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Cyclicity in some classes of H_v-groups

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Abstract

The study of cyclicity in hyperstructures was started very early, almost from the beginning of the introduction of a hypergroup by F. Marty in 1934. New concepts appeared in hyperstructures the main of which are *the period of a generator* and *the single power cyclicity*. These terms have no meaning in the classical structures as groups. We study the cyclicity in special large classes of H_v-groups.

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1 Introduction

In the hyperstructure theory, especially in H_v-structures or in h/v-structures, a lot of new concepts appeared which are not possible to be seen in the classical theory. Defining the powers and cyclicity in H_v-structures, we discover the cyclic hyperoperation, the period of a generator, the single power cyclicity, and so on. The fundamental relations relate the hyperstructures with the corresponding classical structures. Therefore, the cyclicity is connected with the fundamental classes, as well. In this paper, we present some of the theory of cyclicity and we define the concept of cyclicity with respect to the fundamental classes. We give examples on the concept.

2 Preliminaries

The class of hyperstructures called H_v-structures, introduced in 1990 [7], [8] by Vougiouklis, satisfy the *weak axioms*, where the non-empty intersection replaces equality.

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Definition 2.1. *An algebraic hyperstructure* $(H,\cdot)$ *is called a set* H *equipped with a hyperoperation (abbreviated hope)*

$$\cdot : H \times H \rightarrow \mathcal{P}(H) - \{\emptyset\}.$$

We abbreviate by *WASS* the *weak associativity*:

$$(xy)z \cap x(yz) \neq \emptyset,\ \forall x,y,z \in H$$

and by *COW* the *weak commutativity*:

$$xy \cap yx \neq \emptyset,\ \forall x,y \in H.$$

The algebraic structure $(H,\cdot)$ is an H_v-*semigroup* if it is *WASS*, and it is called an H_v-group if it is a reproductive H_v-semigroup, i.e.

$$xH = Hx = H,\ \forall x \in H.$$

The algebraic structure $(R,+,\cdot)$ is called an H_v-*ring* if $(+)$ and $(\cdot)$ are *WASS*; $(+)$ is reproductive and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset \text{ and } (x+y)z \cap (xz+yz) \neq \emptyset, \forall x,y,z \in R.$$

Let $(R,+,\cdot)$ be an H_v-ring, a COW H_v-group $(M,+)$ is called an H_v-*module* over R, if there is an external hope

$$\cdot : R \times M \ \rightarrow P(M) : (a,x) \rightarrow ax$$

such that for any $a,b \in R$ and for any $x,y \in M$, we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset \text{ and } (ab)x \cap a(bx) \neq \emptyset.$$

For more definitions and applications on H_v-structures one can see in books and papers as [1], [3], [4], [8], [12], [13], [14].

Definition 2.2. *Let* $(H,\cdot)$ *and* $(H,\star)$ *be two* H_v*-semigroups. Then the hope* $(\cdot)$ *is smaller than* $(\star)$*, and* $(\star)$ *greater than* $(\cdot)$*, iff there exists an automorphism*

$$f \in Aut(H,\star) \text{ such that } xy \subset f(x \star y), \forall x,y \in H.$$

We write $\cdot \leq \star$ and say $(H,\star)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure, then it is *basic structure* and $(H,\star)$ is an H_b-*structure.*

The following little theorem leads to a partial order on H_v-structures and to posets.

The Little Theorem. *Greater hopes than the ones which are WASS or COW, are WASS or COW, respectively.*

Example 2.3. *In quaternions* $Q=\{1,-1,\ i,-i,\ j,-j,\ k,-k\}$*, with* $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$*, denote* $\underline{i} = \{i,-i\}, \underline{j} = \{j,-j\}, \underline{k} = \{k,-k\}$*, define hopes* $(\star)$ *by enlarging only few products. For example,* $(-1) \star k = \underline{k}$*,* $k \star i = \underline{j}$ *and* $i \star j = \underline{k}$*. Then* $(Q,\star)$ *is an* H_v*-group.*

Let $(G,\cdot)$ be a hypergroupoid [8]. An n-ary hyperproduct can be defined, induced by $(\cdot)$, by inserting $n-2$ parentheses in the sequence of elements $a_1, a_2, \ldots, a_n$ in a standard position. Let us denote by $p(a_1,\ldots,a_n)$ such a pattern of $n-2$ parentheses and by P_n the set of all such patterns. For example, for $n=4$, we have,

$$P_4(a_1,a_2,a_3,a_4)=\{((a_1a_2)a_3)a_4,(a_1(a_2a_3))a_4,(a_1a_2)(a_3a_4),a_1((a_2a_3)a_4),a_1(a_2(a_3a_4))\}.$$

