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Characterization of ordered semihypergroups in terms of uni-soft bi-hyperideals

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Abstract

In this paper, we introduce the concept of union-soft (briefly, uni-soft) bi-hyperideal of an ordered semihypergroup. The notions of prime (strongly prime, semiprime, irreducible, and strongly irreducible) uni-soft bi-hyperideals in ordered semihypergroups are introduced and related properties are investigated. Numerous examples of these notions are given. The relationship between prime and strongly prime, irreducible and strongly irreducible uni-soft bi-hyperideals are considered and characterizations of these concepts are established. Regular and intra-regular ordered semihypergroups are characterized in terms of these notions.

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1 Introduction

The theory of soft sets is introduced by Molodtsov in [24]. The basic aim of this theory is to introduce a new tool to discuss uncertainty. In soft set theory, we have enough number of parameters

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to deal with uncertainty. This quality makes it prominent among its predecessor theories such as probability theory, interval mathematics, fuzzy sets and rough sets. Hybrid soft set theories such as fuzzy soft sets and vague soft sets have been studied in [22, 28]. Theories of fuzzy sets and rough sets are quite different in their nature from soft sets, but there is a strong link among these three theories. During recent years, efforts have been made to establish links among these three theories [2, 10]. Initially, Maji et al. established the theoretical base for soft sets. They defined several operations on soft sets [21]. Later on it was felt that some of the operations defined for soft sets suffer many difficulties. In order to make these operations sensible, Ali et al. defined some new operations on soft sets [3]. Algebras, which appear as a natural consequence of these new operations have been studied in [4]. The theory of soft sets has emerged as a new tool to discuss algebraic structures. Study of soft algebraic structures was initiated by Aktas and Cagman. They studied soft groups in [1]. For application of soft sets in algebraic structures, see [8, 9, 15, 16, 19, 20, 25, 29].

In 1934 during the 8^{th} Congress of Scandinavian Mathematicians, F. Marty [23], introduced the concept of hyperstructures, analyzed its properties and applied them to groups, rational fractions and algebraic functions. After the pioneering work of F. Marty, algebraic hyperstructures have been developed by many researchers and widely studied from theoretical point of view and for their applications to many subjects of pure and applied mathematics. In a classical algebraic structures, the composition of two elements is an element, while in an algebraic hyperstructures, the composition of two elements is a set. Semihypergroups have been found useful for dealing with problems in different areas of algebraic hyperstructures. A short review of hyperstructures and some principal notions about hyperstructures and semihypergroups theory can be found in [5, 6, 7, 11, 12, 13, 14, 18, 26, 27].

Motivated by the study of Naz and Shabir [25], here in this paper, we apply uni-soft set theory to ordered semihypergroups and define the notion of uni-soft bi-hyperideals. Moreover, we define prime, strongly prime, semiprime, irreducible and strongly irreducible uni-soft bi-hyperideals of ordered semihypergrouprs. Different classes of ordered semihypergroups are characterized by the properties of uni-soft bi-hyperideals.

2 Preliminaries

A hypergroupoid is a nonempty set S equipped with a hyperoperation $\circ$, that is a map $\circ : S \times S \longrightarrow P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [23]). We shall denote by $x \circ y$, the hyperproduct of elements x, y of S. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$. Let A, B be the nonempty subsets of S. Then the hyperproduct of A and B is defined as $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$. We shall write $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$.

Definition 2.1. [27] *An algebraic hyperstructure $(S, \circ, \leq)$ is called an ordered semihypergroup (also called po-semihypergroup) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that the monotone condition holds as follows:*

$a \leq b$ implies that $x \circ a \leq x \circ b$ and $a \circ x \leq b \circ x$ for all $x, a, b \in S$. If for every $a \in A$ there exists $b \in B$ such that $a \leq b$, then we can write as $A \preceq B$. If $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Definition 2.2. [5] *A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a subsemihypergroup of S if for all $x, y \in A$, we have $x \circ y \subseteq A$.*

Equivalently: A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a subsemihypergroup of S if $A \circ A \subseteq A$.

Definition 2.3. [27] *Let $(S, \circ, \leq)$ be an ordered semihypergroup and A be a nonempty subset of S. Then A is called a left (right) hyperideal of S if:*

(1) $S \circ A \subseteq A$ $((A \circ S) \subseteq A)$.

(2) *If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.*

If A is both a right hyperideal and a left hyperideal of S, then it is called a hyperideal of S.
For $A \subseteq S$, we denote $(A] = \{t \in S \mid t \leq h$ for some $h \in A\}$.

Definition 2.4. [27] *A subsemihypergroup A of an ordered semihypergroup $(S, \circ, \leq)$ is called a bi-hyperideal of S if:*

(1) $A \circ S \circ A \subseteq A$.

(2) *If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.*

We denote by $B(a)$ the bi-hyperideal of S generated by a $(a \in S)$. We have

$$B(a) = (a \cup (a \circ a) \cup (a \circ S \circ a)].$$

Definition 2.5. [5] (1) *An ordered semihypergroup $(S, \circ, \leq)$ is called regular if for each $a \in S$ there exists $x \in S$ such that $a \leq a \circ x \circ a$.*

(2) *An ordered semihypergroup $(S, \circ, \leq)$ is called intra-regular if for each $a \in S$ there exist $x, y \in S$ such that $a \leq x \circ a \circ a \circ y$.*

Lemma 2.6. [12] *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$B = (B \circ S \circ B]$ for every bi-hyperideal B of S.*

(3) *$B(a) = (B(a) \circ S \circ B(a)]$ for every $a \in S$.*

2.1 Basic concepts of soft sets

In what follows, we take $E = S$ as the set of parameters, which is an ordered semihypergroup, unless otherwise specified.
From now on, U is an initial universe set, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.7. [24] *A soft set f_A over U is defined as $f_A : E \longrightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Hence, f_A is also called an approximation function.*

A soft set f_A over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}.$$

It is clear from Definition 2.7, that a soft set is a parameterized family of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.8. [24] *(i) Let $f_A, f_B \in S(U)$. Then f_A is called a soft subset of f_B, denoted by $f_A \widetilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$. Two soft sets f_A and f_B are said to be equal soft sets if $f_A \widetilde{\subseteq} f_B$ and $f_B \widetilde{\subseteq} f_A$ and is denoted by $f_A \widetilde{=} f_B$.*

(ii) Let $f_A, f_B \in S(U)$. Then the soft union of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B = f_{A \cup B}$, is defined by $\left(f_A \widetilde{\cup} f_B\right)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.

(iii) Let $f_A, f_B \in S(U)$. Then the soft intersection of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B = f_{A \cap B}$, is defined by $\left(f_A \widetilde{\cap} f_B\right)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

(iv) $A \subseteq B$, if for any $x \in A$, we have, $f_A(x) \subseteq f_B(x)$. For $x \in S$, we define $A_x = \{(y,z) \in S \times S \mid x \leq y \circ z\}$.

