Journal of Algebraic Hyperstructures and Logical Algebras
Special Issue (JAHA2020) Dedicated to Professor Piergiulio Corsini
Volume 1, Number 3, (2020), pp. 15-29

A brief survey on algebraic hyperstructures: Theory and applications

B. Davvaz[1]

[1] *Department of Mathematics, Yazd University, Yazd, Iran*

davvaz@yazd.ac.ir

Abstract

I am working on algebraic hyperstructures from 1995. During the last twenty years, I together with my students and co-authors studied and developed the theory of algebraic hyperstructures in many directions. In particular, we tried to find real examples of hyperstructures in nature. In this paper we review some parts of these works such as (1) Fundamental relations on hyperstructures; (2) Fuzzy sets and hyperstructures; (3) Rough sets and hyperstructures; (4) Topology and hyperstructures; (5) Number theory and hyperstructures; (6) n-ary hypergroups and there extension to hyperrings and hypermodules; (7) Applications of hyperstructures in biology, physics and chemistry.

Article Information

Corresponding Author:
B. Davvaz;
Received: July 2020;
Accepted: August 2020.
Paper type: Overview.

Keywords:

Hyperstructures, fundamental relations, fuzzy sets, rough sets, n-ary hypergroups.

1 Fundamental relations on hyperstructures

The main tools connecting the class of hyperstructures with the classical algebraic structures are the fundamental relations. The fundamental relation has an important role in the study of algebraic hyperstructures and especially of hypergroups. For all $n > 1$, the relation β_n on a semihypergroup $(H, \circ)$ is defined as follows: $x \beta_n y$ if there exists $a_1, \ldots, a_n$ in H such that $\{x, y\} \subseteq \prod_{i=1}^{n} a_i$, and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. This relation was introduced by Koskas [53] and studied mainly by Corsini [19], Davvaz [23], Davvaz and Leoreanu-Fotea [36], Freni [44], Vougiouklis [71], and many others. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β. If $(H, \circ)$ is a (semi)hypergroup, then the relation β^* is the smallest equivalence relation on H such that the quotient H/β^* is a (semi)group. Freni in [44] proved that if $(H, \circ)$ is a hypergroup, then the relation β is an equivalence relation

https://doi.org/10.29252/HATEF.JAHLA.1.3.2

on H. It is a natural question that how we can change the definition of β to obtain an abelian group, a cyclic group, a solvable group or a nilpotent group. In order to see the answers of these question we refer the readers to [4, 5, 45, 52, 62].

Several kinds of hyperrings are introduced and analyzed in [36] such as Krasner hyperrings, multiplicative hyperrings, general hyperrings. A multivalued system $(R, +, \cdot)$ is a (general) hyperring if (1) $(R, +)$ is a hypergroup; (2) $(R, \cdot)$ is a semihypergroup; (3) $(\cdot)$ is (strong) distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. The above definition contains the class of multiplicative hyperrings and additive hyperrings as well. In a hyperring, Vougiouklis introduced the equivalence relation γ^*, which is similar to the relation β^*. Let $(R, +, \cdot)$ be a hyperring. The relation γ is defined as follows: $a\gamma b$ if and only if $\{a, b\} \subseteq u$, where u is a finite sum of finite products of elements of R. As usual, we denote the transitive closure of γ by γ^*. Let $(R, +, \cdot)$ be a hyperring. Then, the relation γ^* is the smallest equivalence relation in R such that the quotient R/γ^* is a ring. The structure R/γ^* is called the fundamental ring [70]. The commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. In [41], Davvaz and Vougiouklis defined a new fundamental relation to obtain an ordinary commutative ring from a hyperring. They introduced the following definition. If R is a hyperring, then we set $\alpha_0 = \{(x, x) \mid x \in R\}$ and, for every integer $n \geq 1$, α_n is the relation defined as follows:

$x\alpha_n y \;\Leftrightarrow\; \exists(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n,\ \exists\sigma \in \mathbb{S}_n$ and $[\exists(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i},\ \exists\sigma_i \in \mathbb{S}_{k_i},$ $(i = 1, \ldots, n)]$ such that

$$x \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \text{ and } y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. Obviously, for every $n \geq 1$, the relation α_n is symmetric, and the relation $\alpha = \bigcup_{n\geq 0} \alpha_n$ is reflexive and symmetric. If α^* is the transitive closure of α, then the quotient R/α^* is a commutative ring [41]. Then, this relation is investigated in [59]. Also, a similar relation is defined on hypermodules to obtain an ordinary module [13, 14, 58]. The largest class of hyperstructures called H_v-structures. These structures introduced by Vougiouklis in 1990 in the 4th AHA congress held in Greece. In H_v-groups, H_v-rings and H_v-modules, the fundamental relations are defined and they connect the algebraic hyperstructure theory with the classical one. There is a rich monograph about H_v-structures that published by Davvaz and Vougiouklis in 2019 [42].

2 Fuzzy sets and hyperstructures

In 1971, Rosenfeld introduced the fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. There is a considerable amount of work on the association between fuzzy sets and hyperstructures. This work can be classified into three groups. A first group of works studies crisp hyperoperations defined through fuzzy sets. This study was initiated by Corsini and others. A second group of works concerns the fuzzy hyperalgebras. This is a direct extension of the concept of fuzzy algebras. This idea was applied by Zahedi and his group on polygroups. A third group deals also with fuzzy hyperstructures, but with a completely different approach. This was studied by Corsini, Zahedi and others. The basic idea is the following one: a crisp hyperoperation assigns to every pair of elements a crisp set; a fuzzy hyperoperation assigns to every pair of elements a fuzzy set. In 1999, Davvaz introduced the notion of fuzzy subhypergroup (H_v-subgroup, resp.) of a hypergroup (H_v-group, resp.) [20]. Let $(H, \cdot)$ be a hypergroup (H_v-group) and let μ be a fuzzy subset of H. Then, μ is said to be a fuzzy subhypergroup (H_v-subgroup, resp.) of H if the following axioms hold: (1) $\min\{\mu(x),\mu(y)\} \leq \inf_{\alpha\in x\cdot y}\{\mu(\alpha)\}$, for all $x, y \in H$; (2) For all $x, a \in H$ there exists $y \in H$ such that $x \in a\cdot y$ and $\min\{\mu(a),\mu(x)\} \leq \mu(y)$; (3) For all $x, a \in H$ there exists $z \in H$ such that $x \in z\cdot a$ and $\min\{\mu(a),\mu(x)\} \leq \mu(z)$. Then, he gave the relation between a fuzzy subhypergroup and level subhypergroup. This relation is expressed in terms of a necessary and sufficient condition. After this definitions many authors developed fuzzy hyperstructures based on this view point. We refer the readers to [27], for a deep study on fuzzy hyperstructures.

3 Rough sets and hyperstructures

The concept of rough set was originally proposed by Pawlak in [63]. Since then the subject has been investigated in many papers. Some authors studied algebraic properties of rough sets. Let U be a universe of objects and ρ be an equivalence relation on U. Given an arbitrary set $A \subseteq U$, a concept in U, it may be impossible to describe A precisely using the equivalence classes of ρ. That is, the available information is not sufficient to give a precise representation of A. In this case, one may characterize A by a pair of lower and upper approximations $\underline{app}(A) := \bigcup_{[a]_\rho\subseteq A}[a]_\rho$ and $\overline{app}(A) := \bigcup_{[a]_\rho\cap A\neq\emptyset}[a]_\rho$, where $[a]_\rho = \{b \mid a\rho b\}$ is the equivalence class containing a. The lower approximation $\underline{app}(A)$ is the union of all the elementary sets which are subsets of A. The upper approximation $\overline{app}(A)$ is the union of all the elementary sets which have a non-empty intersection with A. An element in the lower approximation necessarily belongs to A, while an element in the upper approximation possibly belong to A. We can express lower and upper approximations as follows: $\underline{app}(A) = \{a \in U \mid [a]_\rho \subseteq A\}$ and $\overline{app}(A) = \{a \in U \mid [a]_\rho \cap A \neq \emptyset\}$. If $X \subseteq U$ is given by a predicate P and $x \in U$, then (1) $x \in \underline{app}(X)$ means that x certainly has property P; (2) $x \in \overline{app}(X)$ means that x possibly has property P; (3) $x \in U\backslash\overline{app}(X)$ means that x definitely does not have property P.