If it is commutative, the cases are two:

$$P_4(a_1,a_2,a_3,a_4)=\{((a_1a_2)a_3)a_4,(a_1a_2)(a_3a_4)\}$$

In an H_v-semigroup $(H,\cdot)$, we define $(\circ)$ the *n-ary circle hope,* by the union of hyperproducts n times, with all possible patterns of parentheses on them [4], [5], [8]. Thus, $\forall h\in H$, we have

$$h^1=\{h\}$$
$$h^2=h\circ h=h\cdot h$$
$$h^3=h\circ h\circ h=[(h\cdot h)\cdot h]\cup[h\cdot(h\cdot h)]$$
$$h^4=h\circ h\circ h\circ h=[((h\cdot h)\cdot h)\cdot h]\cup[(h\cdot h)\cdot(h\cdot h)]\cup[h\cdot(h\cdot(h\cdot h))]\cup[h\cdot((h\cdot h)\cdot h))]\cup[(h\cdot(h\cdot h))\cdot h]$$
$$\cdots\cdots\cdots\cdots\cdots$$

Definition 2.4. *In an H_v-semigroup $(H,\cdot)$, the powers are defined as follows:*

$$h^1=\{h\},h^2=h\cdot h \text{ and } h^n=h\circ h\circ\cdots\circ h,$$

where $(\circ)$ is the n-ary circle hope. An $(H,\cdot)$ is cyclic of period s, if there is a generator $h\in H$ and the minimum integer s, such that

$$H=h^1\cup h^2\cup...\cup h^s.$$

If there are $h\in H$ and s, the minimum one, $H=h^s$, then $(H,\cdot)$ is a *single-power cyclic of period s.*

Analogously, the cyclicity for infinite period, is defined [5], [8]. Thus, for example, an H_v-semigroup $(H,\cdot)$ is called *single-power cyclic of infinite period* with *generator* $h\in H$, if every element of H belongs to a power of h and there exists $s_o\geq 1$, such that for every $s\geq s_o$ we have

$$h^1\cup h^2\cup...\cup h^{s-1}\subset h^s.$$

Denote $[h]=h^1\cup h^2\cup...\cup h^s\cup...$, which is the set generating by the element $h\in H$.

The main tool in hyperstructures is the *fundamental relation* [7], [8], [11], [12], [4], [14].

Definition 2.5. *The fundamental relations β^*,γ^* and ϵ^* are defined in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.*

Let $(G,\cdot)$ be a group and R be a partition in G, then $(G/R,\cdot)$ is an H_v-group. Therefore, the quotient $(G/R,cdot)/\beta^*$ is a group, the *fundamental* one.

Theorem 2.6. *Let $(H,\cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. Define the relation β in H by: $x\beta y$ iff $\{x,y\}\subset u$, where $u\in U$. Then β^* is the transitive closure of β.*

Analogous theorems are for H_v-rings, H_v-vector spaces and so on.

An element is called *single* if its fundamental class is singleton.

General structures can be defined using fundamental structures.

Definition 2.7. *An H_v-ring $(R,+,\cdot)$ is an H_v-field if R/γ^* is a field. An H_v-module over an H_v-field F, it is called an H_v-vector space.*

3 Large classes of hopes

A class of H_v-structures, introduced in [8], is the following:

Definition 3.1. *An H_v-structure is called very thin if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one. Thus, in a very thin H_v-structure in H there exists a hope $(\cdot)$ and a pair $(a,b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, are singletons.*

From the very thin hopes the *Attach Construction* is obtained [11]:
Let $(H,\cdot)$ be an H_v-semigroup, $v \notin H$. We extend the hope $(\cdot)$ into $\underline{\mathrm{H}} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{\mathrm{H}},\cdot)$ is an H_v-group, where $(\underline{\mathrm{H}},\cdot)/\beta^* \cong Z_2$ and v is a single.
The *enlarged* hopes are the ones where a new element appears in one result. The useful enlargements or reductions are those h/v-structures with the same fundamental structure.