Definition 2.9. [17] *Let f_A and g_B be two soft sets of an ordered semihypergroup S over U. Then, the uni-soft product, denoted by $f_A \widetilde{*} g_B$, for all $x \in S$ is defined by*

$$f_A \widetilde{*} g_B : S \longrightarrow P(U), x \longmapsto (f_A \widetilde{*} g_B)(x) = \begin{cases} \bigcap\limits_{(y,z) \in A_x} \{f_A(y) \cup g_B(z)\}, & \text{if } A_x \neq \emptyset, \\ U, & \text{if } A_x = \emptyset, \end{cases}$$

Definition 2.10. [17] *Let $A \subseteq S$. Then the soft characteristic function $\mathcal{S}_A : S \longrightarrow P(U)$ is defined by*

$$\mathcal{S}_A(x) := \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A. \end{cases}$$

The soft set (U_S, S), where $U_S(x) = U$ for all $x \in S$, is called the identity soft set over U. For the characteristic soft set $\mathcal{S}_A$ over U, the soft set $\mathcal{S}_A^c$ over U given as follows:

$$\mathcal{S}_A^c(x) := \begin{cases} \emptyset & \text{if } x \in A \\ U & \text{if } x \notin A. \end{cases}$$

For an ordered semihypergroup S, the soft set "$\mathcal{S}_S^c$" of S over U is defined as follows:

$$\mathcal{S}_S^c : S \longrightarrow P(U), x \longmapsto \mathcal{S}_S^c(x) = \emptyset \text{ for all } x \in S.$$

$\mathcal{S}_S^c$ is called an empty soft set of S over U.

Definition 2.11. [17] *Let f_A be a soft set of an ordered semihypergroup S over U a subset δ such that $\delta \in P(U)$. The δ-exclusive set of f_A is denoted by $e_A(f_A, \delta)$ and defined to be the set*

$$e_A(f_A, \delta) = \{x \in S \mid f_A(x) \subseteq \delta\}.$$

Definition 2.12. [17] *A soft set f_A of an ordered semihypergroup S over U is called a uni-soft semihypergroup of S over U if for any $x, y \in S$, $\bigcup\limits_{\alpha \in x \circ y} f_A(\alpha) \subseteq f_A(x) \cup f_A(y)$.*

Definition 2.13. [17] *Let f_A be a soft set of an ordered semihypergroup S over U. Then f_A is called a uni-soft left (right) hyperideal of S over U if for all $x, y \in S$, it satisfies the following conditions:*

(1) $\bigcup\limits_{\alpha \in x \circ y} f_A(\alpha) \subseteq f_A(y) \left(\bigcup\limits_{\alpha \in x \circ y} f_A(\alpha) \subseteq f_A(x) \right)$.

(2) *If $x \leq y$, then $f_A(x) \subseteq f_A(y)$.*

A soft set f_A of S over U is called a uni-soft hyperideal of S over U (or uni-soft two sided hyperideal) if it is both a uni-soft left hyperideal of S over U and a uni-soft right hyperideal of S over U.

Definition 2.14. *A uni-soft hyperideal f_A of an ordered semihypergroup S over U is called idempotent if, $f_A \widetilde{*} f_A = f_A$.*

3 Uni-soft bi-hyperideals

Definition 3.1. *A uni-soft semihypergroup f_A of an ordered semihypergroup S over U is called a uni-soft bi-hyperideal of S over U if for all $x, y, z \in S$ it satisfies the following conditions:*

(1) $\bigcup_{\alpha \in x \circ y \circ z} f_A(\alpha) \subseteq f_A(x) \cup f_A(z)$.

(2) *If $x \leq y$, then $f_A(x) \subseteq f_A(y)$.*

Example 3.2. *Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation and order relation are defined:*

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{b\}$	$\{d\}$	$\{b\}$	$\{b\}$
c	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e, f\}$	$\{e, f\}$
d	$\{a\}$	$\{a\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e, f\}$	$\{e, f\}$
f	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e, f\}$	$\{f\}$

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (f,e)\}.$$

Suppose $U = \{p, q, r\}$ and $A = \{b, c, e, f\}$. Let us define $f_A(a) = \emptyset$, $f_A(b) = \{p\}$, $f_A(c) = \{p, q\}$, $f_A(d) = \emptyset$, $f_A(e) = \{p, q, r\}$ and $f_A(f) = \{p, q, r\}$. Then f_A is a uni-soft bi-hyperideal of S over U.

Proposition 3.3. [17] *Let S be an ordered semihypergroup such that $\mathcal{S}^c_{\mathcal{A}}$ and $\mathcal{S}^c_{\mathcal{B}}$ be soft sets of S over U where A and B are nonempty subsets of S. Then the following properties hold:*

(1) $\mathcal{S}^c_{\mathcal{A}} \widetilde{\cup} \mathcal{S}^c_{B} = \mathcal{S}^c_{A \cup B}$.

(2) $\mathcal{S}^c_{\mathcal{A}} \widetilde{*} \mathcal{S}^c_{\mathcal{B}} = \mathcal{S}^c_{(A \circ B]}$.

Proposition 3.4. [17] *An ordered semihypergroup S is intra-regular if and only if for every uni-soft right hyperideal f_A and every uni-soft left hyperideal g_B of S over U, we have*

$$f_A \widetilde{\cup} g_B \widetilde{\supseteq} g_B \widetilde{*} f_A.$$

Proposition 3.5. [17] *Let S be an ordered semihypergroup such that f_A and g_B be soft sets of S over U. If S is regular and f_A is a uni-soft right hyperideal of S over U, g_B is a uni-soft left hyperideal of S over U then*

$$f_A \widetilde{\cup} g_B \widetilde{\supseteq} f_A \widetilde{*} g_B.$$

Equivalently, $f_A \widetilde{\cup} g_B = f_A \widetilde{*} g_B$.

Lemma 3.6. *Let S be an ordered semihypergroup, f_A and g_B be two soft sets of S over U. Then*

$$(f_A \widetilde{*} g_B) \widetilde{\supseteq} f_A \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \left(\textit{resp.}, \ (f_A \widetilde{*} g_B) \widetilde{\supseteq} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} g_B \right).$$

Proof. Let $a \in S$. If $A_a = \emptyset$, then $(f_A \widetilde{*} g_B)(a) = U \supseteq (f_A \widetilde{*} \mathcal{S}^c_{\mathcal{S}})(a)$. Let $A_a \neq \emptyset$. Then

$$(f_A \widetilde{*} g_B)(a) = \bigcap_{(y,z) \in A_a} \{f_A(y) \cup g_B(z)\}.$$

As $g_B(z)\widetilde{\supseteq}\mathcal{S}^c_{\mathcal{S}}(z)$ for all $z \in S$. Thus

$$(f_A \widetilde{*} g_B)(a) = \bigcap_{(y,z)\in A_a} \{f_A(y) \cup g_B(z)\} \supseteq \bigcap_{(y,z)\in A_a} \{f_A(y) \cup \mathcal{S}^c_{\mathcal{S}}(z)\} = (f_A \widetilde{*} \mathcal{S}^c_{\mathcal{S}})(a).$$

□

Theorem 3.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and B be a nonempty subset of S. Then B is bi-hyperideal of S if and only if the soft set $\mathcal{S}^c_{\mathcal{B}}$ is a uni-soft bi-hyperideal of S over U.*

