



## Hyperstructures on bar of $V$ & $V$ in pieces

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### Abstract

An application of hyperstructure theory on social sciences is presented. In social sciences when questionnaires are used, there is a new tool, the bar instead of Likert scale. The bar has been suggested by Vougiouklis & Vougiouklis in 2008, who have proposed the replacement of Likert scales, usually used in questionnaires, with a bar. This new tool, gives the opportunity to researchers to elaborate the questionnaires in different ways, depending on the filled questionnaires and of course on the problem. We study these filled questionnaires using hyperstructure theory. The hyperstructure theory is being related with questionnaires and we study the obtained hyperstructures which are used as an organized device of the problem and we focus on special problems.

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## 1 Introduction

The main object of this paper is the class of hyperstructures called  $H_v$ -structures introduced in 1990, which satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions are the following [17]:

**Definition 1.1.** In a set  $H$  equipped with a hyperoperation (abbreviation hyperoperation = hope)

$$\cdot : H \times H \rightarrow P(H) - \{\emptyset\},$$

we abbreviate by

WASS the weak associativity:  $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$  and by

COW the weak commutativity:  $xy \cap yx \neq \emptyset, \forall x, y \in H$ .

The hyperstructure  $(H, \cdot)$  is called an  $H_v$ -semigroup if it is WASS, it is called  $H_v$ -group if it is reproductive  $H_v$ -semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

## 2 Preliminaries

In the classical theory the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an  $H_v$ -group. This is the motivation to introduce the  $H_v$ -structures [17], [18].

In an  $H_v$ -semigroup the powers of an element  $h \in H$  are defined as follows:  $h^1 = \{h\}$ ,  $h^2 = h \cdot h$ , ...,  $h^n = h \circ h \circ \dots \circ h$ , where  $(\circ)$  denotes the  $n$ -ary circle hope, i.e. take the union of hyperproducts,  $n$  times, with all possible patterns of parentheses put on them. An  $H_v$ -semigroup  $(H, \cdot)$  is called *cyclic of period  $s$* , if there exists an element  $h$ , called *generator*, and a natural number  $s$ , the minimum one, such that  $H = h^1 \cup h^2 \dots \cup h^s$ . Analogously the cyclicity for the infinite period is defined [18]. If there is an element  $h$  and a natural number  $s$ , the minimum one, such that  $H = h^s$ , then  $(H, \cdot)$  is called *single-power cyclic of period  $s$* .

**Definition 2.1.**  $(R, +, \cdot)$  is called an  $H_v$ -ring if  $(+)$  and  $(\cdot)$  are WASS, the reproduction axiom is valid for  $(+)$  and  $(\cdot)$  is weak distributive with respect to  $(+)$ :

$$x(y + z) \cap (xy + xz) \neq \emptyset, \text{ and } (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

Let  $(R, +, \cdot)$  be an  $H_v$ -ring,  $(M, +)$  be a COW  $H_v$ -group and there exists an external hope

$$\cdot : R \times M \rightarrow P(M) : (a, x) \rightarrow ax$$

such that  $\forall a, b \in R$  and  $\forall x, y \in M$  we have

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, \text{ and } (ab)x \cap a(bx) \neq \emptyset,$$

then  $M$  is called an  $H_v$ -module over  $F$ . In the case of an  $H_v$ -field  $F$ , which is defined later, instead of an  $H_v$ -ring  $R$ , then the  $H_v$ -vector space is defined.

For more definitions and applications on  $H_v$ -structures one can see [1], [2], [3], [10], [14], [16], [18].

