Hyper vector spaces over Krasner hyperfields

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Abstract

In this paper, we study hyper vector spaces over Krasner hyperfields. First, we introduce the notions of linearly independence (dependence) and basis for a hyper vector space. Second, we investigate their properties and prove some results for hyper vector spaces that are similar to that of vector spaces over fields. Then, we define linear transformations over hyper vector spaces and investigate their properties. Finally, we prove the dimension theorem for linear transformations.

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1 Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Marty [9] in 1934 at the eighth Congress of Scandinavian Mathematicians. Where he generalized the notion of a group to that of a hypergroup. A hypergroup is a non-empty set equipped with an associative hyperoperation and reproductive hyperoperation. In a group, the composition of two elements is an element whereas in a hypergroup, the composition of two elements is a non-empty set. Since then, many different kinds of hyperstructures (hyperring, hypermodule, hypervector space, ...) were widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics (see [3, 4, 5, 6, 11, 16, 17]). There exist different kinds of hyperrings. A special case of this type is the hyperring introduced by Krasner [8]. Also, Krasner introduced a new class of hyperrings and hyperfields: the quotient

hyperrings and hyperfields. For more details about Krasner hyperrings we refer to [6, 8, 12]. Tallini in [14, 15] introduced the notion of hyper vector spaces over fields and studied basic properties of them. On the other hand, Vougiouklis [18] introduced weak hyper vector spaces, namely H_v-vector spaces. Later Ameri et al. in [1] studied the properties of dimension of hypervector spaces over fields and introduced the notions of linearly independent (respectively linearly dependent), generator and basis of a hypervector space.

The remainder part of this paper is organized as follows: In Section 2, we present some definitions related to hyperstructures that are used throughout the paper. In Section 3 we define hyper vector spaces over Krasner hyperfields and introduce some definitions like linearly independence, basis, ... Moreover, we investigate some properties of such spaces and prove that hyper vector spaces have similar properties to that of vector spaces. In Section 4, we define linear transformations between hyper vector spaces and introduce some definitions like the kernel, range, ... Also, we prove the Dimension theorem for linear transformations over hyper vector spaces.

2   Basic definitions

In this section, we present some definitions related to hyperstructures that are used throughout the paper (see [5, 6]).

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if A and B are two non-empty subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called: a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$; a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (this condition is called the reproduction axiom); a *hypergroup* if it is a semihypergroup and a quasihypergroup. A *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axiom: (1) $(R, +)$ is a commutative hypergroup; (2) there exists $0 \in R$ such that $0 + x = \{x\}$ for all $x \in R$; (3) for every $x \in R$ there exists unique $x' \in R$ such that $0 \in x + x'$; (x' is denoted by $-x$); (4) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$; (5) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$; (6) the multiplication "·" is distributive with respect to the hyperoperation "+". Note that every ring is a Krasner hyperring. Different examples of Krasner hyperrings were constructed. We refer to [2, 5]. Let $(R, +, \cdot)$ be a Krasner hyperring and A be a non-empty subset of R. Then A is said to be a *subhyperring* of R if $(A, +, \cdot)$ is itself a hyperring. A subhyperring A of a Krasner hyperring $(R, +, \cdot)$ is a *hyperideal* of R if $r \cdot a \in A$ $(a \cdot r \in A)$ for all $a \in A, r \in R$. A commutative Krasner hyperring $(R, +, \cdot)$ with identity element "1" is a *Krasner hyperfield* if $(R \setminus \{0\}, \cdot)$ is a group.

3   Dimension of hyper vector spaces

In [1], Ameri et al. defined hyper vector spaces as an abelian group over a field and investigated their properties. In this section, we present a different definition for hyper vector spaces over Krasner hyperfields, study their properties and present an example applying the new defined notions.

Definition 3.1. *Let F be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : F \times V \to V$, is called a hyper vector space over F if for all $a, b \in F$ and $x, y \in V$, the following conditions hold: (1) $a \cdot (x + y) = a \cdot x + a \cdot y$; (2) $(a + b) \cdot x = a \cdot x + b \cdot x$; (3) $a \cdot (b \cdot x) = (ab) \cdot x$; (4) $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$; (5) $x = 1 \cdot x$.*

Definition 3.2. *Let F be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : F \times V \to V$, is called a weak hyper vector space over F if for all $a, b \in F$ and $x, y \in V$, the following conditions hold: (1) $a \cdot (x + y) \subseteq a \cdot x + a \cdot y$; (2) $(a + b) \cdot x \subseteq a \cdot x + b \cdot x$; (3) $a \cdot (b \cdot x) = (ab) \cdot x$; (4) $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$; (5) $x = 1 \cdot x$.*

Example 3.3. *Let F be a Krasner hyperfield and X be the set of all n-tuples with entries from F. Then, (1) F is trivially a hyper vector space over itself; (2) X is a weak hypervector space over F.*

Next, we present a (non trivial) example of a hyper vector space.