Proof. Suppose that B is a bi-hyperideal of S. Let $x, y \in S$ such that $x \leq y$. If $y \notin B$, then $\mathcal{S}^c_{\mathcal{B}}(y) = U$ and so $\mathcal{S}^c_{\mathcal{B}}(x) \subseteq U = \mathcal{S}^c_{\mathcal{B}}(y)$. If $y \in B$, then $\mathcal{S}^c_{\mathcal{B}}(y) = \emptyset$. Since $x \leq y$ and B is a bi-hyperideal of S, we have $x \in B$ and thus $\mathcal{S}^c_{\mathcal{B}}(x) = \emptyset = \mathcal{S}^c_{\mathcal{B}}(y)$. Therefore, B is a subsemihypergroup of S. Let $x, y \in S$. Then $\bigcup_{\alpha\in x\circ y} \mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq \mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(y)$ for every $\alpha \in x \circ y$. Indeed: If $x \circ y \nsubseteq B$, then there exists $\alpha \in x \circ y$ such that $\alpha \notin B$, $\mathcal{S}^c_{\mathcal{B}}(\alpha) = U$, so we have $\bigcup_{\alpha\in x\circ y} \mathcal{S}^c_{\mathcal{B}}(\alpha) = U$. Besides that $x \circ y \nsubseteq B$ implies that $x \notin B$ or $y \notin B$. Then $\mathcal{S}^c_{\mathcal{B}}(x) = U$ or $\mathcal{S}^c_{\mathcal{B}}(y) = U$ and so $\bigcup_{\alpha\in x\circ y} \mathcal{S}^c_{\mathcal{B}}(\alpha) = \mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(y)$. Let $x \circ y \subseteq B$. Then $\mathcal{S}^c_{\mathcal{B}}(\alpha) = \emptyset$ for any $\alpha \in x \circ y$. It implies that $\bigcup_{\alpha\in x\circ y} \mathcal{S}^c_{\mathcal{B}}(\alpha) = \emptyset$. Since $\mathcal{S}^c_{\mathcal{B}}(x) \supseteq \emptyset$ for any $x \in B$, we have $\bigcup_{\alpha\in x\circ y} \mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq \mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(y)$. Therefore, $\mathcal{S}^c_{\mathcal{B}}$ is a uni-soft semihypergroup of S over U. Let x, y and z be any elements of S. If $x, z \in B$, then $\mathcal{S}^c_{\mathcal{B}}(x) = \mathcal{S}^c_{\mathcal{B}}(z) = \emptyset$ and since for every $\alpha \in x \circ y \circ z \subseteq B \circ S \circ B \subseteq B$, we have $\mathcal{S}^c_{\mathcal{B}}(\alpha) = \emptyset = \{\mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(z)\}$. Thus $\bigcup_{\alpha\in x\circ y\circ z} \mathcal{S}^c_{\mathcal{B}}(\alpha) = \emptyset = \{\mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(z)\}$. If $x \notin B$ or $z \notin B$, then $\mathcal{S}^c_{\mathcal{B}}(x) = U$ or $\mathcal{S}^c_{\mathcal{B}}(z) = U$, and so we have $\mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq U = \{\mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(z)\}$. Thus $\bigcup_{\alpha\in x\circ y\circ z} \mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq \{\mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(z)\}$. Therefore, $\mathcal{S}^c_{\mathcal{B}}$ is a uni-soft bi-hyperideal of S over U.

Conversely, let $\emptyset \neq B \subseteq S$ such that $\mathcal{S}^c_{\mathcal{B}}$ is a uni-soft bi-hyperideal of S over U. We claim that $B \circ B \subseteq B$. To prove the claim, let $x, y \in B$. By hypothesis, $\bigcup_{\alpha\in x\circ y} \mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq \mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(y) = \emptyset$ which implies that $\mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq \emptyset$ for any $\alpha \in x \circ y$. On the other hand $\mathcal{S}^c_{\mathcal{B}}(x) \supseteq \emptyset$ for all $x \in S$. Thus for any $\alpha \in x \circ y$, $\mathcal{S}^c_{\mathcal{B}}(\alpha) = \emptyset$ implies that $\alpha \in B$. It follows that $B \circ B \subseteq B$. Hence, B is a subsemihypergroup of S. Let $\alpha \in B \circ S \circ B$, then there exist $x, z \in B$ and $y \in S$ such that $\alpha \in x \circ y \circ z$. Since

$$\bigcup_{\alpha\in x\circ y\circ z} \mathcal{S}^c_{\mathcal{B}}(\alpha) \subseteq \mathcal{S}^c_{\mathcal{B}}(x) \cup \mathcal{S}^c_{\mathcal{B}}(z) = \emptyset \cup \emptyset = \emptyset.$$

Hence, for each $\alpha \in x \circ y \circ z$, we have $\mathcal{S}^c_{\mathcal{B}}(\alpha) = \emptyset$, and so $\alpha \in B$. Thus $B \circ S \circ B \subseteq B$. Let $x \in S$ and $y \in B$ such that $x \leq y$. Then $\mathcal{S}^c_{\mathcal{B}}(x) \subseteq \mathcal{S}^c_{\mathcal{B}}(y) = \emptyset$, and so $x \in B$. Thus B is a bi-hyperideal of S. □

Theorem 3.8. *Let f_A be a soft set of an ordered semihypergroup S over U and $\delta \in P(U)$. Then f_A is a uni-soft bi-hyperideal of S over U if and only if nonempty δ-exclusive set $e_A(f_A, \delta)$ is a bi-hyperideal of S.*

Proof. Assume that f_A is a uni-soft bi-hyperideal of S over U. Suppose $\delta \in P(U)$ with $e_A(f_A,\delta) \neq \emptyset$. Let $a \in e_A(f_A,\delta) \circ S \circ e_A(f_A,\delta)$, for any $a \in b \circ s \circ c$ for some $b, c \in e_A(f_A,\delta)$ and $s \in S$. Since $\delta \supseteq f_A(b) \cup f_A(c) \supseteq \bigcup\limits_{a \in b \circ s \circ c} f_A(a)$, we have $f_A(a) \subseteq \delta$. We get $a \in e_A(f_A,\delta)$. This implies that $e_A(f_A,\delta) \circ S \circ e_A(f_A,\delta) \subseteq e_A(f_A,\delta)$. Let $x \in e_A(f_A,\delta)$ and $y \in S$ with $y \leq x$. Since $\delta \supseteq f_A(x) \supseteq f_A(y)$, we get $y \in e_A(f_A,\delta)$. Therefore, $e_A(f_A,\delta)$ is a bi-hyperideal of S.

Conversely, we assume that for every $\delta \in P(U)$, $e_A(f_A,\delta)$ is a bi-hyperideal of S. We show that $\bigcup\limits_{a \in b \circ s \circ c} f_S(a) \subseteq f_A(b) \cup f_A(c)$ for all $b, s, c \in S$. We choose $\delta = f_A(b) \cup f_A(c)$. By assumption $e_A(f_A,\delta)$ is a bi-hyperideal of S. Since $b, c \in e_A(f_A,\delta)$, $b \circ s \circ c \subseteq e_A(f_A,\delta)$. Then for every $a \in b \circ s \circ c$, we have $f_A(a) \subseteq \delta$ and so $\bigcup\limits_{a \in b \circ s \circ c} f_A(a) \subseteq \delta = f_A(b) \cup f_A(c)$. Let $a, b \in S$ such that $a \leq b$. Since $a \leq b$, $b \in e_A(f_A,\delta)$ let $f_A(b) = \delta$ and $e_A(f_A,\delta)$ is a bi-hyperideal of S, we get $a \in e_A(f_A,\delta)$. So $f_A(a) \subseteq \delta = f_A(b)$. Hence, $f_A(a) \subseteq f_A(b)$. Therefore, f_A is a uni-soft bi-hyperideal of S over U. □

Example 3.9. *Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:*

$\circ$	a	b	c
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$
b	$\{a\}$	$\{a,b\}$	$\{a,c\}$
c	$\{a\}$	$\{a,b\}$	$\{c\}$

$$\leq := \{(a,a),(b,b),(c,c),(a,b)\}.$$

Suppose $U = \{1,2,3\}$ and $A = \{b,c\}$. Here $\{a\}$, $\{a,b\}$, $\{a,c\}$, and S are bi-hyperideals of S. Let us define $f_A(a) = \emptyset$, $f_A(b) = \{1,2\}$, and $f_A(c) = \{1,3\}$. Then

$$e_A(f_A,\delta) = \begin{cases} \{a\} & \text{if } \delta = \{1\} \\ \{a\} & \text{if } \delta = \{2\} \\ \{a\} & \text{if } \delta = \{3\} \\ \{a,b\} & \text{if } \delta = \{1,2\} \\ \{a,c\} & \text{if } \delta = \{1,3\} \\ \{a\} & \text{if } \delta = \{2,3\} \\ S & \text{if } \delta = U \end{cases}$$

So by Theorem 3.8, f_A is a uni-soft bi-hyperideal of S over U.

