and $f(v) = y$. Since $u \oslash v, u \otimes v \in J$, we have $f(u) \oslash f(v) = f(u \oslash v) \in f(J)$ and $f(u) \otimes f(v) = f(u \otimes v) \in f(J)$. Therefore, $f(J)$ is an ideal of Y.

Let J be a normal ideal of X. Then $f(u)^- \odot f(v) \in f(J)$ iff $u^- \odot v \in J$ iff $v \odot u^\sim \in J$ iff $f(v) \odot f(u)^\sim \in f(J)$ for any $u, v \in X$. Thus, $f(J)$ is a normal ideal of Y. □

Let $f : X \to Y$ be a bounded pseudo-hoop homomorphism. Denote $\{x \in X : f(x) = 0\} = f^{-1}(0)$ by $kerf$. Then $kerf$ is an ideal of X.

Proposition 4.8. *Let X, Y be two bounded pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. If Y is good, then $\{0\}$ is a normal ideal of Y and $kerf$ is a normal ideal of X.*

Proof. It is clear that $\{0\}$ is an ideal of Y. Since Y is good, we obtain $x^- \odot y = 0$ iff $y \leq x^{-\sim}$ iff $y \leq x^{\sim-}$ iff $y \odot x^\sim = 0$ for any $x, y \in Y$. Therefore, $\{0\}$ is normal. Hence, $kerf$ is a normal ideal of X by Proposition 4.7(1). □

Let W be a nonempty subset of a bounded pseudo-hoop algebra A. We define

$$W^- = \{x^- : x \in W\} \text{ and } W^\sim = \{x^\sim : x \in W\}.$$

Let X, Y be two good pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. Since X is good and $kerf$ is a normal ideal of X, we know that $X/kerf$ is a bounded pseudo-hoop algebra. Then we have the following result.

Proposition 4.9. *Let X, Y be two good pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. If X is normal, then $X/kerf \cong (Imf)^-$ and $X/kerf \cong (Imf)^\sim$.*

Proof. Define $\varphi : X/kerf \to (Imf)^-$ by $\varphi([x]) = f(x)^{\sim-} = f(x)^{-\sim}$ for all $x \in X$. Then $\varphi([x]) \in (Imf)^-$. Since X is normal, for any $x, y \in X$ we have

$$f(x)^- \odot f(y)^- = f(x^{-\sim-} \odot y^{-\sim-}) = f((x^- \odot y^-)^{\sim-}) \in (Imf)^-.$$

By Proposition 2.3(6), for any $x, y \in X$ we obtain

$$f(x)^- \to f(y)^- = f(x^{-\sim-} \to y^{-\sim-}) = f((x^- \to y^-)^{\sim-}) \in (Imf)^-.$$

Similarly, $f(x)^- \rightsquigarrow f(y)^- \in (Imf)^-$. Thus, the operations $\odot$, $\to$ and $\rightsquigarrow$ are closed on $(Imf)^-$. Also, $1 = f(0)^- \in (Imf)^-$ and $0 = f(1)^- \in (Imf)^-$. Therefore, $(Imf)^-$ is a bounded pseudo-hoop algebra. It is clear that $\varphi([0]) = 0$. Since X is good, for any $x, y \in X$ we have

$$\varphi([x] \to [y]) = \varphi([x \to y]) = f((x \to y)^{-\sim}) = f(x^{-\sim} \to y^{-\sim}) = \varphi([x]) \to \varphi([y]).$$

Similarly, we have $\varphi([x] \rightsquigarrow [y]) = \varphi([x]) \rightsquigarrow \varphi([y])$. Since X is normal, we obtain

$$\varphi([x] \odot [y]) = \varphi([x \odot y]) = f((x \odot y)^{-\sim}) = f(x^{-\sim} \odot y^{-\sim}) = f(x)^{-\sim} \odot f(y)^{-\sim} = \varphi([x]) \odot \varphi([y]).$$

Therefore, φ is a bounded pseudo-hoop homomorphism.