Example 3.4. *Let $(F, +, \cdot)$ be a Krasner hyperfield, E be a non-empty set and F^E be the set of all functions from E to F. Then $(F^E, \oplus)$ is a hyper vector space over F. Where "$\oplus$" and $\star : F \times F^E \longrightarrow F^E$ are defined as follows: For all $f, g \in F^E, k, x \in F$,*

$$(f \oplus g)(x) = f(x) + g(x) \text{ and } (k \star f)(x) = k \cdot f(x).$$

Throughout this section, F is a Krasner hyperfield, V is a hyper vector space over F, "0" is the zero of F and "$\underline{0}$" is the zero of V.

Definition 3.5. *Let F be a Krasner hyperfield and $(V, +)$ be a (weak) hyper vector space over F. A non-empty subset $W \subseteq V$ is called subhyperspace of V if for all $x, y \in W$ and $a \in F$, we have (1) $x - y \subseteq W$; (2) $a \cdot x \in W$.*

Proposition 3.6. *Let F be a Krasner hyperfield and $(V, +)$ be a hyper vector space over F. If $a \in F$ and $x \in V$ then*

(1) $-\underline{0} = \underline{0}$;

(2) $-(-a) = a$;

(3) $a \cdot \underline{0} = \underline{0}$;

(4) $0 \cdot x = \underline{0}$.

Proof. Let $x \in V$ and $a \in F$.

- Proof of 1. Having that $-\underline{0} = -\underline{0} + \underline{0}$ and $\underline{0} \in -\underline{0} + \underline{0}$ implies that $-\underline{0} = \underline{0}$.

- Proof of 2. The proof follows from having $0 \in -a + a$ for all $a \in F$.

- Proof of 3. Since $-\underline{0} = \underline{0}$, it follows that $a \cdot \underline{0} = a \cdot (-\underline{0}) = -a \cdot \underline{0}$. Having $\underline{0} \in a \cdot \underline{0} - a \cdot \underline{0} = a \cdot (\underline{0} - \underline{0}) = \{a \cdot \underline{0}\}$ implies that $a \cdot \underline{0} = \underline{0}$.

- Proof of 4. Having $\underline{0} \in 0 \cdot x - 0 \cdot x = (0 - 0) \cdot x = \{0 \cdot x\}$ implies that $0 \cdot x = \underline{0}$. □

Proposition 3.7. *Let F be a Krasner hyperfield and $(V, +)$ be a hyper vector space over F. A non-empty subset $W \subseteq V$ is subhyperspace of V if and only if $a \cdot x + b \cdot y \subseteq W$ for all $x, y \in W$ and $a, b \in F$.*

Proof. Suppose that W is a subhyperspace of V. Then $a \cdot x \in W$ and $-b \cdot y = (-b) \cdot y \in W$ for all $a, b \in F$. Thus, $a \cdot x + b \cdot y = a \cdot x - (-b \cdot y) \subseteq W$.
Conversely, suppose that $a \cdot x + b \cdot y \subseteq W$. By setting $b = 0$, we get that $\{a \cdot x\} = a \cdot x + 0 \cdot y \subseteq W$ and by setting $a = 1, b = -1$, we get that $x - y \subseteq W$. Thus, W is a subhyperspace of V. □

Definition 3.8. *A subset* $S = \{v_1, v_2, \cdots, v_n\}$ *of a hypervector space* V *over a Krasner hyperfield* F *is called linearly independent if* $c_1, c_2, \cdots, c_n \in F$ *and* $\underline{0} \in c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots + c_n \cdot v_n$ *then* $c_1 = c_2 = \cdots = c_n = 0$. *A subset* S *of* V *is called linearly dependent if it is not linearly independent.*

Remark 3.9. *Let* $S = \{v_1, v_2, \cdots, v_n\}$ *be a subset of a hypervector space* V *over a Krasner hyperfield* F. *Then, by Proposition 3.6; 4.,* $\underline{0} \in c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots + c_n \cdot v_n$ *has at least one solution;* $c_1 = \cdots = c_n = 0$.

Definition 3.10. *Let* $S = \{v_1, v_2, \cdots, v_n\}$ *be a subset of a hyper vector space* V *over a Krasner hyperfield* F *and* $v \in V$. v *is said to be a linear combination of* S *(or* $v \in span(S)$*) if there exist* $c_1, c_2, \cdots, c_n \in F$ *such that* $v \in c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots + c_n \cdot v_n$.

