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Hyper equality ideals: Basic properties

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Abstract

In this paper, the concept of (strong) hyper equality ideals in bounded hyper equality algebras are introduced and several properties and related results are given. Also, the properties of hyper equality ideals of the direct product of bounded hyper equality algebras are investigated; we prove that any (strong) hyper equality ideal of the direct product of hyper equality algebras is representable with respect to the product of (strong) hyper equality ideals of any direct component. In the sequel, we investigate the relationships between hyper equality ideals and hyper deductive systems in good bounded hyper equality algebras. Furthermore, we show how one can construct a hyper congruence relation via a strong hyper equality ideal so that the congruence classes form a hyper equality algebra.

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1 Introduction

Jenei [7] introduced the notion of an equality algebra as a candidated for possible algebraic semantics of fuzzy type theory similar to EQ-algebras [9], but without product. Many researchers have studied on equality algebras and obtained interesting results (see [2, 3, 5, 11, 12]). Paad [10] introduced the notion of an ideal in bounded equality algebras. He gave the relationships between ideals and filters in equality algebras. He also introduced the notions of prime ideals and Boolean ideals and investigated the related properties.

Cheng et. al. introduced the notion of a hyper equality algebra inspired by the work of Marty [8], who, for the first time, introduced the notion of an algebraic hyperstructure. They also introduced the notion of hyper deductive systems and hyper filters and gave some related results. They also introduced the notion of a (strong) hyper congruence in hyper equality algebras and show that the congruence classes of hyper congruence can forms a hyper equality algebra provided that an additional property is assumed; they called it regualrity. But there's still an important

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question. What is the relationship between (strong) hyper equality deductive systems and hyper congruences? More closely, how we can construct a hyper congruence by (strong) hyper deductive systems so that the congruence classes form a hyper equality algebra?

In this paper, inspired by Paad's work ([10]), we introduce the notion of (strong) hyper equality ideals in bounded hyper equality algebras and give several properties and results. We also obtain the relationships between strong hyper equality ideals and hyper deductive systems in bounded hyper equality algebras. In the sequel, we answer to the above mentioned questions. We construct a hyper congruence relation by strong hyper equality ideal and show that the congruence classes together with the induced operations, under suitable conditions, from the previous ones form a hyper equality algebera.

2 Preliminaries

In this section, we give some fundamental results from the literature. We assumed that the reader is familiar to the basic properties of algebraic hyperstructures. For more details, we refer to the references [4, 6].

Definition 2.1. *A hyper equality algebra is a quadraple $H = (H; \wedge, \sim, 1)$ constitutes a nonempty set H, a binary operation $\wedge$, a binary hyperoperation $\sim$ and a constant $1 \in H$ such that for all $x, y, z \in H$ the following properties hold:*

$(HE1)$ $(H; \wedge)$ *is a meet-semilattice with top element 1, where the induced partial ordering is* $x \leq y \Leftrightarrow x \wedge y = x$,

$(HE2)$ $x \sim y \ll y \sim x$,

$(HE3)$ $1 \in x \sim x$,

$(HE4)$ $x \in 1 \sim x$,

$(HE5)$ $x \leq y \leq z$ *implies* $x \sim z \ll y \sim z$ *and* $x \sim z \ll x \sim y$,

$(HE6)$ $x \sim y \ll (x \wedge z) \sim (y \wedge z)$,

$(HE7)$ $x \sim y \ll (x \sim z) \sim (y \sim z)$,

where $A \ll B$ is defined by for all $x \in A$ there exists $y \in B$ such that $x \leq y$.

In any hyper equality algebra H for all $x, y \in H$, the auxiliry hyperoperation $\rightarrow$ is defined as $x \rightarrow y := x \sim (x \wedge y)$. Moreover, for any nonempty subsets $A, B \subseteq H$, we write

$$A \wedge B = \{a \wedge b : a \in A, b \in B\} \text{ and } A * B = \bigcup_{a \in A, b \in B} a * b \text{ for } * \in \{\sim, \rightarrow\}.$$

Hyper equality algebra H is said to be

- *bounded* if it has a bottom element 0 (with respect to the order $\leq$). In this case, the set $x \rightarrow 0 = x \sim 0$ is denoted by $\neg x$,
- *good* if $x = 1 \sim x$, for all $x \in H$,
- *symmetric* if $x \sim y = y \sim x$, for all $x, y \in H$.

Proposition 2.2. *In any hyper equality algebra H, for any $x, y, z \in H$ and $A, B \subseteq H$ the following properties hold:*

(1) $x \ll y \rightarrow x$, $x \sim y \ll x \rightarrow y$ *and* $x \sim y \ll y \rightarrow x$.

(2) $x \leq y$ *implies* $1 \in x \rightarrow y$.

(3) $x \leq y$ *implies* $z \to x \ll z \to y$ *and* $y \to z \ll x \to z$. *Particularly,* $x \leq y$ *implies that* $\neg y \ll \neg x$.

(4) $x \to y \ll (y \to z) \to (x \to z)$. *Particularly,* $x \to y \ll \neg y \to \neg x$.

(5) $A \sim B \ll B \sim A$ *and* $A \ll B \to A$.