In [21], the author applied the concept of rough sets to algebraic hyperstructures. Let R be a hyperring (H_v-ring, resp.). For a subset $A \subseteq R$ we define two approximations of A relative to the fundamental relation γ^* as follows: $\underline{\gamma^*}(A) = \{x \in R \mid \gamma^*(x) \subseteq A\}$ and $\overline{\gamma^*}(A) = \{x \in R \mid \gamma^*(x) \cap A \neq \emptyset\}$. The set $\underline{\gamma^*}(A)$ is called the γ^*-lower approximation of A, and the set $\overline{\gamma^*}(A)$ is called the γ^*-upper approximation of A. It is easy to see that (1) $\underline{\gamma^*}(A) \subseteq A \subseteq \overline{\gamma^*}(A)$, (2) $\underline{\gamma^*}(\underline{\gamma^*}(A)) = \underline{\gamma^*}(A)$ and $\overline{\gamma^*}(\overline{\gamma^*}(A)) = \overline{\gamma^*}(A)$. The difference $\widehat{\gamma^*(A)} = \overline{\gamma^*}(A) - \underline{\gamma^*}(A)$ is called the γ^*-boundary region of A. In the case when $\widehat{\gamma^*(A)} = \emptyset$ the set A is said to be γ^*-exact; otherwise A is γ^*-rough. If A and B are non-empty subsets of R, then (1) $\overline{\gamma^*}(A) + \overline{\gamma^*}(B) \subseteq \overline{\gamma^*}(A + B)$; (2) $\overline{\gamma^*}(A)\cdot\overline{\gamma^*}(B)) \subseteq \overline{\gamma^*}(A\cdot B)$. The lower and upper approximations can be presented in an equivalent form as shown below. Let A be a non-empty subsets of R. Then $\underline{\gamma^*(A)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \subseteq A$ and $\overline{\gamma^*(A)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \cap A \neq \emptyset\}$. Now, we consider these sets as subsets of the fundamental ring R/γ^* of an H_v-ring R, and we recall some results from [21]. Let A and B are non-empty subsets of R, then the following hold:

(1) $\overline{\gamma^*(A\cup B)} = \overline{\gamma^*(A)} \cup \overline{\gamma^*(B)}$; (4) $A \subseteq B$ implies $\underline{\gamma^*(A)} \subseteq \underline{\gamma^*(B)}$;

(2) $\underline{\gamma^*(A\cap B)} = \underline{\gamma^*(A)} \cap \underline{\gamma^*(B)}$; (5) $\underline{\gamma^*(A)} \cup \underline{\gamma^*(B)} \subseteq \underline{\gamma^*(A\cup B)}$;

(3) $A \subseteq B$ implies $\overline{\gamma^*(A)} \subseteq \overline{\gamma^*(B)}$; (6) $\overline{\gamma^*(A\cap B)} \subseteq \overline{\gamma^*(A)} \cap \overline{\gamma^*(B)}$.

If A is an H_v-subgroup of $(R,+)$, then $\overline{\gamma^*(A)}$ is a subgroup of $(R/\gamma^*,\oplus)$. If A and B are non-empty subsets of R, then $\overline{\gamma^*(A)} \oplus \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(A+B)}$.

If A is a non-empty subset of R and B is an H_v-ideal of R, then

$$\overline{\gamma^*(A)} \odot \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(B)}.$$

If A and B are H_v-ideals of R, then $\overline{\gamma^*(A)} \odot \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(A)} \cap \overline{\gamma^*(B)}$.

If A is an H_v-ideal of R, then $\overline{\gamma^*(A)}$ is an ideal of R/γ^*. Let A, B and C be H_v-ideals of R. The sequence of strong homomorphisms $A \overset{f}{\to} B \overset{g}{\to} C$ is said to be *exact* if $g \circ f(x) \in \omega_R$, for all $x \in A$. Let $A \overset{f}{\to} B \overset{g}{\to} C$ be an exact sequence of H_v-ideals of R. Then the sequence

$$\overline{\gamma^*(A)} \overset{F}{\to} \overline{\gamma^*(B)} \overset{G}{\to} \overline{\gamma^*(C)}$$

is an exact sequence of ideals of R/γ^*, where $F(\gamma^*(a)) = \gamma^*(f(a))$ and $G(\gamma^*(b)) = \gamma^*(g(b))$, for all $a \in A$ and $b \in B$. For more study about this subject we refer to [22, 25, 68, 69].

4 Topology and hyperstructures

In [12], Ameri studied the concept of a (pseudo, strong pseudo) topological hypergroup and then he gave some related results. In [47], Heidari et al. introduce the concept of topological hypergroups as a generalization of topological groups. Let (H, τ) be a topological space. In order to construct a topological hypergroup we need a topology on $\mathcal{P}^*(H)$. Let (H, τ) be a topological space. Then, the family $\mathcal{U}$ consisting of all sets $S_V = \{U \in \mathcal{P}^*(H) \mid U \subseteq V, U \in \tau\}$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^* [49]. Let (H, τ) be a topological space. Then, we consider the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$. Let $(H, \circ)$ be a hypergroup and (H, τ) be a topological space. Then, the system $(H, \circ, \tau)$ is called a topological hypergroup [47] if (1) the mapping $(x, y) \mapsto x \circ y$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous; (2) the mapping $(x, y) \mapsto x/y$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous, where $x/y = \{z \in H \mid x \in z \circ y\}$; (3) the mapping $(x, y) \mapsto y\backslash x$, from $H \times H$ to $\mathcal{P}^*(H)$ is continuous, where $y/x = \{z \in H \mid x \in y \circ z\}$. Now, we recall some results from [47]. Let $(H, \circ)$ be a hypergroup and τ be a topology on H. Then, the following assertions hold: (1) The mapping $(x, y) \mapsto x \circ y$ is continuous if and only if for every $x, y \in H$ and $U \in \tau$ such that $x \circ y \subseteq U$, there exist $V, W \in \tau$ such that $x \in V$, $y \in W$ and $V \circ W \subseteq U$; (2) The mapping $(x, y) \mapsto x/y$ is continuous if and only if for every $x, y \in H$ and $U \in \tau$ such that $x/y \subseteq U$, there exist $V, W \in \tau$ such that $x \in V$, $y \in W$ and $V/W \subseteq U$. Evidently, every topological group is a topological hypergroup. Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by open subsets if there exist open subsets U and V of X containing x and y, respectively, such that U and V are disjoint. A Hausdorff space is a topological space in which points can be separated by open subsets. Note that some properties in topological groups do not hold in topological hypergroups. For instance, if G is a topological group and U is an open subset of G, then aU is open in G for all $a \in G$. Let X be a topological space and $\sim$ be an equivalence relation on X. For every $x \in X$, denote by $[x]$ its equivalence class. The quotient space of X modulo $\sim$ is given by the set $X/\sim = \{[x] \mid x \in X\}$. We have the projection map $p : X \to X/\sim$, $x \mapsto [x]$ and we equip $X/\sim$ by the topology: $U \subseteq X/\sim$ is open if and only if $p^{-1}(U)$ is an open subset of X. Let A be a subset of the topological space X and $\sim$ be an equivalence relation on X. Then, the *saturation* of A with respect to $\sim$ is the set $\widehat{A} = \{x \in X \mid \exists a \in A, x \sim a\}$. If $\widehat{A} = A$, then A is called saturated. Let $(H, \circ, \tau)$ be a topological hypergroup such that every open subset of H is a complete part. Then, $(H/\beta^*, \otimes, \overline{\tau})$ is a topological group. Let $(G, \cdot)$ be a topological group and H be a non-normal subgroup of it. Let β^* be the fundamental relation of the hypergroup $(G/H, \circ)$. Then, there exists a normal subgroup N of G such that the topological groups $(G/H)/\beta^*$ and G/N are topological isomorphic.