In what follows and for simplicity, we write ax instead of $a \cdot x$ for all $a \in F$ and $x \in V$.

Proposition 3.11. *Let* $S = \{v_1, v_2, \cdots, v_n\}$ *be a subset of a hyper vector space* V *over a Krasner hyperfield* F. *Then* S *is linearly dependent if and only if one of its vectors is a linear combination of the others.*

Proof. Suppose that $S = \{v_1, v_2, \cdots, v_n\}$ is linearly dependent. Then there exist $c_1, \cdots, c_n \in F$ such that $\underline{0} \in c_1 v_1 + \cdots + c_n v_n$ and at least one $c_i \neq 0$. We get now, by Proposition 3.6, that

$$\underline{0} = c_i^{-1} \cdot \underline{0} \in d_1 v_1 + \cdots d_{i-1} v_{i-1} + v_i + d_{i+1} v_{i+1} + \cdots + d_n v_n,$$

where $d_j = c_i^{-1} c_j$ for all $j = 1, \cdots, n$. The latter implies that there exists $x \in d_1 v_1 + \cdots d_{i-1} v_{i-1} + d_{i+1} v_{i+1} + \cdots + d_n v_n$ such that $\underline{0} \in x + v_i$. Having $v_i \in -x + 0 = \{-x\}$ implies that $v_i \in -d_1 v_1 + \cdots - d_{i-1} v_{i-1} - d_{i+1} v_{i+1} + \cdots - d_n v_n$.

Conversely, without loss of generality, suppose that $v_n \in c_1 \cdot v_1 + \cdots + c_{n-1} \cdot v_{n-1}$. We have that $\underline{0} \in v_n - v_n \subseteq c_1 v_1 + \cdots + c_{n-1} v_{n-1} + (-1) v_n$. Thus, S is linearly dependent. $\square$

Proposition 3.12. *Let* V *be a hyper vector space over a Krasner hyperfield* F. *Then*

(1) *a set with two vectors is linearly dependent if and only if one of them is a scalar multiple of the other*

(2) *a set containing* $\underline{0}$ *is linearly dependent;*

(3) *a set with one vector (that is not the zero vector) is linearly independent.*

Proof. The proof of 1. follows from Proposition 3.11.

- Proof of 2. This is clear by using Proposition 3.6, as $\underline{0} = 1\underline{0}$.

- Proof of 3. Let $S = \{x\} \neq \{\underline{0}\}$ and $c \in F$. If $\underline{0} \in cx$ and $c \neq 0$ then $\underline{0} = c^{-1}\underline{0} = c^{-1}(cx) = x$. $\square$

Definition 3.13. *A subset* $S = \{v_1, v_2, \cdots, v_n\}$ *of a hyper vector space* V *over a Krasner hyperfield* F *is said to span* V *if for every vector* $v \in V$, *there exist* $c_1, c_2, \cdots, c_n \in F$ *such that* $v \in c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots + c_n \cdot v_n$.

Definition 3.14. *A subset* $S = \{v_1, v_2, \cdots, v_n\}$ *of a hyper vector space* V *over a Krasner hyperfield* F *is said to basis for* V *if it is linearly independent and it spans* V. *We say that* V *is finite dimensional if it has a finite basis. Otherwise, it is called infinite dimensional.*

Proposition 3.15. *Let V be a hyper vector space over a Krasner hyperfield F and $S = \{v_1, v_2, \cdots, v_n\}$ be a basis for V. Then every vector $v \in V$ is uniquely represented as linear combination of vectors in S.*

Proof. Suppose that there exist $c_i, d_i \in F$ such that $v \in c_1 v_1 + \cdots + c_n v_n$ and $v \in d_1 v_1 + \cdots + d_n v_n$. Then $\underline{0} \in v - v \subseteq (c_1 - d_1)v_1 + \cdots + (c_n - d_n)v_n$. The latter implies that there exist $t_i \in c_i - d_i$ such that $\underline{0} \in t_1 v_1 + \cdots + t_n v_n$. Since S is linearly independent, it follows that $t_i = 0$ for all $i = 1, \cdots, n$. We get now that $0 \in c_i - d_i$ for all $i = 1, \cdots, n$. Having $(V, +)$ a canonical hypergroup implies that $c_i = d_i$ for all $i = 1, \cdots, n$. $\qquad\square$

Theorem 3.16. Plus/Minus Theorem. *Let S be a non empty subset of a hyper vector space V over a Krasner hyperfield F.*

(1) *If S is linearly independent and $v \in V$ is not in span of S then $S \cup \{v\}$ is linearly independent;*

(2) *If there exist a vector $v \in S$ such that v is linear combination of other vectors in S then $S - \{v\}$ and S span the same space.*

Proof. Let $S = \{v_1, \cdots, v_n\}$.