(6) $A \ll B$ *and* $B \ll C$ *imply* $A \ll C$.

(7) $A \subseteq B$ *implies that* $A \ll B$.

(8) *If* H *is good, then*

(8a) $x \ll (x \to y) \to y$. *Particularly,* $x \ll \neg\neg x$.

(8b) $x \ll y \to z$ *if and only if* $y \ll x \to z$.

(8c) $x \to (y \to z) \ll y \to (x \to z)$.

Definition 2.3. *Let* D *be a nonempty subset of hyper equality algebra* H. D *is called a strong hyper deductive system if for all* $x, y \in H$, *it satisfies in the following condition:*
(Shd) $1 \in D$,
(Shd) *if* $x \in D$ *and* $x \to y \cap D \neq \emptyset$, *then* $y \in D$.

It should be observe that any strong hyper deductive system F is an upper set; i.e., if $x \leq y$ and $x \in F$, then $y \in F$.

Assume that θ is a binary relation in H and $A, B \subseteq H$. Then

- $A\bar{\theta}B$ means that for any $a \in A$ there exists $b \in B$ such that $a\theta b$ and for any $b \in B$ there exists $a \in A$ such that $a\theta b$.
- $A\bar{\bar{\theta}}B$ means that for any $a \in A$ and any $b \in B$, $a\theta b$.

The notion of a congruence (or hyper congruence in [4]) relation in a hyper equality algebra H is defined as an equivalence relation θ which is compatible with respect to $\wedge$ and $\sim$. More closely, if θ is an equivalence relation in hyper equality algebra H, then θ is called a

- hyper congruence relation if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $(x \sim u)\bar{\theta}(y \sim v)$ and $(x \wedge u)\theta(y \wedge v)$.
- strong hyper congruence relation if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $(x \sim u)\bar{\bar{\theta}}(y \sim v)$ and $(x \wedge u)\theta(y \wedge v)$.

Note. It is easy to see that every strong hyper congruence relation is a hyper congruence relation.

Let θ be a hyper congruence relation in hyper equality algebra H, $H/\theta = \{[x]_\theta : x \in H\}$, the set of all congruence classes $[x]_\theta$, and $[A]_\theta = \{[a]_\theta : a \in A\}$, for $A \subseteq H$. Also, assume that the (hyper) operations $\overline{\sim}$, $\overline{\to}$ and $\overline{\wedge}$ on H/θ are defined as follows,

$$[x]_\theta \overline{\sim} [y]_\theta = [x \sim y] = \{[a]_\theta : a \in x \sim y\}, \quad [x]_\theta \overline{\to} [y]_\theta = [x \to y]_\theta = \{[a]_\theta : a \in x \to y\},$$

$$[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta,$$

respectively. Two binary relations are induced; the partial ordering $\leq_\theta$ as

$$[x]_\theta \leq_\theta [y]_\theta \;\Leftrightarrow\; [x]_\theta = [x]_\theta \wedge [y]_\theta \;\Leftrightarrow\; (x \wedge y)\theta x,\; \forall [x]_\theta, [y]_\theta \in H/\theta,$$

and the binary relation $\ll_\theta$ on $P^*(H/\theta)$, the set of all nonempty subsets of H/θ, as

$$\forall A, B \subseteq H/\theta,\; A \ll_\theta B \;\Leftrightarrow\; (\forall [a]_\theta \in A, \exists [b]_\theta \in B)\; [a]_\theta \leq_\theta [b]_\theta.$$

In general, $\leq_\theta$ does not imply $\ll_\theta$, otherwise θ is called regular.

Theorem 2.4. *If θ is a regular hyper congruence relation in hyper equality algebra $(H; \sim, \wedge, 1)$, then $(H/\theta; \overline{\sim}, \overline{\wedge}, [1])$ is a hyper equality algebra.*

In this paper, all hyper equality algebras are bounded and $H = (H, \wedge, \sim, 1)$ will denote such a bounded hyper equality algebra, unless otherwise stated.

3 Hyper equality ideals

In this section, we introduce the concept of a hyper equality ideal in bounded hyper equality algebras and we give some related results.

We first give some properties which follows directly from Definition 2.1 and Proposition 2.2. So, the proofs are left to the reader.

Proposition 3.1. *In any hyper equality algebra H, for all $A, B, C \subseteq H$, the following properties hold:*

(1) *$A \ll B$ implies that $1 \in A \to B$.*

(2) *$A \sim B \ll A \to B$.*

(3) *$A \to B \ll (B \to C) \to (A \to C)$. Particularly, $A \to B \ll \neg B \to \neg A$.*

(4) *If $A \ll B$, then there exists $b \in B$ such that $\neg b \ll \neg A$.*

(5) *$A \sim B \ll (A \sim C) \sim (B \sim C)$. Particularly, $A \sim B \ll \neg A \sim \neg B$.*

(6) *If H is good, then*

  (6a) *$A \ll (A \to B) \to B$,*

  (6b) *$A \to (B \to C) \ll B \to (A \to C)$,*

  (6c) *If $A \ll B \to C$, then there exists $b \in B$ such that $b \ll A \to C$.*

  (6d) *$\neg x \ll \neg\neg(\neg x \to \neg y) \to \neg y$.*

  (6e) *There exists $b \in \neg\neg(\neg x \to \neg y)$ such that $b \ll \neg x \to \neg y$.*

Definition 3.2. *Let I be a down set of H; i.e.,*

$$(\mathit{HI}) \qquad x \leq y \text{ and } y \in I \text{ imply } x \in I, \text{ for all } x, y \in H.$$

I is called a

- hyper equality ideal *if $x, y \in I$ imply $\neg x \to y \subseteq I$, for all $x, y \in H$.*
- strong hyper equality ideal *if $x, y \in I$ imply $(\neg x \to y) \cap I \neq \emptyset$, for all $x, y \in H$.*

From (hi), it is obvious that any (strong) hyper equality ideal contains the bottom element 0.