Then, in [48], Heidari et al. introduced the concept of topological polygroups. By considering the relative topology on subpolygroups they proved some properties of them. Also, the topological isomorphism theorems of topological polygroups are proved. Salehi Shadkami et all in [65] presented some facts about complete parts in polygroups and they used these facts to obtain some new results in topological polygroups. They defined the concept of cp-resolvable topological polygroups. A non-empty subset X of a topological polygroup is called cp-resolvable if there exist disjoint dense subsets A and B such that at least one of them is a complete part. Then, they investigated a few properties of cp-resolvable topological polygroups. Also, in [66], they established various relations between complete parts and open sets. They studied the properties of big subsets in a topological polygroup. Al Tahan et al. [11], showed that there is no relation (in general) between pseudotopolgical and strongly pseudotopolgical hypergroupoids. In particular, they presented a topological hypergroupoid that does not depend on the pseodocontinuity nor on strongly pseodocontinuity of the hyperoperation. To study fuzzy topological hypergroups, we refer to [1, 2, 3, 26].

5 Number theory and hyperstructures

In [17], Asghari and Davvaz introduced a hyperoperation associated to the set of all arithmetic functions and analyzed the properties of this hyperoperation. In [6], Al Tahan and Davvaz defined a new hyperoperation associated to the set G of all arithmetic functions. Here, we review some definitions and results from [6]. An arithmetic function is a function in which its domain of definition is the set of natural numbers and its codomain is the set of complex numbers. An arithmetic function f is said to be additive if whenever m and n are coprime, $f(mn) = f(m)+f(n)$. An arithmetic function f is said to be multiplicative if whenever m and n are coprime, $f(mn) = f(m)f(n)$. If f is an additive function and g is a multiplicative function then $f(1) = 0$ and $g(1) = 1$. Denote by $AF(G)$ the set of all additive functions of G and by $\wp^*(G)$ the set of all non empty subsets of G. Now, we define a hyperoperation $*$ on G. Define a hyperoperation on G as follows: $* : G \times G \to \wp^*(G)$, $(\alpha, \beta) \mapsto \alpha * \beta$ such that

$$(\alpha * \beta)(n) = \Big\{\alpha(d) + \beta(\frac{n}{d}) : d \mid n\Big\} = \bigcup_{d|n} \alpha(d) + \beta(\frac{n}{d}).$$

Let α and β be two elements in G. If $\alpha(n) = \beta(n)$ for all natural numbers n, then $\alpha = \beta$. We observe that $(G, *)$ is a commutative hypergroup and $(AF(G), *)$ is a normal subhypergroup of $(G, *)$. Let G be the set of all arithmetic functions. Define a map '$\star$' on $G * G$ as follows: $\star : (G * G) \times (G * G) \to \wp^*(G)$, $((\alpha_1 * \beta_1), (\alpha_2 * \beta_2)) \mapsto (\alpha_1 * \beta_1) \star (\alpha_2 * \beta_2)$ such that for all natural numbers m and n

$$((\alpha_1 * \beta_1) \star (\alpha_2 * \beta_2))(m, n) = \bigcup_{\alpha \in (\alpha_1 * \beta_1)(m), \beta \in (\alpha_2 * \beta_2)(n)} \alpha + \beta.$$

Let G be the set of all arithmetic functions and m, n be natural numbers. Then

1. $((\alpha_1 * \beta_1) \star (\alpha_2 * \beta_2))(m, n) = ((\alpha_2 * \beta_2) \star (\alpha_1 * \beta_1))(n, m)$ for all α_1, α_2, β_1 and $\beta_2 \in G$.
2. $(G * G, \star)$ is associative.

Let α and $\beta \in G$. Then $\alpha * \beta$ is a multiplicative function in $G * G$ if for all coprime natural numbers m and n the following condition holds: $(\alpha * \beta)(mn) = (\alpha * \beta)(m) \star (\alpha * \beta)(n)$. We denote by $AF(G*G)$ the set of all additive functions in $G*G$. Let α, $\beta \in G$ and m, n be two natural numbers. Then, we have

$$\bigcup_{d|m, D|n} \alpha(d) + \beta(D) = \bigcup_{d|m} \alpha(d) \star \bigcup_{D|n} \beta(D).$$

If α and $\beta \in AF(G)$ then $\alpha * \beta \in AF(G*G)$. We define a hyperstructure on G as follows: $\circ : G \times G \to \wp^*(G)$, with $(\alpha, \beta) \mapsto \alpha \circ \beta$ such that $(\alpha * \beta)(n) = \bigcup_{d|n} \alpha(d)\beta(\frac{n}{d})$. If α is a multiplicative function that admits an inverse in $(G, \circ)$, then $\alpha = \imath$. If $\alpha \in (G, \circ)$ such that $\alpha^{-1} \in G$, then $\alpha(1) \neq 0$ and $\alpha^{-1}(1) = \frac{1}{\alpha(1)}$. If $\alpha \in (G, \circ)$ such that $\alpha(1) = a \neq 0$ and $\alpha^{-1} \in G$, then

$$\alpha(n) = \begin{cases} a, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha \in (G, \circ)$ with $\alpha(1) = a \neq 0$, then $\alpha^{-1} \in G$ if and only if

$$\alpha(n) = \begin{cases} a, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We define $O_* : \mathbb{N} \to \mathbb{C}$ as $O_*(n) = 0$ for all $n \in \mathbb{N}$. Note that $O_* \in AF(G)$. If $\alpha \in G$ then $\alpha \in \alpha * O_*$. An element λ is said to be identity in $(G, *)$ if for all natural numbers n, $\alpha * \lambda(n) = \bigcup_{d|n} \alpha(d)$. An element α^{-1} is said to be an inverse of α in $(G, *)$ if $\alpha * \alpha^{-1} = O_*$. We see that $(G, *)$ has unique identity. If α is an element in $(G, *)$ that admits an inverse α^{-1} in $(G, *)$, then α^{-1} is unique. If $\alpha \in AF(G)$ such that its inverse α^{-1} exists, then $\alpha^{-1} \in AF(G)$.