The main tool to study hyperstructures is the fundamental relation. In 1970 M. Koscas defined in hypergroups the relation  $\beta$  and its transitive closure  $\beta^*$ . This relation connects the hyperstructures with the corresponding classical structures and is defined in  $H_v$ -groups as well. T. Vougiouklis introduced the  $\gamma^*$  and  $\epsilon^*$  relations, which are defined, in  $H_v$ -rings and  $H_v$ -vector spaces, respectively [17]. He also named all these relations  $\beta^*$ ,  $\gamma^*$  and  $\epsilon^*$ , Fundamental Relations because they play very important role to the study of hyperstructures especially in the representation theory of them. For similar relations see [3], [18], [23].

**Definition 2.2.** The fundamental relations  $\beta^*$ ,  $\gamma^*$  and  $\epsilon^*$ , are defined, in  $H_v$ -groups,  $H_v$ -rings and  $H_v$ -vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively.

Specifying the above motivation we remark the following: Let  $(G, \cdot)$  be a group and  $R$  be an equivalence relation (or a partition) in  $G$ , then  $(G/R, \cdot)$  is an  $H_v$ -group, therefore we have the quotient  $(G/R, \cdot)/\beta^*$  which is a group, the fundamental one. Remark that the classes of the fundamental group  $(G/R, \cdot)/\beta^*$  are a union of some of the  $R$ -classes. Otherwise, the  $(G/R, \cdot)/\beta^*$  has elements classes of  $G$  where they form a partition which classes are larger than the classes of the original partition  $R$ .

The way to find the fundamental classes is given by the following [17], [19]:

**Theorem 2.3.** Let  $(H, \cdot)$  be an  $H_v$ -group and denote by  $U$  the set of all finite products of elements of  $H$ . We define the relation  $\beta$  in  $H$  by setting  $x\beta y$  iff  $\{x, y\} \subset u$  where  $u \in U$ . Then  $\beta^*$  is the transitive closure of  $\beta$ .

A large class of  $H_v$ -structures is the following [20], [22]:

**Definition 2.4.** Let  $(G, \cdot)$  be groupoid (resp. hypergroupoid) and  $f : G \rightarrow G$  be a map. We define a hope  $(\partial)$ , called theta-hope, we write  $\partial$ -hope, on  $G$  as follows,

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G. \text{ (resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G)$$

If  $(\cdot)$  is commutative, then  $\partial$  is commutative. If  $(\cdot)$  is COW, then  $\partial$  is COW.

Let  $(G, \cdot)$  be a groupoid (or hypergroupoid) and  $f : G \rightarrow P(G) - \{\emptyset\}$  be any multivalued map. We define the  $(\partial)$  on  $G$  as follows,

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$$

Let  $(G, \cdot)$  be a groupoid,  $f_i : G \rightarrow G, i \in I$ , be a set of maps on  $G$ . The

$$f_{\cup} : G \rightarrow P(G) : f_{\cup}(x) = \{f_i(x) | i \in I\},$$

is the union of  $f_i(x)$ . We have the union  $\partial$ -hope  $(\partial)$ , on  $G$  if we take  $f_{\cup}(x)$ . If  $\underline{f} \equiv f \cup (id)$ , then we have the  $b - \partial - hope$ .

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. The basic property is that if  $(G, \cdot)$  is a semigroup, then for every  $f$ , the  $(\partial)$  is WASS.

A well known and large class of hopes is given as follows [13], [15], [18]:

Let  $(G, \cdot)$  be a groupoid. Then for every  $P \subset G, P \neq \emptyset$ , we define the following hopes called  $P$ -hopes: for all  $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$

$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The  $(G, \underline{P}), (G, \underline{P}_r)$  and  $(G, \underline{P}_l)$  are called  $P$ -hyperstructures. The most usual case is if  $(G, \cdot)$  is semigroup, then  $x\underline{P}y = (xP)y \cup x(Py) = xPy$  and  $(G, \underline{P})$  is a semihypergroup. We do not know what hyperstructures are  $(G, \underline{P}_r)$  and  $(G, \underline{P}_l)$ . In some cases, depending on the choice of  $P$ , the  $(G, \underline{P}_r)$  and  $(G, \underline{P}_l)$  can be associative or WASS. If more operations are defined in  $G$ , then for each operation several  $P$ -hopes can be defined.