Example 3.3. *Let $H = \{0, a, b, 1\}$. Define the operation $\wedge$ and hyper operation $\sim$ on H as the following tables:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{b,1\}$	$\{0,a\}$
a	$\{1\}$	$\{1\}$	$\{a,1\}$	$\{a\}$
b	$\{b,1\}$	$\{a,1\}$	$\{1\}$	$\{b,1\}$
1	$\{0,a\}$	$\{a\}$	$\{b,1\}$	$\{1\}$

Then $(H, \wedge, \sim, 0, 1)$ is a bounded hyper equality algebra (see [4]). Routine calculations show that $I = \{0, a\}$ is a hyper equality ideal and $J = \{0, a, b\}$ is a strong hyper equality ideal of H which is not a hyper equality ideal. Because, $a, b \in J$ while $\neg b \to a = \{a, b, 1\} \nsubseteq J$.

Lemma 3.4. *Let I be a hyper equality ideal and A and B be nonempty subsets of H. Then*

(1) *$A \ll B$ and $B \subseteq I$ imply $A \subseteq I$.*

(2) *$A \ll I$ implies that $A \subseteq I$.*

(3) *If $A, B \subseteq I$, then $\neg A \to B \subseteq I$.*

Proof. Straightforward. □

Theorem 3.5. *In H, the following statements hold.*

(1) *Every hyper equality ideal of H is a strong hyper equality ideal.*

(2) *If I is a (strong) hyper equality ideal of H and $x \in I$, then $\neg\neg x \subseteq I$ ($\neg\neg x \cap I \neq \emptyset$).*

(3) *Assume that H is good and I is a hyper equality ideal of H. If $\neg\neg x \subseteq I$ (or $\neg\neg x \ll I$), then $x \in I$.*

Proof. (1) and (2) follow from Lemma 3.4.
To prove (3) and (4), it suffices to take, $y = 0$, in Definition 3.2. □

3.1 Hyper equality ideals and hyper deductive systems

Definition 3.6. *For nonempty subset X of bounded hyper equality algebra H, we define the set $N(X)$ as follows:*

$$N(X) = \{a \in H \mid \neg a \cap X \neq \emptyset\}.$$

It must be noticed that if I is a (strong) hyper equality ideal, then $N(I) \neq \emptyset$. In fact, from $0 \in I$ and $0 \in 1 \sim 0 = 1 \to 0 = \neg 0$, it follows that $0 \in \neg 1 \cap I$, whence $1 \in N(I)$. By a similar way, it is proved that for a strong hyper deductive system F of H, $0 \in N(F)$. Furthermore, it is obvious that $A \cap N(X) \neq \emptyset$ implies that $\neg A \cap X \neq \emptyset$, for all $A \subseteq H$.

Definition 3.7. *A nonempty subset D of hyper equality algebra H is called*
(*i*) *absorptive if $\neg A \cap D \neq \emptyset$ implies that $\neg A \subseteq D$, for any $A \subseteq H$,*
(*ii*) *$\neg$-absorptive if $(\neg x \to y) \cap D \neq \emptyset$ implies that $\neg x \to y \subseteq D$, for any $x, y \in H$.*

Example 3.8. *Consider the hyper equality algebra $(H; \wedge, \sim, 1)$, where $H = \{0, a, 1\}$ is a chain with $0 < a < 1$ and operation $\wedge$ and hyper operation $\sim$ are defined as follows:*

$x \wedge y = \min\{x, y\}$ *and*

$\sim$	0	a	1
0	$\{1\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{a, 1\}$
1	$\{0\}$	$\{a, 1\}$	$\{1\}$

(see [4]). It is not difficult to verify that the set $D = \{0, 1\}$ is absorptive and also $\neg$-absorptive.

As a direct consequence of Definition 3.7 we conclude that

Theorem 3.9. *Every $\neg$-absorptive strong hyper equality ideal is a hyper equality ideal.*

Proposition 3.10. *Let I be a nonempty subset of H. The followings are equivalent:*

(1) *If $x \in I$ and $\neg(\neg x \to \neg y) \cap I \neq \emptyset$, then $y \in I$, for any $x, y \in H$.*

(2) *If $A \subseteq I$ and $\neg(\neg A \to \neg B) \cap I \neq \emptyset$, then $B \cap I \neq \emptyset$, for any $A, B \subseteq H$.*

Proof. (1) $\Rightarrow$ (2) Assume that $A \subseteq I$ and $\neg(\neg A \to \neg B) \cap I \neq \emptyset$, for $A, B \subseteq H$. Then there exists $i \in I$, $a \in A \subseteq I$ and $b \in B$ such that $i \in \neg(\neg a \to \neg b)$. By (1), we get $b \in I$, hence $B \cap I \neq \emptyset$.