If $\alpha \in (G, *)$, then $\alpha^{-1} \in G$ if and only if α is a constant function. A set W associated to the the hyperoperations $+$ and $\cdot$ is said to be weak distributive if $x \cdot (y+z) \cap x \cdot y + x \cdot z \neq \emptyset$ and $(x+y) \cdot z \cap x \cdot z + y \cdot z \neq \emptyset$ whenever x, y and z are in W. A set W associated to the the hyperoperations $+$ and $\cdot$ is said to be weak hyperring if the following conditions are satisfied: $(W, +)$ is a hypergroup; $(W, \cdot)$ is a semihypergroup; $(W, \cdot)$ is weak distributive. We observe that $(G, +, \cdot)$ is a weak hyperring. Denote by $(M, +, \cdot)$ the set of all constant arithmetic functions under the hyperoperations of G and by $(N, +, \cdot)$ the largest distributive set contained in $(G, +, \cdot)$. If M is the set of all constant arithmetic functions in G, then $(M, +, \cdot)$ is a Krasner hyperring. Moreover, $(M, +)$ is a join space with scalar identity. If $\alpha \in N$ with $\alpha(1) \neq 0$, then $\alpha(n) = \alpha(1)$, for all $n \in \mathbb{N}$. If $\alpha \in N$ with $\alpha(1) = 0$, then $\alpha(n) = 0$, for all $n \in \mathbb{N}$. If $\alpha \in N$, then α is a constant function. The largest hyperring contained in $(G, +, \cdot)$ is $(M, +, \cdot)$. Finally, there is no hyperfield contained in $(G, +, \cdot)$. Also, in [7], Al Tahan and Davvaz determined fundamental groups and fundamental rings of hyperstructures of arithmetic functions. In addition, they investigated their complete parts and strongly regular relations.

6 n-ary hypergroups and there extension to hyperrings and hypermodules

The notion of an n-ary group was introduced by Dornte which is a natural generalization of the notion of a group. n-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Since then many papers concerning various n-ary algebra have appeared in the literature. In [40], Davvaz and Vougiouklis introduced the notion of n-ary hypergroups. Let H be a non-empty set and f be a mapping $f : H \times H \to \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then f is called a *binary hyperoperation* on H. We denote by H^n the cartesian product $H \times \ldots \times H$ where H appears n times.

An element of H^n will be denoted by $(x_1, \ldots, x_n)$ where $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $f : H^n \to \wp^*(H)$ is called an n-ary hyperoperation. Let f be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ subsets of H. We define $f(A_1, \ldots, A_n) = \bigcup\{f(x_1, \ldots, x_n) | \, x_i \in A_i, i = 1, \ldots, n\}$. We use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. In this convention $f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$ is written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. A non-empty set H with an n-ary hyperoperation $f : H^n \to \wp^*(H)$ is called an n-ary hypergroupoid and is denoted by (H, f). An n-ary hypergroupoid (H, f) will be called an n-ary semihypergroup if and only if the following associative axiom holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$. If for all $(a_1, a_2, \ldots, a_n) \in H^n$, the set $f(a_1, a_2, \ldots, a_n)$ is singleton, then f is called an n-ary operation and (H, f) is called an n-ary groupoid (resp. n-ary semigroup). If $m = k(n-1)+1$, then the m-ary hyperoperation g given by

$$g(x_1^{k(n-1)+1}) = \underbrace{f(f(\ldots, f(f}_{k}$$

$(\mathrm{x}_1^n), x_{n+1}^{2n-1}), \ldots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$is denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$, to mean $f_{(k)}$ for some $k = 1, 2, \ldots$. An n-ary semihypergroup (H, f) in which the equation

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \tag{1}$$

has solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \leq i \leq n$, is called an n-ary hypergroup. If f is n-ary operation then (1) is as follows: $b = f(a_1^{i-1}, x_i, a_{i+1}^n)$. In this case (H, f) is an n-ary group. The important question is the solvability of (1). Let (H, f) be an n-ary semihypergroup. Then (H, f) is an n-ary hypergroup if and only if (1) is solvable at the place $i = 1$ and $i = n$ or at least one place $1 < i < n$. The reproduction axiom can be formulated for n-ary hypergroups as follows: $f(H^{i-1}, x, H^{n-i}) = H$, for all $x \in H$ and $i = 1, \ldots, n$. Let (H, f) be an n-ary hypergroup, $a_2^{n-1} \in H$ be fixed and let $x \odot y = f(x, a_2^{n-1}, y)$. Then the hypergroupoid $(H, \odot)$ is a hypergroup and it is called a retract of the n-ary hypergroup (H, f). Let (H, f) be an n-ary hypergroup. If the value of $f(x_1, x_2, \ldots, x_n)$ is independent on the permutation of elements $x_1, x_2, \ldots, x_n$, then (H, f) is called a commutative n-ary hypergroup. The element $a \in H$ is called a scalar if $|f(x_1^i, a, x_{i+2}^n)| = 1$, for all $x_1, \ldots, x_i, x_{i+2}, \ldots, x_n \in H$. Element e of an n-ary hypergroup (H, f) is called neutral (identity) element if $f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})$ includes x, for all $x \in H$ and all $1 \leq i \leq n$. If (H, f) is a commutative n-ary hypergroup and $a \in H$ is a scalar element such that $f(a, e, \ldots, e) = a$ for some $e \in H$, then e is a neutral element. If the set of all neutral elements of a given commutative n-ary hypergroup is non-empty, then it is an n-ary group. Let (H, f) be an n-ary hypergroup and B be a non-empty subset of H. Then B is an n-ary subhypergroup of H if the following conditions hold: (1) B is closed under the n-ary hyperoperation f, i.e., for every $(x_1, \ldots, x_n) \in B^n$ implies that $f(x_1, \ldots, x_n) \subseteq B$; (2) Equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has the solution $x_i \in B$ for every $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n, b \in B$ and $1 \leq i \leq n$.

Let (H, f) be an n-ary hypergroup. An equivalence relation θ on H is called compatible if $a_1\theta b_1, \ldots, a_n\theta b_n$, then for all $a \in f(a_1, \ldots, a_n)$ there exists $b \in f(b_1, \ldots, b_n)$ such that $a\theta b$. An equivalence relation θ is called strongly compatible if $a_1\theta b_1, \ldots, a_n\theta b_n$ implies that $a\theta b$ for all

$a \in f(a_1, \dots, a_n)$ and $b \in f(b_1, \dots, b_n)$. If (H, f) is an n-ary hypergroup and θ a compatible relation on H, then $(H/\theta, f|_\theta)$ is an n-ary hypergroup where

$$f|_\theta(\theta(a_1),\ \dots,\ \theta(a_n)) = \{\theta(a) \mid a \in f(a_1, \dots, a_n)\}.$$