Proposition 3.10. *Let f_B be a uni-soft bi-hyperideal of an ordered semihypergroup S over U. Then:*

(1) $f_B \widetilde{*} f_B \widetilde{\supseteq} f_B$;
(2) $f_B \widetilde{*} \mathcal{S}^c_S \widetilde{*} f_B \widetilde{\supseteq} f_B$.

Proof. (1) Let f_B be a uni-soft bi-hyperideal of an ordered semihypergroup S over U. Then for each $x, y, z \in S$, if $A_x = \emptyset$, then $(f_B \widetilde{*} f_B)(x) = U \supseteq f_B(x)$. If $A_x \neq \emptyset$, then

$$(f_B \widetilde{*} f_B)(x) = \bigcap_{(y,z) \in A_x} \{f_B(y) \cup f_B(z)\}.$$

As f_B is a uni-soft semihypergroup of S over U, we have $\bigcup\limits_{x \in y \circ z} f_B(x) \subseteq f_B(y) \cup f_B(z)$ for each $y, z \in S$, $x \in y \circ z$. Thus $f_B(x) \subseteq f_B(y) \cup f_B(z)$. Hence,

$$(f_B \widetilde{*} f_B)(x) = \bigcap_{(y,z) \in A_x} \{f_B(y) \cup f_B(z)\} \supseteq f_B(x).$$

Thus, $f_B \widetilde{*} f_B \widetilde{\supseteq} f_B$.

(2) Let $a \in S$. If $A_a = \emptyset$, then $(f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(a) = U \supseteq f_B(a)$. Let $A_a \neq \emptyset$. Then

$$\begin{aligned}(f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(a) &= \bigcap_{(y,z)\in A_a} \{f_B(y) \cup (\mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(z)\} \\ &= \bigcap_{(y,z)\in A_a} \left\{ f_B(y) \cup \left\{ \bigcap_{(p,q)\in A_z} (\mathcal{S}^c_{\mathcal{S}}(p) \cup f_B(q)) \right\} \right\} \\ &= \bigcap_{(y,z)\in A_a} \bigcap_{(p,q)\in A_z} \{f_B(y) \cup (\mathcal{S}^c_{\mathcal{S}}(p) \cup f_B(q))\} \\ &= \bigcap_{(y,z)\in A_a} \bigcap_{(p,q)\in A_z} \{f_B(y) \cup (\emptyset \cup f_B(q))\} \\ &= \bigcap_{(y,z)\in A_a} \bigcap_{(p,q)\in A_z} \{f_B(y) \cup f_B(q)\}.\end{aligned}$$

Since $(y,z) \in A_a$ and $(p,q) \in A_z$ we have $a \leq y \circ z$ and $z \leq p \circ q$, respectively. Thus $a \leq y \circ z \leq y \circ (p \circ q)$ implies that $a \leq y \circ p \circ q$. So there exists $\beta \in y \circ p \circ q$ such that $a \leq \beta$. Since f_B is a uni-soft bi-hyperideal of S over U, we get

$$f_B(a) \subseteq f_B(\beta) \subseteq \bigcup_{\beta \in y\circ p\circ q} f_B(\beta) \subseteq f_B(y) \cup f_B(q) \text{ for all } y,p,q \in S.$$

Thus

$$\bigcap_{(y,z)\in A_a} \bigcap_{(p,q)\in A_z} \{f_B(y) \cup f_B(q)\} \supseteq \bigcap_{(y,z)\in A_a} \bigcap_{(p,q)\in A_z} f_B(a) = f_B(a).$$

Therefore, $f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B \widetilde{\supseteq} f_B$. □

Theorem 3.11. *An ordered semihypergroup S is regular if and only if $f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B = f_B$, for every uni-soft bi-hyperideal f_B of S over U.*

Proof. Let S be an ordered semihypergroup, f_B is a uni-soft bi-hyperideal of S over U and $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq a \circ x \circ a \leq a \circ x \circ (a \circ x \circ a) = a \circ (x \circ a \circ x \circ a)$ such that $a \leq a \circ z$ for some $z \in x \circ a \circ x \circ a$. Since $A_a \neq \emptyset$, we get

$$(f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(a) = \bigcap_{(y,z)\in A_a} \{f_B(y) \cup (\mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(z)\} \subseteq \{f_B(a) \cup (\mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(z)\}.$$

Now, since $z \in x \circ a \circ x \circ a$, there exists $\alpha \in x \circ a \circ x$ such that $z \leq \alpha \circ a$ and $A_z \neq \emptyset$. Then

$$(\mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B)(z) = \bigcap_{(p,q)\in A_z} \{\mathcal{S}^c_{\mathcal{S}}(p) \cup f_B(q)\} \subseteq \{\mathcal{S}^c_{\mathcal{S}}(\alpha) \cup f_B(a)\} = \emptyset \cup f_B(a) = f_B(a).$$

Thus,

$$f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B \widetilde{\subseteq} f_B(a) \cup f_B(a) = f_B(a).$$

On the other hand by Proposition 3.10, for every uni-soft bi-hyperideal f_B of S over U, we have $f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B \widetilde{\supseteq} f_B$. Hence, $f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B = f_B$.

Conversely, assume that $f_B \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} f_B = f_B$ for every uni-soft bi-hyperideal f_B of S over U. Then S is regular. In fact, by Lemma 2.6, it is enough to prove that

$$B(a) = (B(a) \circ S \circ B(a)], \text{ for all } a \in S.$$

Let $y \in B(a)$. Then $y \in (B(a) \circ S \circ B(a)]$. Indeed: Since $B(a)$ is the bi-hyperideal of S generated a, by Theorem 3.7, $\mathcal{S}^c_{B(a)}$ is a uni-soft bi-hyperideal of S over U. By hypothesis,

$$\left(\mathcal{S}^c_{B(a)} \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} \mathcal{S}^c_{B(a)}\right)(y) = \mathcal{S}^c_{B(a)}(y).$$

As $y \in B(a)$, we have $\mathcal{S}^c_{B(a)}(y) = \emptyset$. Hence, $\left(\mathcal{S}^c_{B(a)} \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} \mathcal{S}^c_{B(a)}\right)(y) = \emptyset$. By Proposition 3.3, $\mathcal{S}^c_{B(a)} \widetilde{*} \mathcal{S}^c_{\mathcal{S}} \widetilde{*} \mathcal{S}^c_{B(a)} = \mathcal{S}^c_{(B(a) \circ S \circ B(a)]}$. Thus $\mathcal{S}^c_{(B(a) \circ S \circ B(a)]}(y) = \emptyset$ implies $y \in (B(a) \circ S \circ B(a)]$. On the other hand $(B(a) \circ S \circ B(a)] \subseteq (B(a)] = B(a)$. Therefore, $(B(a) \circ S \circ B(a)] = B(a)$. □

Proposition 3.12. *A soft set f_A is a uni-soft semihypergroup of an ordered semihypergroup S over U if and only if $f_A \widetilde{*} f_A \widetilde{\supseteq} f_A$.*

Proof. Straightforward. □

Lemma 3.13. *Let S be an ordered semihypergroup, f_A, g_B be two uni-soft bi-hyperideals of S over U. Then $f_A \widetilde{*} g_B$ is a uni-soft bi-hyperideal of S over U.*

Proof. Straightforward. □

4 Prime and semiprime uni-soft bi-hyperideals of ordered semihypergroups.

In this section, we define prime, strongly prime, semiprime, irreducible and strongly irreducible uni-soft bi-hyperideals of ordered semihypergroups. We characterize ordered semihypergroups by the properties of these notions.