(2) $\Rightarrow$ (1) Obvious. □

Theorem 3.11. *Let H be a good bounded hyper equality algebra and I be an absorptive nonempty subset of H. Then I is a strong hyper equality ideal if and only if*

(1) *$0 \in I$,*

(2) *$x \in I$ and $\neg(\neg x \to \neg y) \cap I \neq \emptyset$ imply $y \in I$, for any $x, y \in H$.*

Proof. Let I be an absorptive strong hyper equality ideal of H. We know that $0 \in I$. Let $x \in I$ and $\neg(\neg x \to \neg y) \cap I \neq \emptyset$, for $x, y \in H$. Then $\neg(\neg x \to \neg y) \subseteq I$. By Proposition 3.1(6b) we have

$$\neg x \to \neg y = \neg x \to (y \to 0) \ll y \to (\neg x \to 0) = y \to \neg\neg x,$$

and so by Proposition 3.1(6c) and (6b) we get

$$y \ll (\neg x \to \neg y) \to \neg\neg x = (\neg x \to \neg y) \to (\neg x \to 0) \ll \neg x \to \neg(\neg x \to \neg y).$$

Now, since $\{x\} \subseteq I$ and $\neg(\neg x \to \neg y) \subseteq I$, by Lemma 3.4(3) we get $\neg x \to \neg(\neg x \to \neg y) \subseteq I$, thus by Lemma 3.4(1) it follows that $y \in I$.

Conversely, assume that I satisfies the given conditions, and $x \leq y$ and $y \in I$, for $x, y \in H$. By Proposition 2.2(3) we have $\neg y \ll \neg x$ and so by Proposition 3.1(1) we have $0 \in \neg 1 \subseteq \neg(\neg y \to \neg x)$, which implies that $\neg(\neg y \to \neg x) \cap I \neq \emptyset$. By (2), we conclude that $x \in I$, proving (HI). Now, let $x, y \in I$. By Proposition 3.1(3) we have $(\neg x \to y) \to y \ll \neg y \to \neg(\neg x \to y)$ and so by Proposition 3.1(4) there exists $b \in \neg y \to \neg(\neg x \to y)$ such that

$$\neg b \ll \neg((\neg x \to y) \to y). \tag{3.1}$$

Also by Proposition 3.1(6a) we have $\neg x \ll (\neg x \to y) \to y$, hence by Proposition 3.1(4),

$$\neg w \ll \neg\neg x, \tag{3.2}$$

for some $w \in (\neg x \to y) \to y$. Again, by Proposition 3.1(6a) we have $\neg x \ll \neg\neg\neg x$ and so $0 \in \neg 1 \subseteq \neg(\neg x \to \neg\neg\neg x)$. This implies that $\neg(\neg x \to \neg\neg\neg x) \cap I \neq \emptyset$ and since $x \in I$, by Proposition 3.10 we get $\neg\neg x \cap I \neq \emptyset$, and so $\neg\neg x \subseteq I$. Combining (3.2) and Lemma 3.4(1), we conclude that $\neg w \subseteq I$, thus $\neg((\neg x \to y) \to y) \cap I \neq \emptyset$. Hence, $\neg((\neg x \to y) \to y) \subseteq I$, and by (3.1) we conclude that $\neg b \subseteq I$. This implies that $\neg(\neg y \to \neg(\neg x \to y)) \cap I \neq \emptyset$. Now, from $y \in I$, by Proposition 3.10, we get $\neg x \to y \cap I \neq \emptyset$, and so $\neg x \to y \subseteq I$. Therefore, I is a hyper equality ideal of H and so is a strong hyper equality ideal. □

Theorem 3.12. *Assume that H is a good bounded hyper equality algebra.*

(1) *If F is an absorptive strong hyper deductive system of H, $N(F)$ is an absorptive strong hyper equality ideal of H.*

(2) *If I is an absorptive strong hyper equality ideal of H, $N(I)$ is an absorptive strong hyper deductive system of H.*

Proof. (1) Assume that F is an absorptive strong hyper deductive system of H. We know that $0 \in N(F)$. Now, let $x \in N(F)$ and $\neg(\neg x \to \neg y) \cap N(F) \neq \emptyset$. Then $\neg\neg(\neg x \to \neg y) \cap F \neq \emptyset$ and since F is absorptive, we obtain that $\neg\neg(\neg x \to \neg y) \subseteq F$. By Proposition 3.1(6e), there exists $b \in \neg\neg(\neg x \to \neg y)$ such that $b \ll \neg x \to \neg y$, whence $(\neg x \to \neg y) \cap F \neq \emptyset$. By hypothesis, we know that $\neg x \cap F \neq \emptyset$ or in other words $\neg x \subseteq F$. By the definition of a strong hyper deductive system, it is easily verified that $\neg y \cap F \neq \emptyset$. This implies that $y \in N(F)$ and so by Theorem 3.11, $N(F)$ is a strong hyper equality ideal of H. To prove absorptivity, let $\neg A \cap N(F) \neq \emptyset$, for $A \subseteq H$. Then $\neg\neg A \cap F \neq \emptyset$ and so $\neg\neg A \subseteq F$. Now, for any $u \in \neg A$ we have $\neg u \subseteq F$, which means that $u \in N(F)$. Hence $\neg A \subseteq N(F)$.