Since $kerf$ is normal, we get $[x] = [y]$ iff $x \sim_{kerf} y$ iff $f(x^- \odot y) = f(y^- \odot x) = 0$ iff $f(x)^- \odot f(y) = f(y)^- \odot f(x) = 0$ iff $f(x)^- \leq f(y)^-$ and $f(y)^- \leq f(x)^-$ iff $f(x)^- = f(y)^-$ iff $\varphi([x]) = \varphi([y])$ for any $x, y \in X$. Thus, φ is injective. Since $f(a)^- = f(a^{-\sim-}) = \varphi([a^-])$ for any $a \in X$, we have φ is surjective. Hence, φ is isomorphic. Therefore, $X/kerf \cong (Imf)^-$. Similarly, $X/kerf \cong (Imf)^\sim$. □

5 Prime ideals

In this section, we introduce the concept of prime ideals in pseudo-hoop algebras and obtain several equivalent conditions of prime ideals.

Definition 5.1. *Let $(A, \odot, \to, \rightsquigarrow, 1)$ be a bounded pseudo-hoop algebra and P an ideal of A. Then P is called a prime ideal if $P \neq A$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in A$.*

Example 5.2. *Let $(A, \odot, \to, \rightsquigarrow, 1)$ be a bounded hoop algebra as in Example 3.2. Then $I_2 = \{0, c\}$ and $I_3 = \{0, a, d\}$ are all prime ideals of A. Since $a \wedge c = 0$ and $a, c \notin \{0\}$, $I_1 = \{0\}$ is not prime.*

Proposition 5.3. *Let X, Y be two bounded pseudo-hoop algebras and $f : X \to Y$ be a bounded pseudo-hoop homomorphism. Then the following statements hold:*
(1) If I is a prime ideal of Y and $f^{-1}(I) \neq X$, then $f^{-1}(I)$ is a prime ideal of X.
(2) If $f : X \to Y$ is a bounded pseudo-hoop isomorphism and J is a prime ideal of X, then $f(J)$ is a prime ideal of Y.

Proof. (1) It is obvious that $f^{-1}(I)$ is a proper ideal of X. For any $x, y \in X$, if $x \wedge y \in f^{-1}(I)$, then $f(x) \wedge f(y) = f(x \wedge y) \in I$. Since I is prime, we obtain $f(x) \in I$ or $f(y) \in I$. Thus, $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. Hence, $f^{-1}(I)$ is prime.

(2) By Proposition 4.7(2), $f(J)$ is an ideal of Y. Since $J \neq X$ and f is bijective, we have $f(J) \neq Y$. Let $x, y \in Y$ such that $x \wedge y \in f(J)$. Since f is surjective, there exist $u, v \in X$ such that $f(u) = x$ and $f(v) = y$. Then $f(u \wedge v) = f(u) \wedge f(v) = x \wedge y \in f(J)$. Thus, $u \wedge v \in J$. Since J is prime, we have $u \in J$ or $v \in J$. Hence, $x \in f(J)$ or $y \in f(J)$. Therefore, $f(J)$ is prime. □

Theorem 5.4. *Let A be a bounded pseudo-hoop algebra with the pre-linear condition and P be an ideal of A. Then the following conditions are equivalent:*
(1) P is prime;
(2) If $x \wedge y = 0$, then $x \in P$ or $y \in P$;
(3) For any $x, y \in A$, $(x \to y)^\sim \in P$ or $(y \to x)^\sim \in P$;
(4) For any $x, y \in A$, $(x \rightsquigarrow y)^- \in P$ or $(y \rightsquigarrow x)^- \in P$.

Proof. (1) $\Rightarrow$ (2) It is obvious by (1).