- Proof of 1. Let $\underline{0} \in c_1 v_1 + \cdots c_n v_n + c_{n+1} v$. Then $c_{n+1} = 0$, otherwise $v \in span(S)$. The latter implies that $\underline{0} \in c_1 v_1 + \cdots c_n v_n$. The linearly independence of S asserts that $c_1 = \cdots = c_n = 0$.

- Proof of 2. Let $w \in span(S)$. Then there exist $d_1, \cdots, d_n \in F$ such that $w \in d_1 v_1 + \cdots d_n v_n$. Without loss of generality, let v_n be a linear combination of other vectors in S. Then there exist $c_1, \cdots, c_{n-1} \in F$ such that $w \in d_1 v_1 + \cdots + d_{n-1} v_{n-1} + d_n(c_1 v_1 + \cdots c_{n-1} v_{n-1}) = (d_1 + c_1 d_n)v_1 + \cdots + (d_{n-1} + c_{n-1} d_n)v_{n-1}$. The latter implies that $w \in span(S - \{v_n\})$. $\quad\square$

Proposition 3.17. *Let V be a hyper vector space over a Krasner hyperfield F, $n \in \mathbb{N}$ and W_i be a subhyperspace of V for $i = 1, \ldots, n$. Then*

(1) $\{\underline{0}\}$ *and V are subhyperspaces of V;*

(2) $W_1 \cap W_2$ *is a subhyperspace of V;*

(3) $W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}$ *is a subhyperspace of V;*

(4) $\bigcap\limits_{i=1}^{n} W_i$ *is a subhyperspace of V;*

(5) $\sum\limits_{i=1}^{n} W_i$ *is a subhyperspace of V.*

Proof. The proof is straightforward. $\qquad\square$

Example 3.18. *Let $F_2 = \{0, 1\}$ and define $(F_2, +)$ and $(F_2, \cdot)$ by the following tables:*

+	0	1
0	0	1
1	1	F_2

$\cdot$	0	1
0	0	0
1	0	1

Define the map $\odot : F_2 \times F_2^3 \to F_2^3$ *as:*

$$a \odot (x, y, z) = (a \cdot x, a \cdot y, a \cdot z)$$

for all $a, x, y, z \in F_2$.

It is clear that F_2^3 *is a weak hyper vector space over the Krasner hyperfield* F_2. *Define the subhyperspaces* W_1, W_2 *as*

$$W_1 = \{(0,0,0), (0,1,0)\} \text{ and } W_2 = \{(0,0,0), (1,0,0), (0,0,1), (1,0,1)\}.$$

One can easily see that $W_1 \cup W_2 = \{(0,0,0), (0,1,0), (1,0,0), (0,0,1), (1,0,1)\}$ *is not a subhyperspace of* F_2^3.

Proposition 3.19. *Let* V *be a (weak) hyper vector space over a Krasner hyperfield* F *and* W_1, W_2 *be subhyperspaces of* V. *Then* $W_1 \cup W_2$ *is a subhyperspace of* V *if and only if* $W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$.

Proof. If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ then it is clear that $W_1 \cup W_2$ is a subhyperspaces of V.
Let $W_1 \cup W_2$ be a subhyperspaces of V. Suppose that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Then there are $x, y \in V$ such that $x \in W_1$, $y \in W_2$, x is not in W_2 and y not in W_1. Having $x + y \subseteq W_1 \cup W_2$ implies that for all $z \in x + y$ we have $z \in W_1 \cup W_2$. Without loss of generality, let $z \in W_1$. Having $z \in x + y$ implies that $y \in z - x \subseteq W_1$. The latter implies contradiction. $\square$

Definition 3.20. *Let* V *be a hyper vector space over a Krasner hyperfield* F *and* $S \subseteq V$. *We define* $L(S)$ *as follows:*

$$L(S) = \Big\{ t \in V : t \in \sum_{i=1}^{n} a_i s_i, a_i \in F, s_i \in S, n \in \mathbb{N} \Big\}.$$

Proposition 3.21. *Let* V *be a hyper vector space over a Krasner hyperfield* F *and* $S \subseteq V$. *Then* $L(S)$ *is the smallest subhyperspace of* V *containing* S.