### 3 V & V bar in questionnaires

Last decades hyperstructures seem to be applied not only in mathematics, but also in a variety of sciences [1], [7], such as the social ones.

An important application which can be used in social sciences is the combination of hyperstructure theory with fuzzy theory, by the replacement of the Likert Scale by the V & V Bar. The suggestion is the following [7]:

**Definition 3.1.** In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 \_\_\_\_\_ 1

*The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point she/he feels expresses her/his answer to the specific question.*

The final suggested length of the bar according to the Golden Ratio is 62mm [24], [25]. This theory is closely related to fuzzy theory, as with this proposal a discrete situation is replaced by a fuzzy one [2], [3], [4], [5], [7], [18]. The use of the bar instead of a scale of Likert has several advantages during both the filling and research processing.

Questionnaire processing using the bar gives the researchers the possibility to 'escalate' the answers, without having to decide in advance how many different grades would be used [1],[11]

Even more so, there is a good degree of flexibility in establishing balanced or imbalanced scales taking into consideration the needs of the specific research each time.

More specifically after the researcher has collected the filled-in questionnaires he/she will be able to process and access them in numerous ways without having to repeat the test putting the subjects in a new time consuming process and risking the reliability of results. Using the bar minimizes such risks, reinforces the objectivity of it as it gives more spaces to mathematical processing of the results [8].

## 4 V & V bar and hyperstructures

In the research processing suppose that we want to use Likert scale dividing the continuum  $[0,62]$  first into equal steps (segments) and, second into equal area division of Gauss distribution [21] or parabola distribution. If we consider that the continuum  $[0,62]$  is divided into  $n$  segments, we can number the  $n$  segments starting with 0.

**Definition 4.1.** [6],[12] *Suppose we divide the V & V bar  $[0,62]$  in  $n$  segments by the points  $t_1, t_2, \dots, t_{n-1}$ . So, we have the following  $n$  segments, starting with 0:*

$$0 = [0, t_1], 1 = (t_1, t_2], \dots, n-1 = (t_{n-1}, 62],$$

*and we consider these segments as elements of  $(Z_n, +)$ . We define a hope  $(\oplus)$  in  $Z_n$  as follows:*

$$\text{For all } i, j \in \{0, 1, \dots, n-1\}, \quad e_i \oplus e_j = \{e_k : x + y \in e_k, \forall x \in e_i, \forall y \in e_j\}$$

*Then, we obtain the  $H_v$ -group  $(Z_n, \oplus)$  which can be considered as an organized device on the segments of the V & V bar.*

**Example 4.2.** [11] *When the 62mm of the bar are divided into 5 equal segments as follows:*

$$0 = [0, 12.4], 1 = (12.4, 24.8], 2 = (24.8, 37.2], 3 = (37.2, 49.6], 4 = (49.6, 62]$$

*the obtained table is the following one:*

$\oplus$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>0</b>	0,1	1,2	2,3	3,4	0,4
<b>1</b>	1,2	2,3	3,4	0,4	0,1
<b>2</b>	2,3	3,4	0,4	0,1	1,2
<b>3</b>	3,4	0,4	0,1	1,2	2,3
<b>4</b>	0,4	0,1	1,2	2,3	3,4

The multiplication table obtained by this hyperoperation, when the bar is divided into  $n$  equal segments ( $n \neq 0$ ) is the following:

$\oplus$	<b>0</b>	<b>1</b>	<b>2</b>	<b>...</b>	<b>n-2</b>	<b>n-1</b>
<b>0</b>	0, 1	1, 2	2, 3	...	n-2, n-1	0, n-1
<b>1</b>	1, 2	2, 3	3, 4	...	0, 5	0, 1
<b>2</b>	2, 3	3, 4	4, 5	...	0, 1	1, 2
<b>...</b>	...	...	...	...	...	...
<b>n-2</b>	n-2, n-1	0, n-1	0, 1	...	n-4, n-3	n-3, n-2
<b>n-1</b>	0, n-1	0, 1	1, 2	...	n-3, n-2	n-2, n-1

The defined hyperoperation can be applied in all kind of subdivisions.