Definition 4.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and f_A be a uni-soft bi-hyperideal of S over U. f_A is called prime (resp., strongly prime, semiprime) if $h_D \widetilde{\supseteq} f_A$ $\left(\text{resp.}, g_C \widetilde{\supseteq} f_A \text{ or } h_D \widetilde{\supseteq} f_A, g_C \widetilde{\supseteq} f_A\right)$ or $g_C \widetilde{*} h_D \widetilde{\supseteq} f_A$ $\left(\text{resp.}, \ (g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) \widetilde{\supseteq} f_A, g_C \widetilde{*} g_C \widetilde{\supseteq} f_A\right)$, then $g_C \widetilde{\supseteq} f_A$ for all uni-soft bi-hyperideals g_C and h_D of S over U.*

Example 4.2. *Let $U = \{h_1, h_2, h_3\}$ be a set of houses under consideration where $E := \{e_1, e_2, e_3\}$ be the set of parameters for selection of the house. Let*
e_1 stands for "expensive houses",
e_2 stands for "wooden houses",
e_3 stands for "houses located in the urban area",
with the following binary operation $\circ$ given in the Cayley table:

$\circ$	e_1	e_2	e_3
e_1	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$
e_2	$\{e_1\}$	$\{e_1, e_2\}$	$\{e_1, e_2\}$
e_3	$\{e_1\}$	$\{e_1, e_3\}$	$\{e_1, e_3\}$

$$\leq := \{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_1, e_2), (e_1, e_3)\}.$$

The covering relation is given as follows, $\prec := \{(e_1, e_2), (e_1, e_3)\}$*. Then* $(E, \circ, \leq)$ *is an ordered semihypergroup.*

Suppose $A = \{e_2, e_3\}$*. Let us define* $f_A(e_1) = \emptyset$*,* $f_A(e_2) = \{h_1, h_2\}$ *and* $f_A(e_3) = \{h_1, h_2, h_3\}$*. Then* f_A *is a uni-soft bi-hyperideal of* S *over* U*. Now, take* $B = \{e_2\}$*. Let us define* $g_B(e_1) = \emptyset$*,* $g_B(e_2) = \{h_1, h_3\}$ *and* $g_B(e_3) = \emptyset$*. Then* g_B *is also a uni-soft bi-hyperideal of* S *over* U*.*

Also, $f_A \widetilde{*} g_B \widetilde{\supseteq} g_B$ *for every uni-soft bi-hyperideals* f_A *and* g_B *of* S *over* U*. Thus every uni-soft bi-hyperideal is prime. Hence, every uni-soft bi-hyperideal is semiprime. Also,* $(f_A \widetilde{*} g_B)(e_2) = \{h_1, h_2, h_3\}$

Proposition 4.3. *Let* $\{f_{B_i} : i \in I\}$ *be a family of prime uni-soft bi-hyperideals of an ordered semihypergroup* S *over* U*. Then* $\bigcup\limits_{i\in I} f_{B_i}$ *is a semiprime uni-soft bi-hyperideal of* S *over* U*.*

Proof. Since union of uni-soft bi-hyperideals is a uni-soft bi-hyperideal, we get $\bigcup\limits_{i\in I} f_{B_i}$ is a uni-soft bi-hyperideal of S over U. Let g_C be any uni-soft bi-hyperideal of S over U such that $g_C \widetilde{*} g_C \widetilde{\supseteq} \bigcup\limits_{i\in I} f_{B_i}$. Then $g_C \widetilde{*} g_C \widetilde{\supseteq} f_{B_i}$ for all $i \in I$. Since each f_{B_i} is prime uni-soft bi-hyperideal of S over U. So $g_C \widetilde{\supseteq} f_{B_i}$ for all $i \in I$. Hence, $g_C \widetilde{\supseteq} \bigcup\limits_{i\in I} f_{B_i}$. Thus $\bigcup\limits_{i\in I} f_{B_i}$ is a semiprime uni-soft bi-hyperideal of S over U. □

Definition 4.4. *Let* $(S, \circ, \leq)$ *be an ordered semihypergroup. A uni-soft bi-hyperideal* f_B *of* S *over* U *is called irreducible (resp., strongly irreducible) uni-soft bi-hyperideal of* S *over* U *if for any uni-soft bi-hyperideals* g_C *and* h_D *of* S *over* U*,* $g_C \widetilde{\cup} h_D = f_B$ $\left(\text{resp., } g_C \widetilde{\cup} h_D \widetilde{\supseteq} f_B\right)$ *implies either* $g_C = f_B$ *or* $h_D = f_B$ $\left(\text{resp., } g_C \widetilde{\supseteq} f_B \text{ or } h_D \widetilde{\supseteq} f_B\right)$*.*

Theorem 4.5. *Every strongly irreducible semiprime uni-soft bi-hyperideal of an ordered semihypergroup* S *over* U *is a strongly prime uni-soft bi-hyperideal of* S *over* U*.*

Proof. Let f_B be a strongly irreducible semiprime uni-soft bi-hyperideal of S over U. Let g_C and h_D be any two uni-soft bi-hyperideals of S over U such that

$$(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) \widetilde{\supseteq} f_B.$$

As $g_C \widetilde{\cup} h_D$ is a uni-soft bi-hyperideal of S over U, and $(g_C \widetilde{\cup} h_D) \widetilde{*} (h_D \widetilde{\cup} g_C) \widetilde{\supseteq} g_C \widetilde{*} h_D$ and

$$(g_C \widetilde{\cup} h_D) \widetilde{*} (h_D \widetilde{\cup} g_C) \widetilde{\supseteq} h_D \widetilde{*} g_C.$$

Thus

$$(g_C \widetilde{\cup} h_D) \widetilde{*} (h_D \widetilde{\cup} g_C) \widetilde{\supseteq} (g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) \widetilde{\supseteq} f_B.$$

Since f_B is a semiprime uni-soft bi-hyperideal of S over U, we obtain $g_C \widetilde{\cup} h_D \widetilde{\supseteq} f_B$. As f_B is strongly irreducible uni-soft bi-hyperideal of S over U, so either $g_C \widetilde{\supseteq} f_B$ or $h_D \widetilde{\supseteq} f_B$. Thus f_B is a strongly prime uni-soft bi-hyperideal of S over U. □

Proposition 4.6. *Let* f_B *be a uni-soft bi-hyperideal of an ordered semihypergroup* S *over* U *with* $f_B(a) = t$ *where* $a \in S$ *and* $t \in P(U)$*. Then there exists an irreducible uni-soft bi-hyperideal* g_C *of* S *over* U *such that* $f_B \widetilde{\supseteq} g_C$ *and* $g_C(a) = t$*.*