Proof. Having $s = 1 \cdot s \in L(S)$ for all $s \in S$ implies that $S \subseteq L(S)$.
Let $x, y \in L(S)$ and $a, b \in F$. Since $c\underline{0} = \underline{0}$ and $\underline{0} + x = x$ for all $c \in F, x \in V$ then we can write $ax + by$ as follows:

$$ax + by = a \sum_{i=1}^{n} a_i s_i + b \sum_{i=1}^{m} b_i s_i = \sum_{i=1}^{\max(m,n)} (aa_i + bb_i) s_i \subseteq L(S).$$

Thus, $L(S)$ is a subhyperspace of V. Let W be a subhyperspace of V containing S. Then $\sum_{i=1}^{n} a_i s_i \subseteq W$ for all $s_i \in S \subseteq W$ and $a_i \in F$. Thus, $L(S) \subseteq W$. $\square$

Proposition 3.22. *Let* V *be a hyper vector space over a Krasner hyperfield* F *and* W_1, W_2 *be subhyperspaces of* V. *Then* $L(W_1 \cup W_2) = W_1 + W_2$.

Proof. Let $x \in W_1 \cup W_2$. Then $x \in W_1$ or $x \in W_2$. Writing x as $x = x + \underline{0}$ or $x = \underline{0} + x$ implies that $W_1 \cup W_2 \subseteq W_1 + W_2$. Propositions 3.17 and 3.21 assert that $L(W_1 \cup W_2) \subseteq W_1 + W_2$. For every $z \in W_1 + W_2$, $z \in x + y$ for some $x \in W_1, y \in W_2$. The latter implies that $x, y \in W_1 \cup W_2$. We get now that $x + y \subseteq L(W_1 \cup W_2)$. Thus, $W_1 + W_2 \subseteq L(W_1 \cup W_2)$. $\square$

Proposition 3.23. *Let* W *be a non trivial subhyperspace of* V *and* S *be a spanning set for* W. *Then* S *contains a linearly independent set that spans* W.

Proof. Let $S = \{v_1, \cdots, v_n\} \subseteq W$. Since every non zero vector in V is linearly independent and S spans $W \neq \{\underline{0}\}$, it follows that S contains a linearly independent subset $S' = \{v_1, \cdots v_k\}$. If S is linearly independent, we are done. If not, remove the first v_j such that $v_j \in span(S)$. Using Plus/Minus Theorem, we get that S and $S - \{v_j\}$ span W. If $S - \{v_j\}$ is linearly independent, we are done. If not, remove the first v_r such that $v_r \in span(S - \{v_j\})$. Continuing on this pattern, we get the required result. $\square$

Corollary 3.24. *Let W be a subhyperspace of V and S be a spanning set for W. Then S contains a basis for W.*

Proof. Proposition 3.23 asserts that there exist a linearly independent subset of S that spans W. Thus, S contains a basis for W. $\square$

Theorem 3.25. *Let V be a hyper vector space of dimension n. Then every linearly independent subset of V contains at most n elements.*

Proof. The proof is similar to that of Theorem 3.2 in [1]. $\square$

Corollary 3.26. *Let B, B' be two basis for a finite dimensional hyper vector space V. Then $|B| = |B'|$.*

Proof. Since B is a basis for V and B' is linearly independent subset of V, it follows by using Theorem 3.25 that $|B'| \leq |B|$.

Since B' is a basis for V and B is linearly independent subset of V, it follows by using Theorem 3.25 that $|B| \leq |B'|$. $\square$

Lemma 3.27. *Let V be a finite dimensional hyper vector space. Then every linearly independent subset S of V is contained in a finite basis of V.*

Proof. Let $dim(V) = n$ and $|S| = k$. Theorem 3.25 asserts that $k \leq n$. If $k = n$ then S is a basis for V. Otherwise if S is not a basis of V then there exist $v_1 \in V$ such that v_1 is not in $span(S)$. Thus, by Plus/Minus Theorem, $S \cup \{v_1\}$ is linearly independent set. The latter contradicts Theorem 3.25. If $k < n$, then there exists $v_1 \in V$ such that v_1 is not in $span(S)$. By Plus/Minus Theorem, we get that $S \cup \{v_1\}$ is linearly independent. Again if $|S \cup \{v_1\}| = n$ then $S \cup \{v_1\}$ is a basis for V. Otherwise, $|S \cup \{v_1\}| = k + 1 < n$ and there exist $v_2 \in V$ such that v_2 is not in $span(S \cup \{v_1\})$. Thus, by Plus/Minus Theorem, $S \cup \{v_1, v_2\}$ is linearly independent set. Continuing on this pattern, we get that $S \cup \{v_1, \cdots, v_{n-k}\}$ is a linearly independent set. Since $|S \cup \{v_1, \cdots, v_{n-k}\}| = n = dim(V)$, it follows that $S \cup \{v_1, \cdots, v_{n-k}\}$ is a basis for V containing S. $\square$

Lemma 3.28. *Let V be a finite dimensional hyper vector space and W be a subhyperspace of V. Then dimension of $W \leq$ dimension of V.*

Proof. Let $dim(W) = k$. Then there exists a basis S for W such that $|S| = k$. Having S a linearly independent set in V implies, by Theorem 3.25, that $k \leq dim(V)$. $\square$

Let V be a hyper vector space over F and W be a subhyperspace of V. Define $V/W = \{v + W : v \in V$ with the hyperoperation: $(x + W) \oplus (y + W) = (x + y) + W$, and define $\star : F \times V/W \to V/W$ as follows: $a \star (x + W) = a \cdot x + W$.