Definition 3.2. [10] *Let $(G,\cdot)$ be a groupoid and $f : G \to G$ be a map. We define a hope (∂), called theta-hope, we write ∂-hope, on G as follows:*

$$x\partial y = \{f(x)\cdot y, x \cdot f(y)\}, \forall x, y \in G.$$

If $(\cdot)$ is commutative, then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.
If $(G,\cdot)$ is a groupoid and $f : G \to P(G)-\{\emptyset\}$ be a multivalued map, then we define the ∂-hope on G as follows:

$$x\partial y = (f(x)\cdot y) \cup (x \cdot f(y)), \forall x, y \in G.$$

Motivation for ∂-hope is the *derivative* where only the product of functions, is used.
Basic property: if $(G,\cdot)$ is a semigroup, then for all f, the ∂-hope is *WASS*.

Example 3.3. *(a) In integers $(Z,+,\cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in Z-\{0\}$. Then $(Z,\partial_+,\partial\cdot)$, where ∂_+ and $\partial\cdot$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with*

$$(Z,\partial_+,\partial\cdot)/\gamma^* \cong\ Z_n.$$

(b) *In $(Z,+,\cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x$, $\forall x \in Z-\{n\}$. Then $(Z,\partial_+,\partial\cdot)$ is an H_v-ring, moreover, $(Z,\partial_+,\partial\cdot)/\gamma^* \cong\ Z_n$.*

Special case of the above is for $n = p$, prime, then $(Z,\partial_+,\partial\cdot)$ is an H_v-field.
Combining the *uniting elements* procedure [2], with the enlarging theory or the ∂-theory, we can obtain analogous results.

Theorem 3.4. *In the ring $(Z_n,+,\cdot)$, with $n = ms$ enlarge the multiplication only in the product of the elements $0\cdot m$ by setting $0 \otimes m = \{0,m\}$ and the rest results remain the same. Then*

$$(Z_n,+,\otimes)/\gamma^* \cong (Z_m,+,\cdot).$$

Remark that we can enlarge other products as well, for example $2\cdot m$ by setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case 0 and 1 are scalars.
In the ring $(Z_n,+,\cdot)$, with $n = ps$ where p is prime, we enlarge only the product $0\cdot p$ by $0 \otimes p = \{0,p\}$ and the rest remain the same. Then $(Z_n,+,\otimes)$ is a very thin H_v-field.
Well known hopes defined on classical structures is the following [6], [8]:

Definition 3.5. *Let $(G, \cdot)$ be a groupoid, then for every $P \subset G, P \neq \emptyset$, we define the following hopes called P-hopes: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),\ \ \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),\ \ \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P}), (G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-hyperstructures.

If $(G, \cdot)$ is a semigroup, then

$$x\underline{\mathrm{P}}y = (xP)y \cup x(Py) = xPy$$

and the $(G, \underline{\mathrm{P}})$ is a semihypergroup.

4 Cyclicity with respect to fundamental classes

A large class of hyperstructures is the following [9], [13]:

Definition 4.1. *The H_v-semigroup $(H, \cdot)$ is called a h/v-group if H/β^* is a group.*

The h/v-group is a generalization of the H_v-group, where the following property is valid: Let $(H, \cdot)$ be an h/v-group and devote $[x]$ the fundamental class of the element $x \in H$, then we have

$$x[y] = [xy] = [x]y, \forall x, y \in H.$$

Definition 4.2. *Let $(H, \cdot)$ be an h/v-group and devote $[x]$ the fundamental class of the element $x \in H$. We define the property of the reproductivity of classes, if the product of any element x, on the left, with all elements y of H gives representatives of all fundamental classes $[y], \forall y \in H$.*

Similarly, *h/v-rings, h/v-fields, h/v-vector spaces* etc. are defined.

Combining the definitions of cyclicity and reproductivity of classes, we give a new definition: *cyclicity with respect to fundamental classes.*

Definition 4.3. *An h/v-group is called cyclic with respect to fundamental classes if there exist an element $h \in H$ and a number s, the minimum one, which is called generator, such that the union of the powers h^s contain elements of all fundamental classes.*

Definition 4.4. *An h/v-group is called single-power cyclic with respect to fundamental classes if there exists an element $h \in H$, which is called generator, and a number s, the minimum one, such that the power h^s contains elements of all fundamental classes.*

According to this definition we remark the cyclicity with respect to fundamental classes is a generalization of the cyclicity. Therefore, an immediately corollary is that all *cyclic h/v*-groups are *cyclic with respect to fundamental classes*, as well. However, the period normally is smaller.