Proof. Let $X = \left\{h_D : h_D \text{ is a uni-soft bi-hyperideal of } S \text{ over } U \text{ with } h_D(a) = t \text{ and } f_B \widetilde{\supseteq} h_D\right\}$. Then $X \neq \emptyset$, since $f_B \in X$. The collection X is partially ordered set under inclusion. Let Y be any totally ordered subset of X, say $Y = \{h_{D_i} : i \in I\}$. Let $x, y, z \in S$ and for every $\alpha \in x \circ y$,

$$\bigcup_{\alpha \in x \circ y} \left(\bigcap_{i \in I} h_{D_i}\right)(\alpha) = \bigcap_{i \in I} \left(\bigcup_{\alpha \in x \circ y} h_{D_i}(\alpha)\right) \subseteq \bigcap_{i \in I} \left(h_{D_i}(x) \cup h_{D_i}(y)\right)$$

$$= \left\{\bigcap_{i \in I} \left(h_{D_i}(x)\right)\right\} \cup \left\{\bigcap_{i \in I} \left(h_{D_i}(y)\right)\right\} = \left\{\bigcap_{i \in I} h_{D_i}\right\}(x) \cup \left\{\bigcup_{i \in I} h_{D_i}\right\}(y).$$

Hence $\bigcap_{i \in I} h_{D_i}$ is a uni-soft semihypergroup of S over U.

Also, for any $\gamma \in x \circ y \circ z$,

$$\bigcup_{\gamma \in x \circ y \circ z} \left(\bigcap_{i \in I} h_{D_i}\right)(\gamma) = \bigcap_{i \in I} \left(\bigcup_{\gamma \in x \circ y \circ z} h_{D_i}(\gamma)\right) \subseteq \bigcap_{i \in I} \left(h_{D_i}(x) \cup h_{D_i}(z)\right)$$
$$= \left\{\bigcap_{i \in I} \left(h_{D_i}(x)\right)\right\} \cup \left\{\bigcap_{i \in I} \left(h_{D_i}(z)\right)\right\} = \left\{\bigcap_{i \in I} h_{D_i}\right\}(x) \cup \left\{\bigcap_{i \in I} h_{D_i}\right\}(z).$$

Let $x, y \in S$ such that $x \leq y$. Since h_{D_i} are uni-soft bi-hyperideal of S over U, we get

$$\left(\bigcap_{i \in I} h_{D_i}\right)(x) = \bigcap_{i \in I} \left(h_{D_i}(x)\right) \subseteq \bigcap_{i \in I} \left(h_{D_i}(y)\right) = \left(\bigcap_{i \in I} h_{D_i}\right)(y).$$

Hence, $\bigcap_{i \in I} h_{D_i}$ is a uni-soft bi-hyperideal of S over U. As $f_B \widetilde{\supseteq} h_{D_i}$ for each $i \in I$, so $f_B \widetilde{\supseteq} \bigcap_{i \in I} h_{D_i}$. Also, $\left(\bigcap_{i \in I} h_{D_i}\right)(a) = \bigcap_{i \in I} \left(h_{D_i}(a)\right) = t$. Thus $\bigcap_{i \in I} h_{D_i}$ is the greatest lower bound of Y. By Zorn's Lemma, there exists a uni-soft bi-hyperideal g_C of S over U, which is minimal with respect to the property that $f_B \widetilde{\supseteq} g_{C_i}$ and $g_C(a) = t$. Now, we show that g_C is an irreducible uni-soft bi-hyperideal of S over U. Suppose $g_C = g_{C_1} \widetilde{\cup} g_{C_2}$ where g_{C_1} and g_{C_2} are uni-soft bi-hyperideal of S over U. Then $g_C \widetilde{\supseteq} g_{C_1}$ and $g_C \widetilde{\supseteq} g_{C_2}$. We claim that $g_C = g_{C_1}$ or $g_C = g_{C_2}$. Suppose on contrary that $g_C \neq g_{C_1}$ and $g_C \neq g_{C_2}$. Since g_C is minimal with respect to the property that $g_C(a) = t$ and since $g_C \neq g_{C_1}$ and $g_C \neq g_{C_2}$, it follows that $g_{C_1}(a) \neq t$ and $g_{C_2}(a) \neq t$. Thus $t = g_C(a) = \left(g_{C_1} \widetilde{\cup} g_{C_2}\right)(a) \neq t$, which is a contradiction. Hence, either $g_C = g_{C_1}$ or $g_C = g_{C_2}$. Thus g_C is an irreducible uni-soft bi-hyperideal of S over U. □

Theorem 4.7. *An ordered semihypergroup S is both regular and intra-regular if and only if for every uni-soft right hyperideal f_A of S over U and every uni-soft left hyperideal g_B of S over U, we have*

$$f_A \widetilde{\cup} g_B \widetilde{\supseteq} \left(g_B \widetilde{*} f_A\right) \widetilde{\cup} \left(f_A \widetilde{*} g\right).$$

Proof. Follows from Propositions 3.4 and 3.5. □

Theorem 4.8. *For an ordered semihypergroup S the following statements are equivalent:*

(1) *S is both regular and intra-regular.*

(2) *$f_B = f_B \widetilde{*} f_B$ for every uni-soft bi-hyperideal f_B of S over U.*

(3) *$g_C \widetilde{\cup} h_D = (g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C)$ for every uni-soft bi-hyperideal g_C and h_D of S over U.*

(4) *Each uni-soft bi-hyperideal of S over U is semiprime.*

(5) *Each proper uni-soft bi-hyperideal of S over U is the union of all irreducible semiprime uni-soft bi-hyperideals of S over U which contained in.*

Proof. $(1) \Leftrightarrow (2)$. Suppose S is both regular and intra-regular ordered semihypergroup and f_B is a uni-soft bi-hyperideal of S over U. Then for each $a \in S$, we have $(f_B \widetilde{*} f_B)(a) \widetilde{\subseteq} f_B(a)$. Indeed: Since S is regular and intra-regular, we obtain there exist $x, y, z \in S$ such that $a \leq a \circ x \circ a$ and $a \leq y \circ a \circ a \circ z$. Thus

$$a \leq a \circ x \circ a \leq a \circ x \circ a \circ x \circ a = a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a = (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a).$$

Then for some $p \in a \circ x \circ y \circ a$ and $q \in a \circ z \circ x \circ a$ we have $a \leq p \circ q$ that is $(p, q) \in A_a$. Since $A_a \neq \emptyset$, we have

$$(f_B \widetilde{*} f_B)(a) = \bigcap_{(p_1, q_1) \in A_a} \{f_B(p_1) \cup f_B(q_1)\} \subseteq \{f_B(p) \cup f_B(q)\}.$$

As f_B is a uni-soft bi-hyperideal of S over U, we have

$$\bigcup_{\substack{p \in a \circ \delta \circ a \\ \delta \in x \circ y}} f_B(p) \subseteq \{f_B(a) \cup f_B(a)\} = f_B(a).$$

and

$$\bigcup_{\substack{q \in a \circ \gamma \circ a \\ \gamma \in z \circ x}} f_B(q) \subseteq \{f_B(a) \cup f_B(a)\} = f_B(a).$$

Thus

$$(f_B \widetilde{*} f_B)(a) \subseteq \{f_B(p) \cup f_B(q)\} \subseteq \{f_B(a) \cup f_B(a)\} = f_B(a).$$

Hence, $f_B \widetilde{\supseteq} f_B \widetilde{*} f_B$ by Proposition 3.10, we have $f_B \widetilde{*} f_B \widetilde{\supseteq} f_B$. Therefore, $f_B \widetilde{*} f_B = f_B$.