Proposition 3.29. *V/W is a hyper vector space over F.*

Lemma 3.30. *Let V be a finite dimensional hyper vector space and W be a subhyperspace of V. Then dimension of V/W = dimension of V - dimension of W.*

Proof. Let $S = \{w_1, \cdots, w_k\}$ be a basis for W and $dim(V) = n$. Then, by Lemma 3.27, S is contained in a basis $B = \{w_1, \cdots, w_k, v_{k+1}, \cdots, v_n\}$ for V. If $x \in V/W$ then there exists $v \in V$ such that $x = v + W$. Having B a basis for V and $v \in V$ implies that there exist $c_1, \cdots, c_n$ such that $x = v + W \in (c_1 w_1 + \cdots + c_k w_k + c_{k+1} v_{k+1} + \cdots + c_n v_n) + W = (c_{k+1} v_{k+1}) + W + \cdots + (c_n v_n + W) = c_{k+1} \star (v_{k+1} + W) + \cdots + c_n \star (v_n + W)$. The latter implies that $\{v_{k+1} + W, \cdots, v_n + W\}$ spans V/W. To prove that $\{v_{k+1} + W, \cdots, v_n + W\}$ is linearly independent, let $\underline{0} + W \in c_{k+1} \star (v_{k+1} + W) + \cdots + c_n \star (v_n + W) = (c_{k+1} v_{k+1} + \cdots + c_n v_n) + W$. Then there exists $z \in W$ with $z \in c_{k+1} v_{k+1} + \cdots + c_n v_n$. Having $z \in W$ implies that there exist $d_1, \cdots, d_k \in F$ such that $z \in d_1 w_1 + \cdots + d_k w_k$. We get now $\underline{0} \in z - z \subseteq d_1 w_1 + \cdots + d_k w_k - c_{k+1} v_{k+1} - \cdots - c_n v_n$. Since B is linearly independent, it follows that $d_1 = \cdots = d_k = c_{k+1} = \cdots = c_n = 0$. $\square$

Next, we present an example on a finite dimensional hyper vector space.

Example 3.31. *Let $E = \{a, b\}$, $K = \{0, 1, 2\}$ and define $(K, +)$ and $(K, \cdot)$ by the following tables:*

+	0	1	2
0	0	1	2
1	1	1	K
2	2	K	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
1	0	2	2

Define $\underline{0}, f, g : E \longrightarrow K$ as follows:

$$\underline{0}(a) = \underline{0}(b) = 0, f(a) = 0, f(b) = 1, g(a) = 1 \text{ and } g(b) = 0.$$

One can easily see the following:

(1) K is a Krasner hyperfield,

(2) $K^E = \{\underline{0}, f, g, 2f, f + g, 2f + g, 2g, f + 2g, 2f + 2g\}$,

(3) $B = \{f, g\}$ is a basis for K^E,

(4) $W_1 = \{\{\underline{0}, f, 2f\}, W_2 = \{\{\underline{0}, g, 2g\}$ are non-trivial subhyperspaces of K^E of dimension 1.

4 Linear transformations

In this section, we define linear transformations between hyper vector spaces and investigate their properties. Moreover, we prove the Dimension theorem for linear transformations over hyper vector spaces.

Throughout this section, F is a Krasner hyperfield, U, V are hyper vector spaces, and we denote by "0" the zero of F, by "$\underline{0}$" the zero of U and by "$\overline{0}$" the zero of V.

Definition 4.1. *Let U, V be two hyper vector spaces over a Krasner hyperfield F and $T : U \to V$. Then T is a linear transformation if for all $x, y \in U$ and $a \in F$: (1) $T(x + y) = T(x) + T(y)$; (2) $T(ax) = aT(x)$.*

Proposition 4.2. *Let U, V be two hyper vector spaces over a Krasner hyperfield F and $T : U \to V$. Then T is a linear transformation if and only if $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in U$ and $a, b \in F$:*

Proof. The proof is straightforward. □

Proposition 4.3. *Let $T : U \to V$ be a linear transformation. Then for all $x \in U$:*

(1) $T(\underline{0}) = T(\overline{0})$;

(2) $T(-x) = -T(x)$.

Proof. Let $x \in U$ and $y \in V$. Then, by Proposition 3.6, $0x = \underline{0}$ and $0y = \overline{0}$.

