Journal of Algebraic Hyperstructures and Logical Algebras
Special Issue (JAHA2020) Dedicated to Professor Piergiulio Corsini
Volume 1, Number 3, (2020), pp. 81-90

Hyperoperations defined on sets of S-helix matrices

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Abstract

A hyperproduct on non-square ordinary matrices can be defined by using *the helix-hyperoperation*. Therefore, the helix-hyperoperation (abbreviated *hope*) is based on a classical operation and was introduced in order to overcome the non-existing cases. We study the helix-hyperstructures on the special type of matrices, the *S-helix matrices*, used on the small dimension representations. In this paper, we introduce and focus our study on the class of S-helix matrices called *k-overlap helix matrices*. The reason is that their hyper-vector spaces can represent n-dimensional spaces which have independent both, single valued dimensions and multivalued dimensions.

Article Information

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S. Vougioukli;
Received: July 2020;
Accepted: August 2020.
Paper type: Original.

Keywords:

Hyperstructures, H_v-structures, hope, helix-hopes.

1 Introduction

The class of the H_v-structures is the largest class of hyperstructures, where in the axioms, the non-empty intersection replaces the equality. The new axioms are called weak. This replacement leads to a partial order on the H_v-structures defined on the same set. Moreover, new hyperoperations can be defined which satisfy the weak axioms. In 2005 T. Vougiouklis and S. Vougiouklis [14], introduced the helix-hyperoperation which is defined on any type of ordinary matrices. The helix-hyperoperations are defined on the sum and on the product of matrices, as well, and all entries of the participating matrices are used and appeared in the results.

We continue our study on helix-hyperoperations focusing on matrices with entries elements from small finite fields. We introduce and study the k-overlap helix matrices which are S_0-helix matrices, and this type of matrices can be used in the representation theory of hyperstructures.

https://doi.org/10.29252/HATEF.JAHLA.1.3.7

2 Preliminaries

We deal with the largest class of hyperstructures called H_v*-structures*, introduced in 1990 [10], satisfying the *weak axioms* where the non-empty intersection replaces the equality.

Definition 2.1. *In a set* H *equipped with a hyperoperation (abbreviate by hope)*

$$\cdot : H \times H \to P(H) - \{\emptyset\} : (x, y) \to x \cdot y \subset H$$

we abbreviate by
WASS the weak associativity: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ *and by*
COW the weak commutativity: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called H_v*-semigroup* if it is WASS and is called H_v*-group* if it is reproductive H_v-semigroup: $xH = Hx = H, \forall x \in H$.

The $(R, +, \cdot)$ is called H_v*-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y + z) \cap (xy + xz) \neq \emptyset \text{ and } (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

For more definitions and applications on H_v-structures, one can see [1], [2], [3], [11]. An H_v-structure is *very thin,* if and only if, all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one.

The fundamental relations β^* and γ^* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [3], [9], [10], [11]. The main theorem is the following:

Theorem 2.2. *Let* $(H, \cdot)$ *be an* H_v*-group and let us denote by* U *the set of all finite products of elements of* H*. We define the relation* β *in* H *as follows:* $x\beta y$ *iff* $\{x, y\} \subset u$ *where* $u \in U$*. Then the fundamental relation* β^* *is the transitive closure of the relation* β*.*

An element is called *single* if its fundamental class is a singleton.

Definition 2.3. *Let* $(H, \cdot)$ *and* $(H, \otimes)$ *be* H_v*-semigroups defined on the same* H*. The* $(\cdot)$ *is smaller than* $(\otimes)$*, and* $(\otimes)$ *greater than* $(\cdot)$*, iff there exists an automorphism*

$$f \in Aut(H, \otimes) \text{ such that } xy \subset f(x \otimes y), \forall x \in H.$$

Then $(H, \otimes)$ *contains* $(H, \cdot)$ *and write* $\cdot \leq \otimes$*. If* $(H, \cdot)$ *is a classical structure, then it is basic and* $(H, \otimes)$ *is an* H_v*-structure.*

The Little Theorem [11]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

Fundamental relations are used for general definitions of hyperstructures [10], [3].

Definition 2.4. *(a) The* H_v*-ring* $(R, +, \cdot)$ *is called* H_v*-field if the quotient* R/γ^* *is a field.*
(b) The H_v*-semigroup* $(H, \cdot)$ *is called h/v-group if* H/β^* *is a group.*

The *h/v-fields* and the other related hyperstructures are defined in a similar way.