$(2) \Leftrightarrow (3)$. Let g_C and h_D be uni-soft bi-hyperideals of S over U. Then $g_C \widetilde{\cup} h_D$ is a uni-soft bi-hyperideal of S over U. By (2),

$$g_C \widetilde{\cup} h_D = \left(g_C \widetilde{\cup} h_D\right) \widetilde{*} \left(g_C \widetilde{\cup} h_D\right) \widetilde{\supseteq} g_C \widetilde{*} h_D.$$

Similarly, we can prove that $g_C \widetilde{\cup} h_D \widetilde{\supseteq} h_D \widetilde{*} g_C$. Thus $g_C \widetilde{\cup} h_D \widetilde{\supseteq} (g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C)$. For the reverse inclusion, by Lemma 3.13, $g_C \widetilde{*} h_D$ and $h_D \widetilde{*} g_C$ are uni-soft bi-hyperideals of S over U. So $(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C)$ is a uni-soft bi-hyperideal of S over U. By (2) we have

$$\begin{aligned}
(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) &= \left((g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C)\right) \widetilde{*} \left((g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C)\right) \\
&\widetilde{\supseteq} g_C \widetilde{*} h_D \widetilde{*} h_D \widetilde{*} g_C = g_C \widetilde{*} (h_D \widetilde{*} h_D) \widetilde{*} g_C \\
&= g_C \widetilde{*} h_D \widetilde{*} g_C \ \text{ (as } h_D \widetilde{*} h_D = h_D \text{ by } (2)) \\
&\widetilde{\supseteq} g_C \widetilde{*} \mathcal{S}_{\mathcal{S}}^{c} \widetilde{*} g_C \ \text{ (as } g_C \widetilde{*} h_D \widetilde{\supseteq} g_C \widetilde{*} \mathcal{S}_{\mathcal{S}}^{c} \text{ by Lemma 3.6)} \\
&\widetilde{\supseteq} g_C \ \text{ (as } g_C \widetilde{*} \mathcal{S}_{\mathcal{S}}^{c} \widetilde{*} g_C \widetilde{\supseteq} g_C \text{ by Proposition 3.10).}
\end{aligned}$$

Hence, $(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) \widetilde{\supseteq} g_C$. Similarly we can prove that $(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) \widetilde{\supseteq} h_D$. Thus $(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) \widetilde{\supseteq} g_C \widetilde{\cup} h_D$. Therefore, $(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) = g_C \widetilde{\cup} h_D$.

$(3) \Leftrightarrow (1)$. Let g_C be a uni-soft right hyperideal and h_D be a uni-soft left hyperideal of S over U, then g_C and h_D are uni-soft bi-hyperideals of S over U. Hence, by hypothesis, $(g_C \widetilde{*} h_D) \widetilde{\cup} (h_D \widetilde{*} g_C) = g_C \widetilde{\cup} h_D$. Thus by Theorem 4.7, S is both regular and intra-regular.

$(3) \Rightarrow (4)$. Let f_A and g_B be uni-soft bi-hyperideals of S over U such that $f_A \widetilde{*} f_A \widetilde{\supseteq} g_B$. By hypothesis

$$f_A = f_A \widetilde{\cup} f_A = (f_A \widetilde{*} f_A) \widetilde{\cup} (f_A \widetilde{*} f_A) = f_A \widetilde{*} f_A.$$

Thus $f_A \widetilde{\supseteq} g_B$. Hence, every uni-soft bi-hyperideal of S over U is semiprime.

$(4) \Rightarrow (5)$. Let f_A be a proper uni-soft bi-hyperideal of S over U and $\{f_{A_i} : i \in I\}$ be the collection of all irreducible uni-soft bi-hyperideals of S over U which contained in f_A. By Proposition 4.6, this collection is nonempty. Hence, $f_A \widetilde{\supseteq} \bigcup_{i \in I} f_{A_i}$. Let $a \in S$, then by Proposition 4.6, there exists an irreducible uni-soft bi-hyperideal f_{A_α} of S over U such that $f_A \widetilde{\supseteq} f_{A_\alpha}$ and $f_A(a) = f_{A_\alpha}(a)$. Thus $f_{A_\alpha} \in \{f_{A_i} : i \in I\}$. Hence, $\bigcup_{i \in I} f_{A_i} \widetilde{\supseteq} f_{A_\alpha}$. So $\bigcup_{i \in I} f_{A_i}(a) \widetilde{\supseteq} f_{A_\alpha}(a) = f_A(a)$. Thus $\bigcup_{i \in I} f_{A_i} \widetilde{\supseteq} f_A$. Consequently $\bigcup_{i \in I} f_{A_i} = f_A$. By hypothesis each uni-soft bi-hyperideal of S over U is semiprime. So each uni-soft bi-hyperideal of S over U is the union of all irreducible semiprime uni-soft bi-hyperideals of S over U which contained in.

$(5) \Rightarrow (2)$. Let f_A be a uni-soft bi-hyperideal of S over U. Then by Lemma 3.13, $f_A \widetilde{*} f_A$ is also uni-soft bi-hyperideal of S over U. Since f_A is a uni-soft semihypergroup of S over U so by Proposition 3.12, $f_A \widetilde{*} f_A \widetilde{\supseteq} f_A$. By hypothesis $f_A \widetilde{*} f_A = \bigcup_{i \in I} f_{A_i}$ where f_{A_i} are irreducible semiprime uni-soft bi-hyperideals of S over U. Thus $f_A \widetilde{*} f_A \widetilde{\supseteq} f_{A_i}$ for all $i \in I$. Hence, $f_A \widetilde{\supseteq} f_{A_i}$ for all $i \in I$, because each f_{A_i} is semiprime. Thus $f_A \widetilde{\supseteq} \bigcup_{i \in I} f_{A_i} = f_A \widetilde{*} f_A$. Hence, $f_A \widetilde{*} f_A = f_A$. □

Proposition 4.9. *Let S be both regular and intra-regular ordered semihypergroup. Then the following statements are equivalent:*

(1) *Every uni-soft bi-hyperideal of S over U is strongly irreducible.*

(2) *Every uni-soft bi-hyperideal of S over U is strongly prime.*

Proof. $(1) \Rightarrow (2)$. Let S be both regular and intra-regular ordered semihypergroup and f_A be a strongly irreducible uni-soft bi-hyperideal of S over U. Let g_B and h_C be uni-soft bi-hyperideals of S over U such that $(g_B \widetilde{*} h_C) \widetilde{\cup} (h_C \widetilde{*} g_B) \widetilde{\supseteq} f_A$. Since S is both regular and intra-regular, by Theorem 4.8, $(g_B \widetilde{*} h_C) \widetilde{\cup} (h_C \widetilde{*} g_B) = g_B \widetilde{\cup} h_C$. Thus $g_B \widetilde{\cup} h_C \widetilde{\supseteq} f_A$. Since f_A is a strongly irreducible, we have either $g_B \widetilde{\supseteq} f_A$ or $h_C \widetilde{\supseteq} f_A$. Thus f_A is a strongly prime uni-soft bi-hyperideal of S over U.