(2) Assume that I is an absorptive strong hyper equality ideal of H. We know that $1 \in N(I)$. Let $x \in N(I)$ and $(x \to y) \cap N(I) \neq \emptyset$. Then $\neg(x \to y) \cap I \neq \emptyset$, and so $\neg(x \to y) \subseteq I$. By Proposition 3.1(3), we know that $x \to y \ll \neg y \to \neg x \ll \neg\neg x \to \neg\neg y$ and by Proposition 3.1(4), there exists $b \in \neg\neg x \to \neg\neg y$ such that $\neg b \ll \neg(x \to y) \subseteq I$, whence by Lemma 3.4(1) we get $\neg b \subseteq I$. This implies that $\neg(\neg\neg x \to \neg\neg y) \cap I \neq \emptyset$. On the other hand, by hypothesis we know that $\neg x \subseteq I$ and so by Proposition 3.10 and Theorem 3.11 we conclude that $\neg y \cap I \neq \emptyset$, which means that $y \in N(I)$. Therefore, $N(I)$ is a strong hyper deductive system of H. Similar to the proof of part (1), it is proved that $N(I)$ is absorptive.

□

4 Direct products

Let $H_1 = (H_1; \wedge_1, \sim_1, 1_1)$ and $H_2 = (H_2; \wedge_2, \sim_2, 1_2)$ be two bounded hyper equality algebras and $H = H_1 \times H_2$. Also, let $1 = (1_1, 1_2)$ and $0 = (0_1, 0_2)$. For $A, C \subseteq H_1$ and $B, D \subseteq H_2$, let $(A, B) = \{(a, b) \in H : a \in A, b \in B\}$ and consider the binary relation $\ll$ on H defined as

$$(A, B) \ll (C, D) \ \Leftrightarrow\ A \ll_1 C,\ B \ll_2 D.$$

Routine verifications show that $(H; \wedge, \sim, 1)$, where the operation $\wedge$ and the hyperoperation $\sim$ on H are defined as

$$(x_1, y_1) \sim (x_2, y_2) := (x_1 \sim_1 x_2, y_1 \sim_2 y_2) \text{ and } (x_1, y_1) \wedge (x_2, y_2) := (x_1 \wedge_1 x_2, y_1 \wedge_2 y_2), \tag{4.1}$$

is a bounded hyper equality algebra. Observe that the partial ordering induced by $\wedge$ is denoted by $\leq$. In fact, $(a, b) \leq (c, d)$ if and only if $a \leq_1 c$ and $b \leq_2 d$.

This definition can be generalized to any arbitrary family of bounded hyper equality algebras. To see this, let $\{H_i\}_{i\in I}$ be a nonempty family of bounded hyper equality algebras and $H = \prod_{i\in I} H_i$. Also, let $1 = \{1_i\}_{i\in I}$ and $0 = \{0_i\}_{i\in I}$ and for $A_i \subseteq H_i$, let $\{A_i\}_{i\in I} = \{\{a_i\}_{i\in I} : a_i \in A_i\}$. Assume that the binary relation $\ll$ on H is defined as

$$\{A_i\}_{i\in I} \ll \{B_i\}_{i\in I} \Leftrightarrow A_i \ll_i B_i, \forall i \in I.$$

Then $(H; \sim, \wedge, 1)$ is a bounded hyper equality algebra, called the direct product of H_i's, where

$$\{x_i\}_{i\in I} \sim \{y_i\}_{i\in I} = \{x_i \sim_i y_i\}_{i\in I}, \{x_i\}_{i\in I} \wedge \{y_i\}_{i\in I} = \{x_i \wedge_i y_i\}_{i\in I}.$$

Example 4.1. *Let $H = \{0, a, 1\}$ be a chain with the ordering $0 < a < 1$, and assume that two hyper operations $\sim_1$ and $\sim_2$ are defined as the following tables:*

$\sim_1$	0	a	1
0	$\{1\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{a,1\}$
b	$\{0\}$	$\{a,1\}$	$\{1\}$

$\sim_2$	0	a	1
0	$\{1\}$	$\{a,1\}$	$\{a,1\}$
a	$\{a,1\}$	$\{0,a,1\}$	$\{a,1\}$
b	$\{0,1\}$	$\{a,1\}$	$\{1\}$

Then $(H; \wedge, \sim_1, 1)$ and $(H; \wedge, \sim_2, 1)$ are hyper equality algebras (see[4]). *Then $H = H_1 \times H_2$ together with the (hyper) operation defined as* (4.1) *is a hyper equality algebra.*

In the sequel, we investigate the structure of hyper equality ideals of the direct product of hyper equality algebras. Because of analogously and brevity, we only consider the direct product of only two hyper equality algebras.