(2) $\Rightarrow$ (3) Since A is a lattice, for any $x, y \in A$ we have

$$(x \to y)^{\sim} \wedge (y \to x)^{\sim} = ((x \to y) \vee (y \to x))^{\sim} = 1^{\sim} = 0.$$

It follows that $(x \to y)^{\sim} \in P$ or $(y \to x)^{\sim} \in P$ by (2).

(3) $\Rightarrow$ (1) Suppose $x \wedge y \in P$ and $(x \to y)^{\sim} \in P$. We obtain $(x \wedge y) \oslash (x \to y)^{\sim} \in P$ by (RI1). Since $(x \wedge y)^{\sim} = ((x \to y) \odot x)^{\sim} = x \rightsquigarrow (x \to y)^{\sim}$, we get

$$x \leq (x \wedge y)^{\sim} \to (x \to y)^{\sim} = (x \wedge y) \oslash (x \to y)^{\sim} \in P.$$

So $x \in P$. Similarly, if $x \wedge y \in P$ and $(y \to x)^{\sim} \in P$, then $y \in P$.

(2) $\Rightarrow$ (4) The proof is similar to (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (1) The proof is similar to (3) $\Rightarrow$ (1). □

Corollary 5.5. *Let A be a bounded pseudo-hoop algebra with the pre-linear condition. If P is a prime ideal of A, then every proper ideal of A containing P is also prime.*

Proof. By Theorem 5.4(3) or (4). □

Corollary 5.6. *Let A be a bounded pseudo-hoop algebra with the pre-linear condition. Then every proper ideal of A is prime if and only if the ideal $\{0\}$ of A is prime.*

Proposition 5.7. *Let A be a good pseudo-hoop algebra and P be a normal ideal of A. If A satisfies the pre-linear condition, then P is prime if and only if A/P is a pseudo-hoop chain.*

Proof. It is enough to prove $[x] \leq [y] \Leftrightarrow (x \to y)^{\sim} \in P$ for $x, y \in A$. Suppose $[x] \leq [y]$, then $[x \to y] = [1]$, i.e. $(x \to y) \sim_P 1$. Therefore, $1 \odot (x \to y)^{\sim} = (x \to y)^{\sim} \in P$. Conversely, suppose $(x \to y)^{\sim} \in P$. We have $1 \odot (x \to y)^{\sim} = (x \to y)^{\sim} \in P$ and $(x \to y) \odot 1^{\sim} = 0 \in P$. Since P is normal, we obtain $(x \to y) \sim_P 1$. Thus, $[x \to y] = [1]$, i.e. $[x] \leq [y]$. So P is prime if and only if $(x \to y)^{\sim} \in P$ or $(y \to x)^{\sim} \in P$ for any $x, y \in A$ if and only if $[x] \leq [y]$ or $[y] \leq [x]$ for any $[x], [y] \in A/P$ if and only if A/P is a pseudo-hoop chain. □

6 Ideals and filters

In this section, we shall investigate the relationship between ideals and filters in pseudo-hoop algebras. First, some results are obtained by using the set of complement elements of pseudo-hoop algebras. In addition, the notion of $\odot$-prime ideals in pseudo-hoop algebras is given and the relationship between $\odot$-prime ideals and maximal filters is discussed.

Definition 6.1. *Let $(A, \odot, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo-hoop algebra and X be a subset of A. The sets of complement elements are denoted by $M(X)$ and $N(X)$, where $M(X) = \{x \in A \mid x^{-} \in X\}$ and $N(X) = \{x \in A \mid x^{\sim} \in X\}$.*

Example 6.2. [1] *Let $A = \{0, a, b, c, d, e, f, 1\}$. Define $\to$, $\rightsquigarrow$ and $\odot$ as follows:*

$\to = \rightsquigarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

Then $(A, \odot, \to, \rightsquigarrow, 1)$ is a bounded hoop algebra. Let $F_1 = \{d, e, f, 1\}$ and $F_2 = \{c, 1\}$. Then $M(F_1) = N(F_1) = \{0, a, b, c\}$ and $M(F_2) = N(F_2) = \{0, d\}$.