An H_v-group is called *cyclic*, if there is an element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator.

Definition 2.5. [11, 3] *Let* $(R, +, \cdot)$ *be an* H_v*-ring,* $(M, +)$ *be COW* H_v*-group and there exists an external hope* $\cdot : R \times M \to P(M) : (a, x) \to ax$*, such that,* $\forall a, b \in R$ *and* $\forall x, y \in M$*, we have*

$$a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset \text{ and } (ab)x \cap a(bx) \neq \emptyset,$$

then M is called an H_v-module over R. In the case of an H_v-field F instead of H_v-ring R, then the H_v-vector space is defined.

Definition 2.6. [3, 5, 17] *Let $(L,+)$ be an H_v-vector space on $(F,+,\cdot)$, $\Phi : F \to F/\gamma^*$, be the canonical map and $\omega_F = \{x \in F : \Phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\Phi\prime : L \to L/\epsilon^*$ and denote again 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x,y) \to [x,y],$$

then L is an H_v-Lie algebra over F if the following axioms are satisfied:
(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \emptyset,$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \emptyset, \forall x, x_1, x_2, y, y_1, y_2 \in L \text{ and } \lambda_1, \lambda_2 \in F,$$

(L2) $[x,x] \cap \omega_L \neq \emptyset, \forall x \in L$
(L3) $([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \emptyset, \forall x, y \in L.$

A large class of hopes is given as follows [9], [11]:

Definition 2.7. *Let $(G,\cdot)$ be a groupoid, then for every subset $P \subset G, P \neq \emptyset$, we define the following hopes, called P-hopes: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G,\underline{P}), (G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called P-hyperstructures.

If $(G,\cdot)$ is a semigroup, then $x\underline{\mathrm{P}}y = (xP)y \cup x(Py) = xPy$ and $(G,\underline{\mathrm{P}})$ is a semihypergroup.

An application of H_v-structures in Nuclear Physics, in the Santilli's isotheory, a generalization of P-hopes is used [3]: Let $(G,\cdot)$ be an abelian group and P a subset of G with more than one element. We define the hope $\times_P$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \cdot | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e, \end{cases}$$

we call this hope, P_e-*hope*. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

The H_v-matrix representations are defined as follows: [9], [11], [12], [13]:

Definition 2.8. *H_v-matrix is a matrix with entries of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r H_v$-matrices. The sum of products of elements of the H_v-ring is the n-ary circle hope on the hypersum.*

Let $(H,\cdot)$ be an H_v-group (or h/v-group). Find an H_v-ring $(R,+,\cdot)$, a set $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$ and a map $T : H \to M_R : h \mapsto T(h)$ such that

$$T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

T is *H_v-matrix (or h/v-matrix) representation.* If $T(h_1h_2) \subset T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is called *inclusion*. If $T(h_1h_2) = T(h_1)T(h_2) = \{T(h) \mid h \in h_1h_2\}, \forall h_1, h_2 \in H$, then T is *good* and then an induced representation T^* for the hypergroup algebra is obtained. If T is one to one and good, then it is *faithful*.

Several H_v-fields are used in representation theory and applications in applied sciences. We present some of them in the finite small case [3].

Construction 2.9. (i) On the ring $(Z_4, +, \cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are, (a) very thin minimal, (b) COW, (c) they have the elements 0 and 1, scalars.
Then, we have only the following isomorphic cases $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$.

Fundamental classes: $[0] = \{0, 2\}, [1] = \{1, 3\}$ and we have $(Z_4, +, \otimes)/\gamma^* \cong (Z_2, +, \cdot)$.

Thus it is isomorphic to $(Z_2 \times Z_2, +)$. In this H_v-group, there is only one unit and the elements have a unique double inverse.
(ii) On the ring $(Z_6, +, \cdot)$ we define, up to isomorphism, all multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are: (a) very thin minimal, (b) COW, (c) they have the elements 0 and 1, scalars
Then we have the following cases, by giving the only one hyperproduct,

$$\begin{array}{ll} (I) & 2 \otimes 3 = \{0,3\} \ or \ 2 \otimes 4 = \{2,5\} \ or \ 2 \otimes 5 = \{1,4\} \\ & 3 \otimes 4 = \{0,3\} \ or \ 3 \otimes 5 = \{0,3\} \ or \ 4 \otimes 5 = \{2,5\}, \end{array}$$