- Proof of 1. $T(\underline{0}) = T(0x) = 0T(x) = \overline{0}$.

- Proof of 2. Since $\underline{0} \in x - x$, it follows that $T(\underline{0}) \in T(x - x) = T(-x) + T(x)$. Using 1., we get that $\overline{0} \in T(-x) + T(x)$. Thus, $T(-x) = -T(x)$. □

Example 4.4. *Let U, V be two hyper vector spaces over a Krasner hyperfield F. Then the following are linear transformations: (1) (Zero transformation) $T : U \to V$ defined by $T(x) = \overline{0}$ for all $x \in U$; (2) (Inclusion map) $T : U \to V$ defined by $T(x) = x$ for all $x \in U$. Here, U is a subhyperspace of V; (3) $T : U \to V$ defined by $T(x) = kx$ for all $x \in U$ and a fixed $k \in F$. Here, U is a subhyperspace of V.*

Definition 4.5. *Let $T : U \to V$ be a linear transformation. Then*

(1) the kernel of T is defined as: $ker(T) = \{x \in U : T(x) = \overline{0}\}$;

(2) the range of T is defined as: $R(T) = \{T(x) : x \in U\}$.

Example 4.6. *Let $T : V \to V/W$ defined by $T(x) = x + W$. Then T is an onto linear transformation with W as its kernel.*

Let $T : U \to V$ be any transformation. Then, T is *one-to-one* (1-1) if: $T(x) = T(y)$ implies $x = y$; T is *onto* if $R(T) = V$; T is an *isomorphism* if T is linear, one to one and onto.

Proposition 4.7. *Let $T : U \to V$ be a linear transformation. Then T is one to one if and only if $ker(T) = \{\underline{0}\}$.*

Proof. Suppose that T is one to one and let $x \in ker(T)$. Using Proposition 4.3, we get that $T(x) = T(\underline{0}) = \overline{0}$. Having T one to one implies that $x = \underline{0}$.

For the converse, suppose that $ker(T) = \{\underline{0}\}$ and let $x, y \in U$ such that $T(x) = T(y)$. Then $\overline{0} \in T(x) - T(y) = T(x - y)$. The latter implies that there exists $z \in x - y$ such that $T(z) = \overline{0}$. But $ker(T) = \{\underline{0}\}$, then $z = \underline{0}$. Since $\underline{0} \in x - y$, it follows that $x = y$. □

Proposition 4.8. *Let $T : U \to V$ be a linear transformation. Then:*

(1) $ker(T)$ is a subhyperspace of U;

(2) $R(T)$ is a subhyperspace of V.

Proof. Suppose that $a, b \in F$.

- Proof of 1. Let $x, y \in ker(T)$. Then $T(ax + by) = aT(x) + bT(y) = a\bar{0} + b\bar{0} = \bar{0}$. Thus, $ax + by \subseteq ker(T)$.

- Proof of 2. Let $v_1, v_2 \in R(T)$. Then there exist $x, y \in U$ such that $T(x) = v_1, T(y) = v_2$. Since T is a linear transformation, it follows that $av_1 + bv_2 = aT(x) + bT(y) = T(ax + by)$. The latter implies that for every $v \in av_1 + bv_2$, there exist $z \in ax + by$ such that $T(z) = v$. Thus, $av_1 + bv_2 \subseteq R(T)$. $\square$

Proposition 4.9. *Let $T : U \to V$ be a linear transformation and X, W be subhyperspaces of U, V respectively. Then*

(1) $T(X) = \{T(x) : x \in U\}$ *is a subhyperspace of V;*

(2) $T^{-1}(W) = \{x \in U : T(x) \in W\}$ *is a subhyperspace of U.*

Proof. Let $a, b \in F$.

- Proof of 1. Let $T(x), T(y) \in T(X)$ for some $x, y \in X$. Then $aT(x) + bT(y) = T(ax + by)$. Since X is a subhyperspace of U, it follows that $ax + by \subseteq X$. Thus, $aT(x) + bT(y) \subseteq T(X)$.