Example 4.5. *Consider the h/v-group, based on h/v-matrices based $(Z_6, \otimes)$, which is given by the following table* [13]*:*

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2},\underline{5}$	$\underline{0}, \underline{3}$	$\underline{1},\underline{4}$	$\underline{2}, \underline{5}$	$\underline{0},\underline{3}$	$\underline{1}, \underline{4}$

This h/v-group is a cyclic where the element $\underline{1}$ *is a generator of period* 5 *and the element* $\underline{5}$ *is a generator of period* 4. *The elements* $\underline{2}, \underline{3}$ *and* $\underline{4}$ *are not generators.*
We denote $[0], [1]$ *and* $[2]$ *the classes of the elements* $0, 1$ *and* 2, *which are the sets* $\{0, 3\}, \{1, 4\}$ *and* $\{2, 5\}$ *respectively. Thus the fundamental classes are*

$$[\underline{0}] = \{\underline{0}, \underline{3}\}, [\underline{1}] = \{\underline{1}, \underline{4}\}, [\underline{2}] = \{\underline{2}, \underline{5}\}.$$

We remark that the re-productivity from the left is satisfied, for example:

$$1 \otimes 0 = 4, 1 \otimes 1 = 5, 1 \otimes 2 = 0, 1 \otimes 3 = 1, 1 \otimes 4 = 2, 1 \otimes 5 = 3.$$

The re-productivity from the right is not satisfied, however representatives of all fundamental classes are given, for example:

$$0 \otimes 1 = 1, 1 \otimes 1 = 5, 2 \otimes 1 = 3, 3 \otimes 1 = 1, 4 \otimes 1 = 5, 5 \otimes 1 = \{0, 3\}.$$

The $(Z_6, \otimes)$ *is an h/v-group which is cyclic with respect to fundamental classes, where the elements* $\underline{1}, \underline{2}, \underline{4}$ *and* $\underline{5}$ *are generators of period* 3 *with respect to fundamental classes:*

$$\underline{2}^1 = \underline{2}, \underline{2}^2 = \underline{4}, \underline{2}^3 = \underline{0}.$$

The element $\underline{3}$ *is not a generator.*

5 Conclusion

We introduced the concept of cyclicity with respect to the fundamental classes. The cyclicity, generally in hyperstructures, is related to the fundamental classes, but the above one is based on them. Some examples are presented as well, in order to see the new field which is opened.

References

[1] P. Corsini, V. Leoreanu, *Application of hyperstructure theory*, Kluwer Academic Publishers, 2003.

[2] P. Corsini, T. Vougiouklis, *From groupoids to groups through hypergroups*, Rendiconti Mathematics, S. VII, 9 (1989), 173–181.

[3] B. Davvaz, V. Leoreanu, *Hyperring theory and applications*, International Academic Press, 2007.

[4] B. Davvaz, T. Vougiouklis, *A walk through weak hyperstructures,* H_v-Structures, World Scientific, 2018.

[5] T. Vougiouklis, *Cyclicity in a special class of hypergroups*, Acta Universitatis Carolinae. Mathematica et Physica, 22(1) (1981), 3–6.

[6] T. Vougiouklis, *Generalization of P-hypergroups*, Rendiconti del Circolo Matematico di Palermo Series 2, 36 (1987), 114–121.

[7] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, 4^{th}AHA, Xanthi 1990, World Scientific, (1991), 203–211.

[8] T. Vougiouklis, *Hyperstructures and their representations*, Monographs in Mathematics, Hadronic, 1994.

[9] T. Vougiouklis, *The h/v-structures*, Journal of Discrete Mathematics Sciences and Cryptography, 6(2-3) (2003), 235–243.

[10] T. Vougiouklis, *∂-operations and H_v-fields*, Acta Mathematics Sinica, (Engl. Ser.), 24(7) (2008), 1067–1078.

[11] T. Vougiouklis, *Enlarged fundamentally very thin H_v-structures*, Journal Algebraic structures and their applications (ASTA), 1(1) (2014), 11–20.

[12] T. Vougiouklis, *On the hyperstructure theory*, Southeast Asian Bulletin of Mathematics, 40 (2016), 603–620.

[13] T. Vougiouklis, *H_v-fields, h/v-fields*, Ratio Mathematica, 33 (2017), 181–201.

[14] T. Vougiouklis, *Fundamental relations in H_v-structures. The 'Judging from the Results' proof*, Journal of Algebraic Hyperstrucures and Logical Algebras (JAHLA), 1(1) (2020), 21–36.