$(2) \Rightarrow (1)$. Suppose f_A is a strongly prime uni-soft bi-hyperideal of S over U and g_B and h_C be uni-soft bi-hyperideals of S over U such that $g_B \widetilde{\cup} h_C \widetilde{\supseteq} f_A$. As $(g_B \widetilde{*} h_C) \widetilde{\cup} (h_C \widetilde{*} g_B) \widetilde{\supseteq} g_B \widetilde{\cup} h_C \widetilde{\supseteq} f_A$. Since f_A is strongly prime, we get either $g_B \widetilde{\supseteq} f_A$ or $h_C \widetilde{\supseteq} f_A$. Thus f_A is a strongly irreducible uni-soft bi-hyperideal of S over U. □

Theorem 4.10. *Each uni-soft bi-hyperideal of an ordered semihypergroup S is strongly prime if and only if S is both regular and intra-regular and the set of uni-soft bi-hyperideals of S over U is totally ordered under inclusion.*

Proof. Suppose that each uni-soft bi-hyperideal of S over U is strongly prime. Then each uni-soft bi-hyperideal of S over U is semiprime. Thus by Theorem 4.8, S is both regular and intra-regular. To prove that the set of uni-soft bi-hyperideals of S over U is totally ordered under inclusion. Let g_B and h_C be any two uni-soft bi-hyperideals of S over U. Then by Theorem 4.8, $(g_B \widetilde{*} h_C) \widetilde{\cup} (h_C \widetilde{*} g_B) = g_B \widetilde{\cup} h_C$. As each uni-soft bi-hyperideal of S is strongly prime, we obtain $g_B \widetilde{\cup} h_C$ is strongly prime. Hence, either $g_B \widetilde{\supseteq} g_B \widetilde{\cup} h_C$ or $h_C \widetilde{\supseteq} g_B \widetilde{\cup} h_C$. If $g_B \widetilde{\supseteq} g_B \widetilde{\cup} h_C$, then $g_B \widetilde{\supseteq} h_C$ and if $h_C \widetilde{\supseteq} g_B \widetilde{\cup} h_C$, then $h_C \widetilde{\supseteq} g_B$.

Conversely, assume that S is both regular and intra-regular and the set of uni-soft bi-hyperideals of S over U is totally ordered under inclusion. Then each uni-soft bi-hyperideal of S over U is strongly prime. Indeed: Let f_A be an arbitrary uni-soft bi-hyperideal of S over U and g_B, h_C be any uni-soft bi-hyperideals of S over U such that $(g_B \widetilde{*} h_C) \widetilde{\cup} (h_C \widetilde{*} g_B) \widetilde{\supseteq} f_A$. Since S is both regular and intra-regular, by Theorem 4.8, $(g_B \widetilde{*} h_C) \widetilde{\cup} (h_C \widetilde{*} g_B) = g_B \widetilde{\cup} h_C$. Thus $g_B \widetilde{\cup} h_C \widetilde{\supseteq} f_A$. Since the set of uni-soft bi-hyperideals of S over U is totally ordered, we have either $g_B \widetilde{\supseteq} h_C$ or $h_C \widetilde{\supseteq} g_B$ that is either $g_B \widetilde{\cup} h_C = g_B$ or $g_B \widetilde{\cup} h_C = h_C$. Thus either $g_B \widetilde{\supseteq} f_A$ or $h_C \widetilde{\supseteq} f_A$. Hence, f_A is a strongly prime uni-soft bi-hyperideal of S over U. □

Theorem 4.11. *If the set of uni-soft bi-hyperideals of an ordered semihypergroup S over U is totally ordered under inclusion, then S is both regular and intra-regular if and only if each uni-soft bi-hyperideal of S over U is prime.*

Proof. Suppose S is both regular and intra-regular. Let f_B be any uni-soft bi-hyperideal of S over U, and g_C, h_D are uni-soft bi-hyperideals of S over U such that $g_C \widetilde{*} h_D \widetilde{\supseteq} f_B$. Since the set of uni-soft bi-hyperideals of S over U is totally ordered under inclusion, we get that either $g_C \widetilde{\supseteq} h_D$ or $h_D \widetilde{\supseteq} g_C$. Suppose $g_C \widetilde{\supseteq} h_D$, then $g_C \widetilde{*} g_C \widetilde{\supseteq} g_C \widetilde{*} h_D \widetilde{\supseteq} f_B$. By Theorem 4.8, f_B is semiprime, so $g_C \widetilde{\supseteq} f_B$. Hence, f_B is a prime uni-soft bi-hyperideals of S over U.

Conversely, assume that every uni-soft bi-hyperideal of S over U is prime. Since every prime uni-soft bi-hyperideal is semiprime, so by Theorem 4.8, S is both regular and intra-regular. □

Theorem 4.12. *Let S be an ordered semihypergroup. Then the following assertions are equivalents:*

(1) *The set of uni-soft bi-hyperideals of S over U is totally ordered under inclusion.*
(2) *Each uni-soft bi-hyperideal of S over U is strongly irreducible.*
(3) *Each uni-soft bi-hyperideal of S over U is irreducible.*

Proof. (1) $\Rightarrow$ (2). Let f_B, g_C and h_D be uni-soft bi-hyperideals of S over U such that $g_C \widetilde{\cup} h_D \widetilde{\supseteq} f_B$. Since the set of uni-soft bi-hyperideals of S over U is totally ordered, we get that either $g_C \widetilde{\supseteq} h_D$ or $h_D \widetilde{\supseteq} g_C$. Thus either $g_C \widetilde{\cup} h_D = g_C$ or $g_C \widetilde{\cup} h_D = h_D$. Hence, $g_C \widetilde{\cup} h_D \widetilde{\supseteq} f_B$ implies either $g_C \widetilde{\supseteq} f_B$ or $h_D \widetilde{\supseteq} f_B$. Thus f_B is strongly irreducible.

(2) $\Rightarrow$ (3). Let f_B be an arbitrary uni-soft bi-hyperideal of S over U and g_C, h_D be two uni-soft bi-hyperideals of S over U such that $g_C \widetilde{\cup} h_D = f_B$. Then $f_B \widetilde{\supseteq} g_C$ and $f_B \widetilde{\supseteq} h_D$. By hypothesis either $g_C \widetilde{\supseteq} f_B$ or $h_D \widetilde{\supseteq} f_B$. So either $g_C = f_B$ or $h_D = f_B$. That is f_B is an irreducible uni-soft bi-hyperideal of S over U.

(3) $\Rightarrow$ (1). Let g_C, h_D be any two uni-soft bi-hyperideals of S over U. Then $g_C \widetilde{\cup} h_D$ is a uni-soft bi-hyperideal of S over U. Also $g_C \widetilde{\cup} h_D = g_C \widetilde{\cup} h_D$. So by hypothesis either $g_C \widetilde{\cup} h_D = g_C$ or $g_C \widetilde{\cup} h_D = h_D$, that is either $g_C \widetilde{\supseteq} h_D$ or $h_D \widetilde{\supseteq} g_C$. Hence, the set of all uni-soft bi-hyperideals of S over U is totally ordered under inclusion. □

5 Conclusion

In this paper, we have applied uni-soft set theory in hyperstructure, in particular, in ordered semihypergroups. We introduced the notions of uni-soft bi-hyperideals in ordered semihypergroups. Moreover we introduced the notions of prime, (strongly prime, semiprime, irreducible, and strongly irreducible) uni-soft bi-hyperideals of ordered semihypergroups. We considered characterizations of different classes in terms of these newly defined uni-soft hyperideals. Seems that the obtained characterizations will be very useful for future study of ordered semihypergroups and their applications. In our future study of soft hyperstructures, we will apply the above new idea to other algebraic structures.

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