In all 6 cases, the fundamental classes are $[0] = \{0, 3\}, [1] = \{1, 4\}, [2] = \{2, 5\}$ and we have $(Z_6, +, \otimes)/\gamma^* \cong (Z_3, +, \cdot)$.

$$\begin{array}{ll} (II) & 2 \otimes 3 = \{0,2\} \ or \ 2 \otimes 3 = \{0,4\} \ or \ 2 \otimes 4 = \{0,2\} \ or \ 2 \otimes 4 = \{2,4\} \ or \\ & 2 \otimes 5 = \{0,4\} \ or \ 2 \otimes 5 = \{2,4\} \ or \ 3 \otimes 4 = \{0,2\} \ or \ 3 \otimes 4 = \{0,4\} \ or \\ & 3 \otimes 5 = \{1,3\} \ or \ 3 \otimes 5 = \{3,5\} \ or \ 4 \otimes 5 = \{0,2\} \ or \ 4 \otimes 5 = \{2,4\}. \end{array}$$

In all 12 cases, the fundamental classes are [0]={0,2,4}, [1]={1,3,5} and we have $(Z_6, +, \otimes)/\gamma^* \cong (Z_2, +, \cdot)$.

In representations, several new classes are used:

Definition 2.10. *Let $M = M_{m\times n}$ be the module of $m \times n$ matrices over R and $P = \{P_i : i \in I\} \subseteq M$. We define a P-hope $\underline{P}$ on M as follows,*

$$\underline{P} : M \times M \to P(M) : (A, B) A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq M,$$

where P^t denotes the transpose of P.

The hope $\underline{\mathrm{P}}$ is bilinear map, is strong associative and inclusion distributive:

$$A\underline{\mathrm{P}}(B + C) \subseteq A\underline{\mathrm{P}}B + A\underline{\mathrm{P}}C, \ \forall A, B, C \in M.$$

Let $M = M_{m\times n}$ the $m \times n$ matrices over R and let us take sets

$$S = \{s_k : k \in K\} \subseteq R, \ Q = \{Q_j : j \in J\} \subseteq M, \ P = \{P_i : i \in I\} \subseteq M.$$

Define three hopes as follows,

$$\begin{gathered} \underline{\mathrm{S}} : R \times M \to P(M) : (r, A) \to r\underline{\mathrm{S}}A = \{(rs_k)A : k \in K\} \subseteq M, \\ \underline{\mathrm{Q}}_+ : M \times M \to P(M) : (A, B) \to A\underline{\mathrm{Q}}_+B = \{A + Q_j + B : j \in J\} \subseteq M, \\ \underline{\mathrm{P}} : M \times M \to P(M) : (A, B) \to A\underline{\mathrm{P}}B = \{AP_i^tB : i \in I\} \subseteq M. \end{gathered}$$

Then $(M, \underline{\mathrm{S}}, \underline{\mathrm{Q}}_+, \underline{\mathrm{P}})$ is hyperalgebra on R called *general matrix P-hyperalgebra*.

3 The helix-hopes

Basic definitions on helix-hopes [4], [6], [7], [8], [14], [15], [16]:

Definition 3.1. *Let $A = (a_{ij}) \in M_{m\times n}$ be $m \times n$ matrix and $s, t \in N$ be naturals such that $1 \le s \le m$ and $1 \le t \le n$. We define the map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to the matrix A, the matrix $A\underline{cst} = (a_{ij})$, where $1 \le i \le s$ and $1 \le j \le t$. We call this map cut-projection of type $\underline{st}$. Thus, $A\underline{cst}$ is matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.*

We use cut-projections on all types of matrices to define sums and products.

Definition 3.2. *Let $A = (a_{ij}) \in M_{m\times n}$ be an $m \times n$ matrix and $s, t \in N$ be naturals such that $1 \le s \le m$ and $1 \le t \le n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (\underline{a}_{ij})$ which has as entries the sets*

$$\underline{a}_{ij} = \{a_{i+ks,j+\lambda t} \mid 1 \le i \le s, 1 \le j \le t \text{ and } k, \lambda \in N, i + ks \le m, j + \lambda t \le n\}.$$

Thus, we have the map

$$\underline{st} : M_{m\times n} \to M_{s\times t} : A \to A\underline{st} = (\underline{a}_{ij}).$$

We call this multivalued map helix-projection of type $\underline{st}$. $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in \underline{a}_{ij}, \forall i, j$. Obviously $A\underline{mn} = A$.

Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in N$ such that $1 \le s \le m$ and $1 \le t \le n$. Then, we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-projection on both, rows and columns. Thus, we have

$$(A\underline{\mathrm{sn}})\underline{\mathrm{st}} = (A\underline{\mathrm{mt}})\underline{\mathrm{st}} = A\underline{\mathrm{st}}.$$

Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$ such that $1 \le s \le m$ and $1 \le t \le n$. If $A\underline{\mathrm{st}}$ is not a set but one single matrix, then we call A *cut-helix matrix* of type $s \times t$. In other words, the matrix A is a helix matrix of type $s \times t$, if $A\underline{\mathrm{cst}} = A\underline{\mathrm{st}}$.

Definition 3.3. ***a.*** *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be two matrices and $s = min(m, u), t = min(n, u)$. We define a hope, called helix-addition or helix-sum, as follows:*

$$\oplus : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A, B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where

$$(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}.$$

b. *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be two matrices and $s = min(n, u)$. We define a hope, called helix-multiplication or helix-product, as follows:*

$$\otimes : M_{m\times n} \times M_{u\times v} \to P(M_{m\times v}) : (A, B) \to A \otimes B = A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m\times v},$$

where

$$(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (a_{it}b_{tj}) a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}.$$

The helix-sum is an external hope and the commutativity is valid. For the helix-product we remark that we have $A \otimes B = A\underline{ms} \cdot B\underline{sv}$ so we have either $A\underline{ms} = A$ or $B\underline{sv} = B$, that means the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark 3.4. *In $M_{m\times n}$ the addition is ordinary operation, thus we are interested only in the "product". From the fact that the helix-product on non-square matrices is defined, the definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined [17], [3], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ϵ^* gives, by a quotient, a Lie algebra, from which a classification is obtained.*

In the following, we restrict ourselves on the matrices $M_{m\times n}$ where $m < n$. We have analogous results if $m > n$ and for $m = n$ we have the classical theory.

Notation. For given $k \in N - \{0\}$, we denote by $\underline{k}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{k} = m$ if the remainder is zero. Thus a matrix $A = (a_{k\lambda}) \in M_{m\times n}$, $m < n$ is a *cut-helix matrix* if we have $a_{k\lambda} = a_{k\underline{\lambda}}$, $\forall k, \lambda \in N - \{0\}$.

Moreover, let us denote by $I_c = (c_{k\lambda})$ the *cut-helix unit matrix* which the cut matrix is the unit matrix I_m. Therefore, since $I_m = (\delta_{k\lambda})$, where $\delta_{k\lambda}$ is the Kronecker's delta, we obtain that, $\forall k, \lambda$, we have $c_{k\lambda} = \delta_{k\underline{\lambda}}$. We remind that a matrix X has inverses X^{-1}, with respect to I_c, if we have $I_c \in (X \otimes X^{-1}) \cap (X^{-1} \otimes X)$.

Proposition 3.5. *For $m < n$ in $(M_{m\times n}, \otimes)$ the cut-helix unit matrix $I_c = (c_{k\lambda})$, where $c_{k\lambda} = \delta_{k\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.*

Proof. Let $A, B \in M_{m\times n}$. Then in the helix-multiplication, since $m < n$, we take the helix projection of the matrix A, therefore, the result $A \otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m \times m$. Moreover, in order to have $A \otimes B = A\underline{mm} \cdot B = B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c = (c_{k\lambda})$, where $c_{k\lambda} = \delta_{k\underline{\lambda}}, \forall k, \lambda \in N - \{0\}$, is necessarily the left scalar unit.

Let $A = (a_{uv}) \in M_{m\times n}$ and consider the hyperproduct $A \otimes I_c$. In the entry $k\lambda$ of this hyperproduct there are sets, for all $1 \leq k \leq m$ and $1 \leq \lambda \leq n$, of the form,

$$\Sigma \underline{a}_{ks} c_{s\lambda} = \Sigma \underline{a}_{ks} \delta_{s\underline{\lambda}} = \underline{a}_{k\underline{\lambda}} \ni a_{k\lambda}.$$

Therefore, $A \otimes I_c \ni A, \forall A \in M_{m\times n}$. □

Definition 3.6. *A matrix $A = (a_{ij}) \in M_{m\times n}$ is called a 1-overlap helix matrix if $n = 2m$, and is called a k-overlap helix matrix if $n = (k + 1)m$.*

4 The class of S-helix matrices

Definition 4.1. *Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in N$ such that $1 \leq s \leq m$ and $1 \leq t \leq n$. If $A\underline{st}$ is a set of upper triangular matrices with the same diagonal, then we call A an S-helix matrix of type $s \times t$. Therefore, in an S-helix matrix A of type $s \times t$, the $A\underline{st}$ has on the diagonal entries which are not sets but elements.*

In the following, we restrict our study on the case of $A = (a_{ij}) \in M_{m\times n}$ with $m < n$.