The natural map $\pi : H \to H/\theta$ where $\pi(x) = \theta(x)$ is an onto homomorphism. Let (A, f) and (B, g) be two n-ary hypergroups and let $\varphi : A \to B$ be a homomorphism. Then the kernel of φ, written $ker\varphi$, is defined by $ker\varphi = \{(a, b) \in A^2 \mid \varphi(a) = \varphi(b)\}$. It is easy to see that $ker\varphi$ is a compatible relation. Let (A, f) and (B, g) be two n-ary hypergroups and let $\varphi : A \to B$ be a homomorphism. Then there exists a compatible relation θ on A and a monomorphism $\psi : A/\theta \to B$ such that $\psi \circ \pi = \varphi$. If ρ and θ are compatible relations on an n-ary hypergroup (H, f) such that $\rho \subseteq \theta$, then there exists a compatible relation μ on $(H/\rho, f/\rho)$ such that $(H/\rho)/\mu$ is isomorphic to H/θ. The diagonal relation $\triangle$ on H is the set $\{(a, a) \mid a \in H\}$ and the full relation H^2 is denoted by ∇. The set of all equivalence relations on a set H, with $\subseteq$ as the partial ordering, is a complete lattice. Let θ_1 and θ_2 are two equivalence relations on H. It is clear that $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$. Also, we have $\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \dots$. Let (A_1, f_1) and (A_2, f_2) be two n-ary hypergroups. Define the direct hyperproduct $(A_1 \times A_2, f_1 \times f_2)$ to be the n-ary hypergroup whose universe is the set $A_1 \times A_2$ and such that for $a_i \in A_1$, $a'_i \in A_2$, $1 \leq i \leq n$, $(f_1 \times f_2)((a_1, a'_1), \dots, (a_n, a'_n)) = \{(a, a') \mid a \in f_1(a_1, \dots, a_n),\ a' \in f_2(a'_1, \dots, a'_n)\}$. The mapping $\pi_i : A_1 \times A_2 \to A_i$, $i = 1, 2$, defined by $\pi_i((a_1, a_2)) = a_i$, is called the *projection map* on the ith coordinate of $A_1 \times A_2$. For $i = 1, 2$, the mapping $\pi_i : A_1 \times A_2 \to A_i$ is an onto homomorphism. Furthermore, we have (1) $ker\pi_1 \cap ker\pi_2 = \triangle$; (2) $ker\pi_1$ and $ker\pi_2$ permute; (3) $ker\pi_1 \wedge ker\pi_2 = \nabla$, where $ker\pi_i = \{((a_1, a_2), (b_1, b_2)) \mid \pi_i(a_1, a_2) = \pi_i(b_1, b_2)\}$, $(i = 1, 2)$. Note that $((a_1, a_2), (b_1, b_2)) \in ker\pi_i \Leftrightarrow \pi_i((a_1, a_2)) = \pi_i((b_1, b_2)) \Leftrightarrow a_i = b_i$. Thus $ker\pi_1 \cap ker\pi_2 = \triangle$. Also, if (a_1, a_2), (b_1, b_2) are any two elements of $A_1 \times A_2$, then $(a_1, a_2)\ ker\pi_1\ (a_1, b_2)$ and $(a_1, b_2)\ ker\pi_2\ (b_1, b_2)$, and so $\nabla = ker\pi_1 \circ ker\pi_2$. But, then $ker\pi_1$ and $\ker \pi_2$ permute, and their join is ∇. Let (H, f) be an n-ary hypergroup. A compatible relation θ on H is a *factor compatible relation* if there is a compatible relation θ^* on H such that $\theta \cap \theta^* = \triangle$, $\theta \wedge \theta^* = \nabla$ and θ permutes with θ^*. The pair θ, θ^* is called a pair of factor compatible relations on H. If θ, θ^* is a pair of factor compatible relations on H, then $H \cong H/\theta \times H/\theta^*$ under the map $\psi(a) = (\theta(a), \theta^*(a))$. If (H, f) is an n-ary hypergroup, then $\widehat{\beta}$ denotes the transitive closure of the relation $\beta = \bigcup_{k \geq 1} \beta_k$, where β_1 is the diagonal relation, i.e., $\beta_1 = \{(x, x) |\ x \in H\}$ and for every integer $k > 1$, β_k is the relation defined as follows: $x\beta_k y$ if and only if $\{x, y\} \subseteq f_{(\cdot)}$, where $f_{(\cdot)}$ means that $f_{(k)}$ for some $k = 1, 2, \dots$. When $x\beta_1 y$ (i.e., $x = y$) then we write $\{x, y\} \subseteq f_{(0)}$, we define β^* as the smallest equivalence relation such that the quotient $(H/\beta^*, f/\beta^*)$ is an n-ary group, where H/β^* is the set of all equivalence classes. The fundamental relation β^* is the transitive closure of the relation β, i.e., $(\beta^* = \widehat{\beta})$. For more details about n-ary hypergroups, we refer the reader to [35, 34, 54, 55, 56, 57, 64].

Mirvakili and davvaz in [60, 61] introduced the concept of (m, n)-hyperrings. Here, we present some definitions and results. An (m, n)-*hyperring* is an algebraic hyperstructure (R, f, g), which satisfies the following axioms:

(1) (R, f) is an m-ary hypergroup,

(2) (R, g) is an n-ary hypersemigroup,

(3) the n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R,\ 1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

(R, f, g) is called an n-ary hyperring if $n = m$. If (R, f) is an m-ary semihypergroup, then (R, f, g) is called an (m, n)-semihyperring. In (m, n)-hyperring (R, f, g), if f is an m-ary operation then (R, f, g) is called an (m, n)-multiplicative hyperring and if g be an n-ary operation then (R, f, g) is called an additive (m, n)-hyperring. A multiplicative and additive (m, n)-hyperring is called an (m, n)-ring. A non-empty subset $S \subseteq R$ is called an (m, n)-subhyperring if (S, f, g) is an (m, n)-hyperring. Let $i \in \{1, \ldots, n\}$. A i-hyperideal I of R is an (m, n)-subhyperring of R such that for every $r_1^n \in R$, $g(r_1^{i-1}, I, r_{i+1}^n) \subseteq I$. If I is an i-hyperideal and for every $r_1^n \in R$, $g(r_1^{i-1}, I, r_{i+1}^n) = I$, then I called a strong i-hyperideal. A non-empty subset I of R is called (a) an (strong) (m, n)-hyperideal if I is (a) an (strong) i-hyperideal of R for every $i \in \{1, \ldots, n\}$. For any (m, n)-hyperring (R, f, g) and $I \subseteq R$ the following conditions are equivalent: (1) I is a strong (m, n)-hyperideal of R; (2) I is a strong i-hyperideal of R for $i = 1$ and $i = n$; (3) I is a strong i-hyperideal of R for some $1 < i < n$. An element o is called a (scalar) zero of (R, f, g) if it is a (scalar) identity of (R, f) and for every $x_2^n \in R$ we have $(o = f(o, x_2^n) = f(x_2, o, x_3^n) = \ldots = f(x_2^n, o))$ $\quad o \in f(o, x_2^n) \cap f(x_2, o, x_3^n) \cap \ldots \cap f(x_2^n, o)$. For any (m, n)-semihyperring (R, f, g) and $o \in R$ the following conditions are equivalent: (1) o is a scalar zero of R; (2) o is a scalar i-zero for some $1 < i < n$, i.e, for every $x_1^n \in R$, $g(x_1^{i-1}, o, x_{i+1}^n) = o$; (3) o is a scalar i-zero for $i = 1$ and $i = n$, i.e, $g(o, x_2^n) = o = g(x_1^{n-1}, o)$, for all $x_1^n \in R$., Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two (m, n)-hyperrings. A homomorphism from R_1 to R_2 is a mapping $\phi : R_1 \to R_2$ such that $\phi(f_1(a_1^m)) = f_2(\phi(a_1), \ldots, \phi(a_m))$ and $\phi(g_1(b_1^n)) = g_2(\phi(b_1), \ldots, \phi(b_n))$ hold, for all $a_1^n, b_1^m \in R_1$. If ϕ is injective, then is called embedding. The map ϕ is an isomorphism if ϕ is injective and onto. We say that R_1 is isomorphic to R_2, denote $R_1 \cong R_2$, if there is an isomorphism from R_1 to R_2. Let $\phi : R_1 \to R_2$ be a homomorphism and S_1 be an (m, n)-subhyperring of R_1 and S_2 be an (m, n)-subhyperring of R_2, then $\phi(S_1)$ is an (m, n)-subhyperring of R_2 and if $\phi^{-1}(S_2)$ is non-empty, then $\phi^{-1}(S_2)$ is an (m, n)-subhyperring of R_1. Let $\phi : R_1 \to R_2$ be a homomorphism, then the kernel ϕ, is defined by $\ker \phi = \{(a, b) \in R_1 \times R_1 \mid \phi(a) = \phi(b)\}$. If $b, c \in R$ then we say that an (m, n)-hyperringoid (R, f, g) is a (b, c)-derived from a hyperringoid $(R, +, \cdot)$ and denote this fact by $(R, f, g) = der_c^b(R, +, \cdot)$ if two m-ary hyperoperation and n-ary hyperoperation f and g respectively, have the form