- Proof of 2. Let $x, y \in T^{-1}(W)$. The there exist $v, w \in W$ such that $T(x) = v, T(y) = w$. Since $av + bw \subseteq W$, it follows that $T(ax + by) = av + bw \subseteq W$. Thus, $ax + by \subseteq T^{-1}(W)$. $\square$

Proposition 4.10. *Let $T : U \to V, J : V \to W$ be linear transformations. Then the composition of J and T, $J \circ T$ is a linear transformation. Moreover, if J, T are one to one then $J \circ T$ is one to one.*

Proof. The proof is straightforward. $\square$

Proposition 4.11. *Let $T : U \to V$ be an isomorphism. Then T^{-1} is an isomorphism.*

Proof. The proof is straightforward. $\square$

Proposition 4.12. *Let $T : U \to V$ be a one to one linear transformation and $S \subseteq U$ be a linearly independent subset of U. Then $T(S)$ is a linearly independent subset of V.*

Proof. Let $S = \{u_1, \cdots, u_n\}$ be a linearly independent subset of U. Suppose that $\bar{0} \in c_1 T(u_1) + \cdots + c_n T(u_n)$. Then $\bar{0} \in T(c_1 u_1 + \cdots + c_n u_n)$. The latter implies that there exist $z \in c_1 u_1 + \cdots + c_n u_n$ such that $T(z) = \bar{0}$. Having T one to one implies that $z = \underline{0}$. We get now that $\underline{0} \in c_1 u_1 + \cdots + c_n u_n$. The independency of S implies that $c_1 = \cdots = c_n = 0$. $\square$

Theorem 4.13. Dimension Theorem. *Let U be a finite dimensional hyper vector space and $T : U \to V$ be a linear transformation. Then*

$$dim(ker(T)) + dim(R(T)) = dim(U).$$

Proof. Let $B = \{u_1, \cdots, u_n\}$ be a basis for U. Since $ker(T)$ is a subhyperspace of U, it follows that $dim(ker(T)) = k \leq n$. We consider the following cases:

Case $k = 0$. Since $ker(T) = \{\underline{0}\}$ and B is a basis for U, it follows by Proposition 4.12 that $T(B) = \{T(u_1), \cdots, T(u_n)\} \subseteq R(T)$ is linearly independent. Let $w \in R(T)$. Then there exists $u \in U$ such that $w = T(u) \subseteq T(c_1 u_1 + \cdots + c_n u_n)$ for some $c_1, \cdots, c_n \in F$. The latter implies that $w \in c_1 T(u_1) + \cdots + c_n T(u_n)$. Thus $T(B)$ is a basis for $R(T)$ and hence, $dim(R(T)) = n$.

Case $k = n$. Since $ker(T)$ has a basis S of n linearly independent elements, it follows that S is a basis for U. Having $T(S) = \bar{0}$ and S a basis for U imply that $T(U) = \bar{0}$. Thus, $dim(R(T)) = 0$.

Case $0 < k < n$. Let $S = \{s_1, \cdots .s_k\}$ be a basis for $ker(T)$. By Lemma 3.27, S is contained in a basis B' for U. Let $B' = \{s_1, \cdots .s_k, w_{k+1}, \cdots, w_n\}$ and $v \in U$. Then there exist $c_1, \cdots, c_n \in F$ such that $v \in c_1 s_1 + \cdots + c_k s_k + c_{k+1} w_{k+1} + \cdots + c_n w_n$. Having T a linear transformation implies that $T(v) \in T(c_1 s_1 + \cdots + c_k s_k + c_{k+1} w_{k+1} + \cdots + c_n w_n) = c_1 \bar{0} + \cdots + c_k \bar{0} + c_{k+1} T(w_{k+1}) + \cdots + c_n T(w_n)$. We get that $T(v) \in c_{k+1} T(w_{k+1}) + \cdots + c_n T(w_n)$. Thus, $\{T(w_{k+1}), \cdots, T(w_n)\}$ spans $R(T)$. To prove that $\{T(w_{k+1}), \cdots, T(w_n)\}$ is linearly independent, let $\bar{0} \in c_1 T(w_{k+1}) + \cdots + c_{n-k} T(w_n)$. Then $\bar{0} \in T(c_1 w_{k+1} + \cdots + c_{n-k} w_n)$. The latter implies that there exists $u \in c_1 w_{k+1} + \cdots + c_{n-k} w_n$ for some $c_1, \cdots, c_{n-k} \in F$ such that $T(u) = \bar{0}$. We get now that $u \in ker(T)$. The latter implies $u \in d_1 s_1 + \cdots + d_k s_k$ for some $d_1, \cdots, d_k \in F$. Since $\underline{0} \in u - u \subseteq d_1 s_1 + \cdots + d_k s_k - c_1 w_{k+1} - \cdots - c_{n-k} w_n$ and having B' a basis for U, it follows that $c_1 = \cdots = c_{n-k} = -d_1 = \cdots = -d_k = 0$. Thus, $dim(R(T)) = n - k$. $\qquad\square$

5 Conclusion

In this paper, we dealt with hyper vector spaces over Krasner hyperfields and linear transformations over them. We studied their properties, presented some results that are similar to that of vector spaces.

For future work, it will be interesting to introduce a norm over hyper vector spaces and study normed hyper vector spaces.

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