Remark 4.2. *According to the cut-helix notation, we have,*

$$a_{k\lambda} = a_{k\underline{\lambda}} = 0, \text{ for all } k > \lambda \text{ and } a_{k\lambda} = a_{k\underline{\lambda}}, \text{ for } k = \underline{\lambda}.$$

Proposition 4.3. *The set of S-helix matrices $A = (a_{ij}) \in M_{m\times n}$ with $m < n$, is closed under the helix product. Moreover, it has a unit the cut-helix unit matrix I_c, which is left scalar.*

Proof. The helix product of two S-helix matrices, $X = (x_{ij}), Y = (a_{ij}) \in M_{m\times n}$, $X \otimes Y$, contains matrices $Z = (z_{ij})$, which are upper diagonals. Moreover, for every z_{ii}, the entry ii is singleton since it is product of only $z_{(i+km),(i+km)} = z_{ii}$, entries.

The unit is, from Proposition 2.4, the matrix $I_c = I_{m\times n}$, where we have $I_{m\times n} = I_{\underline{\mathrm{mm}}} = I_m$. □

An example of hyper-matrix representation, seven dimensional, with helix-hope is the following:

Example 4.4. *Consider the special case of the matrices of the type* 3×5 *on the field of real or complex. Then we have*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & x_{22} & x_{23} & 0 & x_{22} \\ 0 & 0 & x_{33} & 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} X \otimes Y &= \begin{pmatrix} x_{11} & \{x_{12}, x_{15}\} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22} & x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + x_{13}y_{33} & x_{11}y_{11} & x_{11}y_{15} + \{x_{12}x_{15}\}y_{22} \\ 0 & x_{22}y_{22} & x_{22}y_{23} + x_{23}y_{33} & 0 & x_{22}y_{22} \\ 0 & 0 & x_{33}y_{33} & 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore, the helix product is a set with cardinality up to 8. The unit of this type is

$$I_c = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Definition 4.5. *We call a matrix $A = (a_{ij}) \in M_{m\times n}$ an S_o-helix matrix if it is an S-helix matrix where the condition $a_{11}a_{22}...a_{mm} \neq 0$, is valid. Therefore, an S_0-helix matrix has no zero elements on the diagonal and the set S_0 is a subset of the set S of all S-helix matrices. We notice that this set is closed under the helix product not in addition. Therefore, it is interesting only when the product is used not the addition.*

In the following, we focus on k-overlap helix matrices, especially on 1-overlap helix matrices, which are S_0-helix matrices.

Example 4.6. *Consider the special case of the 1-overlap S_0-helix matrices of the type* 3×6 *on the field of real or complex. Then we have*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & x_2 & x_{23} & 0 & x_2 & x_{26} \\ 0 & 0 & x_3 & 0 & 0 & x_3 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} \\ 0 & y_2 & y_{23} & 0 & y_2 & y_{26} \\ 0 & 0 & y_3 & 0 & 0 & y_3 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} x_1 & \{x_{12}, x_{15}\} & \{x_{13}, x_{16}\} \\ 0 & x_2 & \{x_{23}, x_{26}\} \\ 0 & 0 & x_3 \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} \\ 0 & y_2 & y_{23} & 0 & y_2 & y_{26} \\ 0 & 0 & y_3 & 0 & 0 & y_3 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & 0 & a_{25} & a_{26} \\ 0 & 0 & a_{33} & 0 & 0 & a_{36} \end{pmatrix}$$

where
$a_{11} = x_1 y_1, a_{12} = x_1 y_{12} + \{x_{12}, x_{15}\} y_2, a_{13} = x_1 y_{13} + \{x_{12}, x_{15}\} y_{23} + \{x_{13}, x_{16}\} y_3, a_{14} = x_1 y_1,$
$a_{15} = x_1 y_{15} + \{x_{12}, x_{15}\} y_2, a_{16} = x_1 y_{16} + \{\{x_{12}, x_{15}\} y_{26} + \{x_{13}, x_{16}\} y_3,$
$a_{22} = x_2 y_2, a_{23} = x_2 y_{23} + \{x_{23}, x_{26}\} y_3, a_{25} = x_2 y_2, a_{26} = x_2 y_{26} + \{x_{23}, x_{26}\} y_3$
$a_{33} = x_3 y_3, a_{36} = x_3 y_3$