$$f(x_1^m) = \sum_{i=1}^{m} x_i + b, \text{ for all } x_1^m \in R,$$

and

$$g(x_1^n) = \prod_{j=1}^{n} y_j \cdot c, \text{ for all } y_1^n \in R.$$

In this case, when b is a zero scalar of $(R, +)$ and c is an identity scalar of $(R, \circ)$ we say that (R, f, g) is derived ' from $(R, +, \cdot)$ and denote this fact by $(R, f, g) = der(R, +, \cdot)$. It is clear that if b belongs to the center of a semihypergroup $(R, +)$ and c belongs to the center of a semihypergroup $(R, \cdot)$ then two m-ary hyperoperation and n-ary hyperoperation f and g are associative and (R, f) and (R, g) are m-ary semihypergroup and n-ary semihypergroup. Now, if b is zero scalar or f define by $f(x_1^m) = \sum_{i=1}^m x_i$ then denote $(R, f, g) = der_c(R, +, \cdot)$ and say (R, f, g) is c-derived from $(R, +, \cdot)$. Now, if $(R, +, \cdot)$ be a hyperring and $c \in Z(R, \cdot)$ Then, the c-derived (R, f, g) is an (m, n)-hyperring.

If (R, f, g) is an (m, n)-hyperring and the relation ρ be a strongly compatible relation on both m-ary hypergroup (R, f) and n-ary semihypergroup (R, g), then the quotient $(R/\rho, f/\rho, g/\rho)$ is an (m, n)-ring.

Let (R, f, g) be an (m, n)-hyperring. For every $k \in \mathbb{N}^*$ and $l_1^s \in \mathbb{N}$, where $s = k(m-1)+1$, we define the relation $\gamma_{k;l_1^s}$, as follows: $x \, \gamma_{k;l_1^s} \, y$ if and only if there exist $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$,

$i = 1, \ldots, s$ such that $\{x, y\} \subseteq f_{(k)}(u_1, \ldots, u_s)$, where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Now, set $\gamma_k = \bigcup_{l_1^s \in \mathbb{N}} \gamma_{k;l_1^s}$ and $\gamma = \bigcup_{k \in \mathbb{N}^*} \gamma_k$. Then, the relation γ is reflexive and symmetric. Let γ^* be the transitive closure of relation γ. It easy to see that $\beta_f \subseteq \gamma$, $\beta_f^* \subseteq \gamma^*$, $\beta_g \subseteq \gamma$ and $\beta_g^* \subseteq \gamma^*$. If (R, f, g) is an (m, n)-hyperring, then for every $k \in \mathbb{N}^*$ we have $\gamma_k \subseteq \gamma_{k+1}$. If (R, f, g) is an (m, n)-hyperring, then for every $k \in \mathbb{N}^*$ we have $\gamma_k^* \subseteq \gamma_{k+1}^*$. The relation γ^* is a strongly compatible relation on both m-ary hypergroup (R, f) and n-ary semihypergroup (R, g). The quotient $(R/\gamma^*, f/\gamma^*, g/\gamma^*)$ is an (m, n)-ring. The relation γ^* is the smallest equivalence relation such that the quotient $(R/\gamma^*, f/\gamma^*, g/\gamma^*)$ is an (m, n)-ring. For all additive (m, n)-hyperrings, we have $\gamma^* = \beta_f^*$. For every additive (m, n)-hyperring, the relation γ is an equivalence relation, i.e. $\gamma = \gamma^*$. If (R, f, g) is an (m, n)-hyperring, then

(1) $(R/\beta_f^*, f/\beta_f^*, g/\beta_f^*)$ is an (m, n)-multiplicative hyperring;

(2) $(R/\beta_g^*, f/\beta_g^*, g/\beta_g^*)$ is an additive (m, n)- hyperring.

If (R, f, g) is an (m, n)-hyperring, then $R/\gamma^* \cong (R/\beta_g^*)/\beta_{f/\beta_g^*}^*$. Let (R_1, f, g) and (R_2, f, g) be two (m, n)-hypersemirings. We define $(f_1, f_2) : (A \times B)^m \to \wp^*(A \times B)$ by $(f, g)((a_1, b_1), \ldots, (a_n, b_n)) = \{(a, b) \mid a \in f(a_1, \ldots, a_n), b \in g(b_1, \ldots, b_n)\}$. Clearly $(R_1 \times R_2, (f_1, f_2), (g_1, g_2))$ is an (m, n)-semihyperring and we call this (m, n)-semihyperring the direct hyperproduct of R_1 and R_2. Let (R_1, f, g) and (R_2, f, g) be two (m, n)-hypersemirings, $a, c \in R_1$ and $b, d \in R_2$. If $\gamma_{R_1}^*$, $\gamma_{R_2}^*$ and $\gamma_{R_1 \times R_2}^*$ are the γ^*-relations on R_1, R_2 and $R_1 \times R_2$ respectively. Then, $(a, b)\ \gamma_{R_1 \times R_2}^*\ (c, d)$ implies $a\ \gamma_{R_1}^*\ c$ and $b\ \gamma_{R_2}^*\ d$. Let $a, c \in R_1$ and $b, d \in R_2$. If $\gamma_{R_1}^*$, $\gamma_{R_2}^*$ and $\gamma_{R_1 \times R_2}^*$ are γ^*-relations on R_1, R_2 and $R_1 \times R_2$ respectively: $(a, b)\gamma_{R_1 \times R_2}^*(c, d)$ if and only if $a\gamma_{R_1}^* c$ and $b\gamma_{R_2}^* d$. Then, we have $(R_1 \times R_2)/\gamma_{R_1 \times R_2}^* \cong R_1/\gamma_{R_1}^* \times R_2/\gamma_{R_2}^*$.. To extend the concept of n-ary hyperstructures to hypermodules, we refer to [15, 16].

7 Applications of hyperstructures in biology, physics and chemistry

Mendel, the father of genetics took the first steps in defining "contrasting characters, genotypes in F_1 and F_2 ... and setting different laws". The genotypes of F_2 is dependent on the type of its parents genotype and it follows certain roles. In [46], Ghadiri, Davvaz and Nekouian analyzed the second generation genotypes of monohybrid and a dihybrid with a mathematical structure. They used the concept of H_v-semigroup structure in the F_2-genotypes with cross operation and proved that this is an H_v-semigroup. They determined the kinds of number of the H_v-subsemigroups of F_2-genotypes. lso, in [30], inheritance issue based on genetic information is looked at carefully via a new hyperalgebraic approach. Several examples are provided from different biology points of view, and it is shown that the theory of hyperstructures exactly fits the inheritance issue. In [8], Al Tahan and Davvaz presented examples of five different types of Non- Mendelian inheritance and studied their relation with hyperstucture theory. They made some hypothetical crosses for the n- hybrid case for both simple and incomplete inheritances and studied their relations with hyperstructures. In [9], the authors considered n-ary hyperstructures associated to the genotypes of the second generation F_2 for $n = 2, 3, 4$. They defined a hyperoperation $\times$ (mating) on F_2 and proved that it is a cyclic H_v-semigroup under the defined hyperoperation. Then they defined a ternary hyperstructure f associated to the genotypes of F_2 and proved that (F_2, f) is a ternary H_v-semigroup. Finally, they defined a 4-ary hyperstructure g associated to the genotypes of F_2 and proved that (F_2, g) is a 4-ary H_v-semigroup.