**Example 4.3.** [9] When the 62mm of the bar are divided into 5 equal area segments as follows:

$$0 = [0, 24], 1 = (24, 29], 2 = (29, 33], 3 = (33, 38], 4 = (38, 62]$$

$\oplus$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>0</b>	0,1,2,3,4	1,2,3,4	2,3,4	3,4	0,4
<b>1</b>	1,2,3,4	4	4	0,4	0,1
<b>2</b>	2,3,4	4	0,4	0	0,1,2
<b>3</b>	3,4	0,4	0	0	0,1,2,3
<b>4</b>	0,4	0,1	0,1,2	0,1,2,3	0,1,2,3,4

So, the multiplication table obtained by the hyperoperation defined on the segments, when the bar is divided into  $n$  equal-area segments ( $n \leq 8$ ) following the Gauss distribution are the following:

$\oplus$	<b>0</b>	<b>1</b>	<b>2</b>	<b>...</b>	<b>n-2</b>	<b>n-1</b>
<b>0</b>	0,1,...,n-2,n-1	1,2,...,n-2,n-1	2,3,...,n-2,n-1	...	n-2,n-1	n-1,0
<b>1</b>	1,2,...,n-2,n-1	n-1	n-1	n-1	0,n-1	0,1
<b>2</b>	2,3,...,n-2,n-1	n-1	n-1	0,n-1	0	0,1,2
<b>...</b>	...	...	...	...	...	...
<b>n-2</b>	n-2,n-1	0,n-1	0	...	0	0,1,...,n-2
<b>n-1</b>	0,n-1	0,1	0,1,2	...	0,1,...,n-2	0,1,...,n-2,n-1

**Properties 4.4.** When the bar is divided into equal-area segments, we have the following properties:

1. The multiplication tables are following the same pattern in the division until 8 equal area segments. In greater division, the pattern is changing.
2. The elements 0 and  $n - 1$  are unit elements.
3. It is an  $H_v$ -group, single-power cyclic of period 2, with 0 and  $n - 1$  be the generators, until 8 equal-area segments. The reste elements are also single-power generators with period greater than 2.

The multiplication table obtained by the hyperoperation defined on the segments, when the bar is divided into  $n$  equal-area segments ( $n \geq 9$ ) following the Gauss distribution are the following:

$\oplus$	<b>0</b>	<b>1</b>	<b>2</b>	<b>...</b>	<b>n-2</b>	<b>n-1</b>
<b>0</b>	0,1,...,n-2	1,2,...,n-2,n-1	2,3,...,n-2,n-1	...	n-2,n-1	n-1,0
<b>1</b>	1,2,...,n-2,n-1	n-1	n-1	n-1	0,n-1	0,1
<b>2</b>	2,3,...,n-2,n-1	n-1	n-1	0,n-1	0	0,1,2
<b>...</b>	...	...	...	...	...	...
<b>n-2</b>	n-2,n-1	0,n-1	0	...	0	0,1,...,n-2
<b>n-1</b>	0,n-1	0,1	0,1,2	...	0,1,...,n-2	1,2,...,n-2,n-1

For more than 8-equal area segments the  $H_v$ -group is also a single-power cyclic with the same generators but period 3.

## 5 Conclusion

The class of hyperstructures called  $H_v$ -structures has been studied from several aspects as well as in connection with many other topics of mathematics. Here we present applications obtained from social sciences mainly the ones used questionnaires. We apply a hyperoperation on questionnaires, and study the obtained hyperstructures, based on different types of separation.

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