Theorem 4.2. *Let $(H_1; \wedge_1, \sim_1, 1_1)$ and $(H_2; \wedge_2, \sim_2, 1_2)$ be two bounded hyper equality algebras. If I_1 and I_2 are two (strong) hyper equality ideals of H_1 and H_2, respectively, then $I_1 \times I_2$ is a (strong) hyper equality ideal of $H_1 \times H_2$.*

Proof. Let I_1 and I_2 be two hyper equality ideals of H_1 and H_2, respectively and $(a, b), (x, y) \in H_1 \times H_2$ such that $(a, b) \leq (x, y)$ and $(x, y) \in I_1 \times I_2$. Then $a \leq x \in I_1$ and $b \leq y \in I_2$, whence $a \in I_1$ and $b \in I_2$ and so $(a, b) \in I_1 \times I_2$. Now, let $(x_1, x_2), (y_1, y_2) \in I_1 \times I_2$. Then $x_1, y_1 \in I_1$ and $x_2, y_2 \in I_2$ and so $\neg x_1 \rightarrow_1 y_1 \subseteq I_1$ and $\neg x_2 \rightarrow_2 y_2 \subseteq I_2$. This implies that

$$\neg(x_1, x_2) \rightarrow (y_1, y_2) = (\neg x_1 \rightarrow_1 y_1, \neg x_2 \rightarrow_2 y_2) \subseteq I_1 \times I_2.$$

Therefore, $I_1 \times I_2$ is a hyper equality ideal of $H_1 \times H_2$.

The proof for strong hyper equality ideals is similar. □

Theorem 4.3. *Let $(H_1; \wedge_1, \sim_1, 1_1)$ and $(H_2; \wedge_2, \sim_2, 1_2)$ be two good bounded hyper equality algebras and I be a ($\neg$-absorptive strong) hyper equality ideal of $H_1 \times H_2$. Then there are two unique ($\neg$-absorptive strong) hyper equality ideals I_1 and I_2 of H_1 and H_2, respectively, such that $I = I_1 \times I_2$.*

Proof. Assume that

$$I_1 = \{x \in H_1 : (x, z) \in I, \text{ for some } z \in H_2\},$$
$$I_2 = \{y \in H_2 : (w, y) \in I, \text{ for some } w \in H_1\}.$$

First of all, we observe that since $(0, 0) \in I$, so $0_1 \in I_1$ and $0_2 \in I_2$, means that I_1 and I_2 are nonempty. Now, we prove that I_1 and I_2 are hyper equality ideals of H_1 and H_2, respectively. Let $x, y \in H_1$ such that $x \leq y$ and $y \in I_1$. Then there exists $z \in H_2$ such that $(y, z) \in I$ and so

$(x, z) \in I$, whence $x \in I_1$. Now, for $x, y \in I_1$ there exist $z_1, z_2 \in H_2$ such that $(x, z_1), (y, z_2) \in I$ and so

$$(\neg x \to y, \neg z_1 \to z_2) = \neg(x, z_1) \to (y, z_2) \subseteq I.$$

Now, for any $a \in \neg x \to y$ there exists $c \in \neg z_1 \to z_2$ such that $(a, c) \in I$ and so $a \in I_1$. This implies that $\neg x \to y \subseteq I_1$, proving I_1 is a hyper equality ideal of H_1. Similarly, we can prove that I_2 is a hyper equality ideal of H_2.

Now, assume that $(t, w) \in I_1 \times I_2$. Then there exist $z_1 \in H_1$ and $z_2 \in H_2$ such that $(t, z_2), (z_1, w) \in I$. From $(t, 0) \leq (t, z_2)$ and $(0, w) \leq (z_1, w)$ we have $(t, 0), (0, w) \in I$ and so

$$(\neg\neg t, \neg 0 \to w) = \neg(t, 0) \to (0, w) \subseteq I.$$

On the other hand, since $t \ll \neg\neg t$ we get $t \leq k$, for some $k \in \neg\neg t$, and so $(t, w) \leq (k, w) \in (\neg\neg t, \neg 0 \to w) \subseteq I$, whence $(t, w) \in I$. Thus $I_1 \times I_2 \subseteq I$. Obviously, $I \subseteq I_1 \times I_2$. Hence $I = I_1 \times I_2$. To prove uniqueness, let J_1 and J_2 be two hyper equality ideals of H_1 and H_2, respectively, such that $I = J_1 \times J_2$. We show that $I_1 = J_1$ and $I_2 = J_2$. For this, let $x \in J_1$. From $(x, 0) \in J_1 \times J_2 = I$ we have $x \in I_1$, i.e., $J_1 \subseteq I_1$. If $x \in I_1$, then there exists $z \in H_2$ such that $(x, z) \in I = J_1 \times J_2$, and so $x \in J_1$. Hence $I_1 \subseteq J_1$. Thus $I_1 = J_1$. By similar way, it is proved that $I_2 = J_2$.