In 1996, R. M. Santilli and T. Vougiouklis [67] point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. e-hyperstructures are a special kind of hyperstructures and they can be interpreted as a generalization of two important concepts for physics: Isotopies and Genotopies. In [37], Davvaz, Santilli and Vougiouklis studied multi-valued hyperstructures following the apparent existence in nature of a realization of two-valued hyperstructures with hyperunits characterized by matter-antimatter systems and their extensions, where matter is represented with conventional mathematics and antimatter is represented with isodual mathematics, Also see [38]. In [39], the authors presented Ying's twin universes, Santilli's isodual theory of antimatter, and Davviaz-Santilli-Vougiouklis two-valued hyperstructures representing matter and antimatter in two distinct but co-existing space times. They identified a seemingly new map for both matter and antimatter providing a mathematical prediction of Ying's twin universes, and introduced a four-fold hyperstructure representing matter-antimatter as well as Ying's twin universes, all co-existing in distinct space times. Another motivation for the study of hyperstructures comes from physical phenomenon as the nuclear fission. This motivation and the results were presented by S. Hošková, J. Chvalina and P. Račková (see [50], HCR2). In [43], the authors provided, for the first time, a physical example of hyperstructures associated with the elementary particle physics, Leptons. They have considered this important group of the elementary particles and shown that this set along with the interactions between its members can be described by the algebraic hyperstructures. The Standard Model (SM) of particle physics is a gauge theory including the Higgs boson, which plays a unique role in the SM. In the SM, all the elementary particles are classified into three generations of matter, i.e., Hadrons, Leptons and Gauge Bosons. In [33], Davvaz et al. showed that the leptons and gauge bosons along with the interactions between their members construct a weak algebraic hyperstructure. This new sight to the elementary particles would make a new arrangement to the elementary particles.

Another motivation for the study of hyperstructures comes from chemical reactions. In [24], Davvaz presented an introduction to some of the results, methods and ideas about chemical examples of weak algebraic hyperstructures. Some of these examples include

(1) Weak algebraic hyperstructures associated with chain reactions [28];

(2) Weak algebraic hyperstructures associated with dismutation reactions [29];

(3) Weak algebraic hyperstructures associated with redox reactions [31].

Also, see [10, 18, 25, 32].

References

[1] N. Abbasizadeh, I. Cristea, B. Davvaz, J. Zhan, *Subpolygroups related to a fuzzy topology*, J. Multiple-Valued Logic and Soft Computing, 31 (2018), 155-173.

[2] N. Abbasizadeh, B. Davvaz, *Topological polygroups in the framework of fuzzy sets*, Journal of Intelligent and Fuzzy Systems, 30 (2016), 2811-2820.

[3] N. Abbasizadeh, B. Davvaz, V. Leoreanu-Fotea, *Studies on fuzzy topological Polygroups*, Journal of Intelligent and Fuzzy Systems, 32(1) (2017), 1101-1110.

[4] H. Aghabozorgi, B. Davvaz, M. Jafarpour, *Solvable polygroups and derived subpolygroups*, Communications in Algebra, 41 (2013), 3098-3107.

[5] H. Aghabozorgi, B. Davvaz, M. Jafarpour, *Nilpotent groups derived from hypergroups*, Journal of Algebra, 382 (2013), 177-184.

[6] M. Al Tahan, B. Davvaz, *On the existence of hyperrings associated to arithmetic functions*, Journal of Number Theory, 174 (2017), 136-149.

[7] M. Al Tahan, B. Davvaz, *Strongly regular relations of arithmetic functions*, Journal of Number Theory, 187 (2018), 391-402.

[8] M. Al Tahan, B. Davvaz, *Hyperstructures associated to Biological inheritance*, Mathematical Biosciences, 285 (2017), 112-118.

[9] M. Al Tahan, B. Davvaz, N*-ary hyperstructures associated to the genotypes of* F_2*-offspring*, International Journal of Biomathematics, 10(8) (2017), 17 pages.

[10] M. Al Tahan, B. Davvaz, *Weak chemical hyperstructures associated to electrochemical cells*, Iranian Journal of Mathematical Chemistry, 9(1) (2018), 65-75.

[11] M. Al Tahan, Š. Hošková, B. Davvaz, *An overview of topological hypergroupoids*, Journal of Intelligent and Fuzzy Systems, 34 (2018), 1907-1916.

[12] R. Ameri, *Topological (transposition) hypergroups*, Italian Journal of Pure and Applied Mathematics, 13 (2003), 171-176.

[13] S.M. Anvariyeh, S. Mirvakili, B. Davvaz, θ^**-Relation on hypermodules and fundamental modules over commutative fundamental rings*, Communications in Algebra, 36(2) (2008), 622-631.

[14] S.M. Anvariyeh, B. Davvaz, *Strongly transitive geometric spaces associated to hypermodules*, Journal of Algebra, 322 (2009), 1340-1359.

[15] S.M. Anvariyeh, B. Davvaz, *Strongly transitive geometric spaces associated to* (m, n)*-ary hypermodules*, Journal of Algebra, 322(4) (2009), 1340-1359.

[16] S.M. Anvariyeh, S. Mirvakili, B. Davvaz, *Commutative fundamental* (m, n)*-hypermodules*, Journal of the Egyptian Mathematical Society, 22 (2014), 167-173.

[17] M. Asghari-Larimi, B. Davvaz, *Hyperstructures associated to arithmetic functions*, ARS Combinatoria, 97 (2010), 51-63.

[18] S.C. Chung, K.M. Chun, N.J. Kim, S.Y. Jeong, H. Sim, J. Lee, H. Maeng, *Chemical hyperalgebras for three consecutive oxidation states of elements*, MATCH Communications in Mathematical and in Computer Chemistry, 72 (2014), 389-402.

[19] P. Corsini, *Prolegomena of hypergroup theory*, Aviani editore, Second edition, 1993.

[20] B. Davvaz, *Fuzzy* H_v*-groups*, Fuzzy Sets and Systems, 101 (1999), 191-195.

[21] B. Davvaz, *Rough sets in a fundamental ring*, Bulletin of the Iranian Mathematical Society, 24(2) (1998), 49-61.

[22] B. Davvaz, *Approximations in hyperrings*, Journal of Multiple-Valued Logic and Soft Computing, 15(5-6) (2009), 471-488.

[23] B. Davvaz, *Polygroup theory and related systems*, World Scientific, 2013.

[24] B. Davvaz, *Weak algebraic hyperstructures as a model for interpretation of chemical reactions*, Iranian Journal of Mathematical Chemistry, 7(2) (2016), 267-283.

[25] B. Davvaz, *Rough algebraic structures: Corresponding to ring theory*, Algebraic Methods in General Rough Sets, Trends in Mathematics, Springer Nature Switzerland AG, 2018.

[26] B. Davvaz, N. Abbasizadeh, *Fuzzy topological F-polygroups*, Journal of Intelligent and Fuzzy Systems, 33(6) (2017), 3433-3440.

[27] B. Davvaz, I. Cristea, *Fuzzy algebraic hyperstructures-an introduction*, Springer International Publishing, 2015.

[28] B. Davvaz, A. Dehghan Nezad, A. Benvidi, *Chain reactions as experimental examples of ternary algebraic hyperstructures*, MATCH Communications in Mathematical and in Computer Chemistry, 65(2) (2011), 491-499.

[29] B. Davvaz, A. Dehghan Nezhad, A. Benvidi, *Chemical hyperalgebra: Dismutation reactions*, MATCH Communications in Mathematical and in Computer Chemistry, 67 (2012), 55-63.

[30] B. Davvaz, A. Dehghan Nezad, M. M. Heidari, *Inheritance examples of algebraic hyperstructures*, Information Sciences, 224 (2013), 180-187.