It is easy to check that F_1 and F_2 are filters of A. Also, $J_1 = \{0, a, b, c\}$ is an ideal of A. Since $b \leq c \in F_1^-$ and $b \notin F_1^-$, $F_1^- = F_1^\sim = \{c, 0\}$ is not an ideal of A. Since $e \geq d \in J_1^-$ and $e \notin J_1^-$, $J_1^- = J_1^\sim = \{1, d\}$ is not a filter of A.

The above example shows that ideals and filters are not dual under complement. Then we have the following results.

Theorem 6.3. *Let F be a filter of a good pseudo-hoop algebra A. Then $M(F)$ is an ideal generated by $F^\sim$ and $N(F)$ is an ideal generated by F^-.*

Proof. Suppose $x, y \in A$ such that $x^- \odot y \in M(F)$ and $x \in M(F)$. Then $(x^- \odot y)^- = x^- \to y^- \in F$ and $x^- \in F$. Since F is a filter of A, we have $y^- \in F$, and so $y \in M(F)$. Thus, $M(F)$ is an ideal of A by Theorem 3.8. For any $x \in F^\sim$, there exists $y \in F$ such that $x = y^\sim$. Since $y \leq y^{\sim-} = x^-$, we have $x^- \in F$, i.e. $x \in M(F)$. Hence, $F^\sim \subseteq M(F)$. Suppose I is an ideal of A containing $F^\sim$. If $x \in M(F)$, i.e. $x^- \in F$, then $x^{-\sim} \in F^\sim \subseteq I$. Since $x \leq x^{-\sim}$, we have $x \in I$. Thus, $M(F) \subseteq I$. Therefore, $M(F)$ is an ideal generated by $F^\sim$. Similarly, $N(F)$ is an ideal generated by F^-. □

Theorem 6.4. *Let A be a bounded pseudo-hoop algebra and I an ideal of A. If A is good, then $M(I)$ and $N(I)$ are filters of A such that $I^\sim \subseteq M(I)$ and $I^- \subseteq N(I)$.*

Proof. If $x \leq y$ and $x \in M(I)$, then $y^- \leq x^-$ and $x^- \in I$. Using (I2), we obtain $y^- \in I$, i.e. $y \in M(I)$. For any $x, y \in M(I)$, we have $x^-, y^- \in I$, and so by Proposition 2.3(7),

$$(x \odot y)^- = x \to y^- = x^{-\sim} \to y^- = x^- \oslash y^- \in I.$$

That is $x \odot y \in M(I)$. Hence, $M(I)$ is a filter of A. Suppose $x \in I^\sim$. There exists $y \in I$ such that $x = y^\sim$. Since $y \in I \Leftrightarrow y^{\sim-} \in I$, we have $x^- = y^{\sim-} \in I$, i.e. $x \in M(I)$. Hence, $I^\sim \subseteq M(I)$.

Similarly, we can show that $N(I)$ is a filter of A and $I^- \subseteq N(I)$. □

Theorem 6.5. *If I is an ideal of a bounded pseudo-hoop algebra A, then $I = M(N(I)) = N(M(I))$.*

Proof. For any $x \in A$, we obtain $x \in I$ iff $x^{-\sim} \in I$ iff $x^- \in N(I)$ iff $x \in M(N(I))$. So $I = M(N(I))$. Analogously, we can show $I = N(M(I))$. □

Theorem 6.6. *If F is a filter of a bounded pseudo-hoop algebra A, then $F \subseteq M(N(F))$ and $F \subseteq N(M(F))$.*

Proof. Let $x \in F$. Since $x \le x^{-\sim}$ and F is a filter of A, we have $x^{-\sim} \in F$. So $x^- \in N(F)$. Then $x \in M(N(F))$. Thus, $F \subseteq M(N(F))$. Similarly, $F \subseteq N(M(F))$. □