Therefore, the helix product is a set with cardinality up to 2^8. The unit of this type is

$$I_c = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Example 4.7. *Consider the special case of the above Example 3.5, of the matrices of the type 3×6. Suppose we want to find a right inverse matrix $Y = X^{-1} = (y_{ij})$ of the matrix X. Then we have $I \in X \otimes Y$, from which we must have*

$$y_{21} = y_{24} = y_{31} = y_{32} = y_{34} = y_{35} = 0,$$
$$y_{11} = y_{14} = y_1 = \frac{1}{x1}, y_{22} = y_{25} = y_2 = \frac{1}{x2}, y_{33} = y_{36} = y_3 = \frac{1}{x3},$$

and

$$y_{12}, y_{15} \in -\{x_{12}, x_{15}\} \frac{1}{x1x2}, y_{23}, y_{26} \in -\{x_{23}, x_{26}\} \frac{1}{x2x3},$$
$$y_{13}, y_{16} \in \{x_{12}, x_{15}\} \cdot \{x_{23}, x_{26}\} \frac{1}{x1x2x3} - \{x_{13}, x_{16}\} \frac{1}{x1x3}.$$

If we want to find a left inverse matrix $Y = X^{-1} = (y_{ij})$ of the matrix X, then we have $I \in Y \otimes X$, from which we must have

$$y_{21} = y_{24} = y_{31} = y_{32} = y_{34} = y_{35} = 0,$$
$$y_{11} = y_{14} = y_1 = \frac{1}{x1},\ y_{22} = y_{25} = y_2 = \frac{1}{x2},\ y_{33} = y_{36} = y_3 = \frac{1}{x3},$$

and the following relations are valid

$$0 \in \frac{x12}{x1} + \{y_{12}, y_{15}\} x_2,\ 0 \in \frac{x15}{x1} + \{y_{12}, y_{15}\} x_2,$$
$$0 \in \frac{x23}{x2} + \{y_{23}, y_{26}\} x_3,\ 0 \in \frac{x26}{x2} + \{y_{23}, y_{26}\} x_3,$$
$$0 \in \frac{x13}{x1} + \{y_{12}, y_{15}\} x_{23} + \{y_{13}, y_{16}\} x_3,\ 0 \in \frac{x16}{x1} + \{y_{12}, y_{15}\} x_{26} + \{y_{13},\ y_{16}\} x_3.$$

An interesting study, for applications, is on small finite H_v-fields [7], [8], [15], [17].

Example 4.8. *On the type* 3×6 *of matrices using the Construction 2.9(I), on* $(Z_4, +, \cdot)$ *we take the small* H_v*-field* $(Z4, +, \otimes)$*, where only* $2 \otimes 3 = \{0, 2\}$ *and fundamental classes* $\{0, 2\}, \{1, 3\}$*. We consider the set of all* S_0*-helix matrices and we take the* S_0*-helix matrix:*

$$X = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 2 \\ 0 & 3 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

Then the powers of X are

$$X^2 = \begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} & \{0,2\} \\ 0 & 1 & \{0,2\} & 0 & 1 & \{0,2\} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } X^3 = \begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} & \{0,2\} \\ 0 & 3 & \{1,3\} & 0 & 3 & \{1,3\} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

and so on. We obtain that the generating set is the following

$$\begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} & \{0,2\} \\ 0 & \{1,3\} & Z_4 & 0 & \{1,3\} & Z_4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

where in the 22 and 25 entries appears simultaneously 1 or 3 and in 23 and 26 entries, we select entries from the same fundamental class.

5 Conclusions

The representation theory of hyperstructures by matrices or hypermatrices is more complicated than the ordinary representation theory. Only some special classes of hyperstructures can be represented and some special hopes can be used. The helix-hopes defined on the k-overlap helix matrices which, moreover, are S_0-helix matrices, can represent finite hyperstructures of low dimension. This can be achieved because the set of the k-overlap helix matrices is closed under the helix-hope.

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