[31] B. Davvaz, A. Dehghan Nezad, M. Mazloum-Ardakani, *Chemical hyperalgebra: Redox reactions*, MATCH Communications in Mathematical and in Computer Chemistry, 71 (2014), 323-331.

[32] B. Davvaz, A. Dehghan Nezad, M. Mazloum-Ardakani, *Describing the algebraic hyperstructure of all elements in radiolytic processes in cement medium*, MATCH Communications in Mathematical and in Computer Chemistry, 72(2) (2014), 375-388.

[33] B. Davvaz, A Dehghan Nezhad, S.M. Moosavi Nejad, *Algebraic hyperstructure of observable elementary particles including the Higgs boson*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 90(1) (2020), 169-176.

[34] B. Davvaz, W.A. Dudek, S. Mirvakili, *Neutral elements, fundamental relations and n-ary hypersemigroups*, International Journal of Algebra and Computation, 19(4) (2009), 567-583.

[35] B. Davvaz, W.A. Dudek, T. Vougiouklis, *A generalization of n-ary algebraic systems*, Communications in Algebra, 37 (2009) 1248-1263.

[36] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press, USA, 2007.

[37] B. Davvaz, R.M. Santilli, T. Vougiouklis, *Studies of multivalued hyperstructures for the characterization of matter-antimatter systems and their extension*, Algebras Groups and Geometries, 28 (2011) 105-116.

[38] B. Davvaz, R.M. Santilli, T. Vougiouklis, *Multi-valued hypermathematics for characterization of matter and antimatter systems*, Journal of Computational Methods in Sciences and Engineering (JCMSE), 13 (2013), 37-50.

[39] B. Davvaz, R.M. Santilli, T. Vougiouklis, *Mathematical prediction of Ying's twin universes*, American Journal of Modern Physics, 4(3) (2015), 5-9.

[40] B. Davvaz, T. Vougiouklis, *n-Ary hypergroups*, Iranian Journal of Science and Technology, Transaction A, 30 (A2) (2006), 165-174.

[41] B. Davvaz, T. Vougiouklis, *Commutative rings obtained from hyperrings (H_v-rings) with α^*-relations*, Communications in Algebra, 35 (2007), 3307-3320.

[42] B. Davvaz, T. Vougiouklis, *A walk through weak hyperstructures- H_v-structure*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019. xi+334 pp.

[43] A. Dehghan Nezhad, S.M. Moosavi Nejad, M. Nadjafikhah, B. Davvaz, *A physical example of algebraic hyperstructures: Leptons*, Indian Journal of Physics, 86(11) (2012), 1027-1032.

[44] D. Freni, *A note on the core of a hypergroup and the transitive closure β^* of β*, Riv. Matematica Pura ed Applicata, 8 (1991), 153-156.

[45] D. Freni, *A new characterization of the derived hypergroup via strongly regular equivalences*, Communications in Algebra, 30 (2002), 3977-3989.

[46] M. Ghadiri, B. Davvaz, R. Nekouian, *H_v-Semigroup structure on F_2-offspring of a gene pool*, International Journal of Biomathematics, 5(4) (2012), 13 pages.

[47] D. Heidari, B. Davvaz, S.M.S. Modarres, *Topological hypergroups in the sense of Marty*, Communications in Algebra, 42 (2014), 4712-4721.

[48] D. Heidari, B. Davvaz, S.M.S. Modarres, *Topological polygroups*, Bulletin of the Malaysian Mathematical Sciences Society, 39 (2016), 707-721.

[49] Š. Hošková-Mayerova, *Topological hypergroupoids*, Computers and Mathematics with Applications, 64(9) (2012), 2845-2849.

[50] Š. Hošková, J. Chvalina, P. Račková, *Transposition hypergroups of Fredholm integral operators and related hyperstructures. Part I*, Journal of Basic Science, 4 (2008), 43-54.

[51] Š. Hošková, J. Chvalina, P. Račková, *Transposition hypergroups of Fredholm integral operators and related hyperstructures. Part II*, Journal of Basic Science, 4 (2008), 55-60.

[52] M. Jafarpour, H. Aghabozorgi, B. Davvaz, *On nilpotent and solvable polygroups*, Bulletin of Iranian Mathematical Society, 39 (2013), 487-499.

[53] M. Koskas, *Groupoides, demi-hypergroupes et hypergroupes*, Journal of Mathematics and Pure Applications, 49(9) (1970), 155-192.

[54] V. Leoreanu-Fotea, B. Davvaz, *n-hypergroups and binary relations*, European Journal of Combinatorics, 29(5) (2008), 1207-1218.

[55] V. Leoreanu-Fotea, B. Davvaz, *Roughness in n-ary hypergroups*, Information Sciences, 178(21) (2008), 4114-4124.

[56] V. Leoreanu-Fotea, B. Davvaz, *Join n-spaces and lattices*, Journal of Multiple-Valued Logic and Soft Computing, 15(5-6) (2009), 421-432.

[57] V. Leoreanu-Fotea, I. Rosenberg, B. Davvaz, T. Vougiouklis, *A new class of n-ary hyperoperations*, European Journal of Combinatorics, 44 (2015), 265-273.

[58] S. Mirvakili, S.M. Anvariyeh, B. Davvaz, *On α-relation and transitivity conditions of α*, Communications in Algebra, 36 (2008), 1695-1703.

[59] S. Mirvakili, S.M. Anvariyeh, B. Davvaz, *Transitivity of γ-relation on hyperfields*, Bulletin Mathematics Society Science Mathematics Roumanie (N.S.), 51(99) (2008), 233-243.

[60] S. Mirvakili, B. Davvaz, *Constructions of (m, n)-hyperrings*, Matematicki Vesnik, 67(1) (2015), 1-16.

[61] S. Mirvakili, B. Davvaz, *Relations on Krasner (m, n)-hyperrings*, European Journal of Combinatorics, 31 (2010), 790-802.

[62] S.S. Mousavi, V. Leoreanu-Fotea, M. Jafarpour, *Cyclic groups obtained as quotient hypergroups*, Analele Stiintifice ale Universitatii Al I Cuza din Iasi Mathematics (N.S.), 61(1) (2015), 109-122.

[63] Z. Pawlak, *Rough sets*, International Journal of Information Computer Science, 11 (1982), 341-356.

[64] M.B. Safari, B. Davvaz, V. Leoreanu-Fotea, *Enumeration of 3- and 4-hypergroups on sets with two elements*, European J. Combinatorics, 44 (2015), 298-306.

[65] M. Salehi Shadkami, M.R. Ahmadi Zand, B. Davvaz, *The role of complete parts in topological polygroups*, International Journal of Analysis and Applications, 11(1) (2016), 54-60.

[66] M. Salehi Shadkami, M.R. Ahmadi Zand, B. Davvaz, *Left big subsets of topological polygroups*, Filomat, 30(12) (2016), 3139-3147.

[67] R.M. Santilli, T. Vougiouklis, *Isotopies, genotopies, hyperstructures and their applications*, New frontiers in hyperstructures (Molise, 1995), 1-48, Ser. New Front. Advance Mathematics Ist. Ric. Base, Hadronic Press, Palm Harbor, FL, 1996.

[68] S. Yamak, O. Kazancı, B. Davvaz, *Generalized lower and upper approximations in a ring*, Information Sciences, 180 (2010), 1759-1768.

[69] S. Yamak, O. Kazancı, B. Davvaz, *Approximations in a module by using set-valued homomorphisms*, International Journal of Computer Mathematics, 88(14) (2011), 2901-2914.

[70] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfields*, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990), World Sci. Publ., 1991, 203-211.

[71] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Inc, 115, Palm Harber, USA, 1994.