Finally, notice that if I is a $\neg$-absorptive strong hyper equality ideal, then it is a hyper equality ideal and so there exist two hyper equality ideals (and so strong hyper equality ideals, by Theorem 3.5(1)) I_1 and I_2 such that $I = I_1 \times I_2$. It remains to prove that I_1 and I_2 are $\neg$-absorptive. For this, let $A_1 \subseteq H_1$ such that $\neg A_1 \cap I_1 \neq \emptyset$. Then there exist $i \in I_1$ and $a \in A_1$ such that $i \in \neg a$. Now, $(i, 0_2) \in (\neg a, 1 \to_2 0) \cap I$, whence $(\neg A_1, 1 \to_2 0) \cap I \neq \emptyset$ and so $(\neg A_1, 1 \to_2 0) \subseteq I = I_1 \times I_2$. This implies that $\neg A_1 \subseteq I_1$. Similarly, we can prove that I_2 is $\neg$-absorptive. □

5 Congruences

We start this section by giving a result about hyper congruence relations.

Proposition 5.1. *Assume that H is a hyper equality algebra and θ is a (strong) hyper congruence in H. Then $[0]_\theta$ is a strong hyper ideal of H.*

Proof. Since every strong hyper congruence is a hyper congruence, it is enough to prove the proposition for hyper congruences. Suppose that θ is a hyper congruence in H and $x, y \in H$ such that $x \leq y$ and $y \in [0]_\theta$. Then $y\theta 0$ and since $x\theta x$ we get $x = x \wedge y\theta x \wedge 0 = 0$; i.e., $x \in [0]_\theta$. Hence, $[0]_\theta$ satisfies (HI). Now, let $x, y \in [0]_\theta$. Then $x \sim 0\overline{\theta}0 \sim 0$ and since $1 \in 0 \sim 0$, there exists $a \in \neg x$ such that $a\theta 1$ and so $a \wedge y\theta 1 \wedge y = y$. On the other hand, from $a\theta 1$ we have $a \to y = a \sim (a \wedge y)\overline{\theta}1 \sim y$. Since $y \in 1 \sim y$, there exists $b \in a \to y \subseteq \neg x \to y$ such that $b\theta y$, whence $b\theta 0$; i.e., $b \in [0]_\theta$. This implies that $(\neg x \to y) \cap [0]_\theta \neq \emptyset$. Therefore, $[0]_\theta$ is a strong hyper ideal of H. □

Definition 5.2. *A bounded hyper equality algebra $(H; \sim, \wedge, 1)$ is called involutive, if $x \in \neg\neg x$, for all $x \in H$.*

Example 5.3. *Consider the hyper equality algebra $(H; \sim, \wedge, 1)$, where $H = \{0, a, 1\}$ is a chain (with $0 < a < 1$). Define the operation $\wedge$ and the hyper operation $\sim$ on H as follows:*

$x \wedge y = \min\{x, y\}$ *and*

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{0\}$
a	$\{0, a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

(see [4]). *Obviously,* $(H; \sim, \wedge, 1)$ *is an involutive hyper equality algebra.*

Theorem 5.4. *Assume that* H *is a symmetric good involutive hyper equality algebra and* I *is an absorptive strong hyper ideal of* H*. Then the binary relation* $\equiv_I$ *on* H *which is defined by*

$$x \equiv_I y \quad \textit{if and only if} \ \neg(\neg x \sim \neg y) \cap I \neq \emptyset$$

is a strong hyper congruence on H*.*

Proof. We first prove that $\equiv_I$ is an equivalence relation. By (HE3) $1 \in \neg x \sim \neg x$ and so $0 \in \neg(\neg x \sim \neg x)$, whence $\neg(\neg x \sim \neg x) \cap I \neq \emptyset$. This proves the reflexivity of $\equiv_I$. Symmetry is obvious, because H is symmetric.

Now, let $x \equiv_I y$ and $y \equiv_I z$, for $x, y, z \in H$. Then $\neg(\neg x \sim \neg y) \subseteq I$ and $\neg(\neg y \sim \neg z) \subseteq I$. By Proposition 3.1(2) we have

$$\neg\neg(\neg y \sim \neg x) \sim \neg\neg(\neg z \sim \neg x) \ll \neg\neg(\neg y \sim \neg x) \to \neg\neg(\neg z \sim \neg x),$$

and so by Proposition 3.1(4), there exists $b \in \neg\neg(\neg y \sim \neg x) \to \neg\neg(\neg z \sim \neg x)$ such that

$$\neg b \ll \neg(\neg\neg(\neg y \sim \neg x) \sim \neg\neg(\neg z \sim \neg x)).$$

By Proposition 3.1(5), we have

$$(\neg y \sim \neg x) \sim (\neg z \sim \neg x) \ll \neg\neg(\neg y \sim \neg x) \sim \neg\neg(\neg z \sim \neg x).$$