Remark 6.7. *In Theorem* 6.6*, we do not necessarily have $F = M(N(F))$ and $F = N(M(F))$. For instance, we have $M(N(F_1)) = \{d, e, f, 1\} = F_1$ and $M(N(F_2)) = \{a, b, c, e, f, 1\} \supseteq F_2$ in Example* 6.2*. Also, the converse of Theorem* 6.6 *is not true in general. Let $D = \{c\}$. Then $N(M(D)) = M(N(D)) = \{a, b, c\} \supseteq D$. But D is not a filter of A.*

In order to further discuss the relationship between ideals and filters of a pseudo-hoop algebra, we introduce the notion of $\odot$-prime ideals in pseudo-hoop algebras.

Definition 6.8. *Let $(A, \odot, \to, \rightsquigarrow, 1)$ be a bounded pseudo-hoop algebra and P an ideal of A. Then P is called a $\odot$-prime ideal of A if $P \neq A$ and $x \odot y \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in A$.*

Example 6.9. *Let A be the pseudo hoop algebra as in Example* 3.2*. Then it is easy to show that $I_3 = \{0, a, d\}$ is a $\odot$-prime ideal of A.*

Proposition 6.10. *Let A be a bounded pseudo-hoop algebra. Then every $\odot$-prime ideal of A is a prime ideal of A. The converse may not hold.*

Proof. Let P be a $\odot$-prime ideal of A. If P is not prime, there exist $x, y \in A$ such that $x \wedge y \in P$, but $x, y \notin P$. We obtain $x \odot y \in P$ by $x \odot y \le x \wedge y$. Then $x \in P$ or $y \in P$, which is a contradiction. Therefore, P is a prime ideal of A.

In Example 3.2, $I_2 = \{0, c\}$ is a prime ideal of A. Since $b \odot d = 0 \in I_2$ and $b, d \notin I_2$, we get I_2 is not a $\odot$-prime ideal of A. Therefore, the converse may not hold. □

Proposition 6.11. *Let A be a bounded pseudo-hoop algebra and P an ideal of A. Then P is a $\odot$-prime ideal of A if and only if P is a prime ideal of A and $x \odot y \in P$ implies $x \wedge y \in P$ for any $x, y \in P$.*

Proof. Let P be a $\odot$-prime ideal of A. Then P is a prime ideal of A by Proposition 6.10. Suppose $x \odot y \in P$. We obtain $x \in P$ or $y \in P$ by Definition 6.8. Since $x \wedge y \le x, y$, we obtain $x \wedge y \in P$. Therefore, $x \odot y \in P$ implies $x \wedge y \in P$ for any $x, y \in P$.

Conversely, if $x \odot y \in P$, then $x \wedge y \in P$. By the notion of prime ideals, we know that $x \in P$ or $y \in P$. Therefore, P is a $\odot$-prime ideal of A. □

Let X be a subset of a pseudo-hoop algebra A. We denote $A - X$ by $\overline{X}$. The following results study the relationship between ideals and filters in pseudo-hoop algebras.

Theorem 6.12. *Let A be a bounded pseudo-hoop algebra and P an ideal of A. If P is a $\odot$-prime ideal of A, then $\overline{P}$ is a maximal filter of A.*

Proof. Suppose P is a $\odot$-prime ideal of A. Since $P \neq A$, we obtain $\overline{P} \neq \emptyset$. Since $0 \in P$, i.e. $0 \notin \overline{P}$, we have $\overline{P} \neq A$. Let $x, y \in \overline{P}$. If $x \odot y \in P$, then $x \in P$ or $y \in P$, which is a contradiction. Thus, $x \odot y \in \overline{P}$. Suppose $x, y \in A$ such that $x \le y$ and $x \in \overline{P}$. It follows that $y \in \overline{P}$, i.e. $y \notin P$. If not, since P is an ideal of A and $x \le y$, we have $x \in P$, which is a contradiction. Therefore, $\overline{P}$ is a filter of A.

Let Q be a filter of A strictly containing $\overline{P}$. Then there exists $a \in A$ such that $a \in Q$ and $a \notin \overline{P}$. So $a \in P \cap Q$. It follows that $a^-, a^\sim \notin P$. If not, then $a^- \oslash a = a^- \rightsquigarrow a^- = 1 \in P$ and $a \otimes a^\sim = a^\sim \to a^\sim = 1 \in P$, which is a contradiction. So $a^\sim \in \overline{P} \subseteq Q$. Using (F1), we have $0 = a \odot a^\sim \in Q$. Then $Q = A$. Hence, $\overline{P}$ is a maximal filter of A. □

Remark 6.13. *By the previous proof, if P is a proper ideal of A and $a \in P$, then $a^-, a^\sim \notin P$.*

Theorem 6.14. *Let A be a bounded pseudo-hoop algebra and P be an ideal of A. If $\overline{P}$ is a normal and maximal filter of A, then P is a $\odot$-prime ideal of A.*

Proof. Let $\overline{P}$ be a normal and maximal filter of A. Then $P \neq \emptyset$. Since $1 \in \overline{P}$, i.e. $1 \notin P$, we have $P \neq A$. Suppose $x, y \in A$ such that $x \odot y \in P$, i.e. $x \odot y \notin \overline{P}$. Therefore, $\overline{P}$ is strictly contained in $(\overline{P} \cup \{x \odot y\}]$. So $(\overline{P} \cup \{x \odot y\}] = A$. By Proposition 2.4, there exists $n \in \mathbb{N}$ and $h \in \overline{P}$ such that $h \odot (x \odot y)^n \leq 0$. That is $h \leq ((x \odot y)^n)^-$. So $((x \odot y)^n)^- \in \overline{P}$. Suppose $x, y \notin P$. Since $\overline{P}$ is a filter of A, we obtain $(x \odot y)^n \in \overline{P}$. It follows that $0 = ((x \odot y)^n)^- \odot (x \odot y)^n \in \overline{P}$. Using (F2), we have $\overline{P} = A$, which is a contradiction. Therefore, $x \odot y \in P$ implies $x \in P$ or $y \in P$. Thus, P is a $\odot$-prime ideal of A. □

7 Conclusions

We defined ideals in pseudo-hoop algebras using two kinds of addition operations. We gave some equivalent characterizations of ideals of good pseudo-hoop algebras. Also, the congruence relation on a pseudo-hoop algebra is induced by ideals are defined. Using ideals, we constructed the quotient pseudo-hoop algebras and got an isomorphism theorem. We proved that if a pseudo-hoop algebra A satisfies condition (pDN), then there is a one-to-one correspondence between the set of all congruence relation on A and the set of all normal ideals of A. The notion of prime ideals in pseudo-hoop algebras is introduced. We showed that the normal ideal of a good pseudo-hoop algebra with the pre-linear condition is prime if and only if the corresponding quotient pseudo-hoop algebra is a pseudo-hoop chain. In addition, we discussed the relationship between ideals and filters in pseudo-hoop algebras. We found that ideals and filters behave differently in pseudo-hoop algebras. Also, we discussed the relationship between $\odot$-prime ideals and maximal filters.

For future works, we will study other types of ideals in pseudo-hoop algebras and discuss the relationships between these ideals. The notion of implicative ideals of hoop algebras was studied in [1]. We shall investigate the notion of implicative ideals in pseudo-hoop algebras. Similarly to the notion of nodal filters in hoop algebras in [15], we shall define the notion of nodal ideals in pseudo-hoop algebras. In this paper, we can observe that the operators M and N defined in Definition 6.1 transform filters into ideals and vice versa. We shall further study other properties of M and N. In addition, stabilizers in hoop algebras were introduced in [3]. We shall study stabilizers in pseudo-hoop algebras. Furthermore, we shall discuss the relationship between ideals and stabilizers in pseudo-hoop algebras.

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