Now, by Proposition 3.1(4), there exists $w \in \neg\neg(\neg y \sim \neg x) \sim \neg\neg(\neg z \sim \neg x)$ such that

$$\neg w \ll \neg((\neg y \sim \neg x) \sim (\neg z \sim \neg x)). \tag{5.1}$$

From Proposition 3.1(5) we have

$$\neg y \sim \neg z \ll (\neg y \sim \neg x) \sim (\neg z \sim \neg x)$$

and so there exists $p \in (\neg y \sim \neg x) \sim (\neg z \sim \neg x)$ such that $\neg p \ll \neg(\neg y \sim \neg z)$. Now, since $\neg(\neg y \sim \neg z) \subseteq I$ we get $\neg p \subseteq I$, and by Lemma 3.4(1), $\neg((\neg y \sim \neg x) \sim (\neg z \sim \neg x)) \cap I \neq \emptyset$ and so

$$\neg((\neg y \sim \neg x) \sim (\neg z \sim \neg x)) \subseteq I. \tag{5.2}$$

Combining (5.1), (5.2) and Lemma 3.4(1) we get $\neg w \subseteq I$, whence $\neg(\neg\neg(\neg y \sim \neg x) \sim \neg\neg(\neg z \sim \neg x)) \cap I \neq \emptyset$. Since $\neg(\neg y \sim \neg x) \subseteq I$, by Proposition 3.10, we conclude that $\neg(\neg z \sim \neg x) \cap I \neq \emptyset$ and so $x \equiv_I z$, proving $\equiv_I$ is transitive. Thus $\equiv_I$ is an equivalence relation.

Now, we prove that $\equiv_I$ is compatible. Let $x, y, z \in H$ and $x \equiv_I y$. Then $\neg(\neg x \sim \neg y) \cap I \neq \emptyset$ and so $\neg(\neg x \sim \neg y) \subseteq I$. Since H is involutive, we have $x \in \neg\neg x$ and $y \in \neg\neg y$ and so $x \sim y \subseteq \neg\neg x \sim \neg\neg y$. Hence,

$$\neg(x \sim y) \subseteq \neg(\neg\neg x \sim \neg\neg y). \tag{5.3}$$

By Proposition 3.1(5) we have

$$(x \wedge z) \sim (y \wedge z) \ll \neg(x \wedge z) \sim \neg(y \wedge z)$$

and so by Proposition 3.1(4), there exists $b \in \neg(x \wedge z) \sim \neg(y \wedge z)$ such that

$$\neg b \ll \neg((x \wedge z) \sim (y \wedge z)). \tag{5.4}$$

On the other hand, we know that $x \sim y \ll (x \wedge z) \sim (y \wedge z)$, by (HE6). Hence, there exists $w \in (x \wedge z) \sim (y \wedge z)$ such that $\neg w \ll \neg(x \sim y)$, then by (5.3) we get

$$\neg w \ll \neg(\neg\neg x \sim \neg\neg y). \tag{5.5}$$

Now, by Proposition 3.1(5) we have $\neg x \sim \neg y \ll \neg\neg x \sim \neg\neg y$, and so there exists $p \in \neg\neg x \sim \neg\neg y$ such that $\neg p \ll \neg(\neg x \sim \neg y)$. Since $\neg(\neg x \sim \neg y) \subseteq I$ we get $\neg p \subseteq I$ and so $\neg(\neg\neg x \sim \neg\neg y) \subseteq I$. By (5.5) we obtain that $\neg w \subseteq I$ and so $\neg((x \wedge z) \sim (y \wedge z)) \cap I \neq \emptyset$. Hence, $\neg((x \wedge z) \sim (y \wedge z)) \subseteq I$ and by (5.4) we get $\neg b \subseteq I$. Since $b \in \neg(x \wedge z) \sim \neg(y \wedge z)$, we have

$$\neg(\neg(x \wedge z) \sim \neg(y \wedge z)) \cap I \neq \emptyset,$$

means that $x \wedge z \equiv_I y \wedge z$. Now, if $a \equiv_I b$ and $c \equiv_I d$, then $a \wedge c \equiv_I b \wedge c$ and $b \wedge c = c \wedge b \equiv_I d \wedge b = b \wedge d$ and so by transitivity we get $a \wedge c \equiv_I b \wedge d$.

Now, let $x, y, z \in H$ and $x \equiv_I y$. From $x \in \neg\neg x$ and $y \in \neg\neg y$, we get $x \sim y \subseteq \neg\neg x \sim \neg\neg y$ and so

$$\neg(x \sim y) \subseteq \neg(\neg\neg x \sim \neg\neg y). \tag{5.6}$$

By Proposition 3.1(7) we have

$$(x \sim z) \sim (y \sim z) \ll \neg(x \sim z) \sim \neg(y \sim z) \tag{5.7}$$

and so by Proposition 3.1(6), there exists $t \in \neg(x \sim z) \sim \neg(y \sim z)$ such that

$$\neg t \ll \neg((x \sim z) \sim (y \sim z)).$$

Now, from (HE7) we know that $x \sim y \ll (x \sim z) \sim (y \sim z)$, and so there exists $k \in (x \sim z) \sim (y \sim z)$ such that $\neg k \ll \neg(x \sim y)$. By (5.6) we get

$$\neg k \ll \neg(\neg\neg x \sim \neg\neg y).$$

From Proposition 3.1(7) we have $\neg x \sim \neg y \ll \neg\neg x \sim \neg\neg y$, and so there exists $v \in \neg\neg x \sim \neg\neg y$ such that $\neg v \ll \neg(\neg x \sim \neg y) \subseteq I$, whence $\neg v \subseteq I$. This implies that $\neg(\neg\neg x \sim \neg\neg y) \cap I \neq \emptyset$ and so $\neg(\neg\neg x \sim \neg\neg y) \subseteq I$. Then $\neg k \subseteq I$, thus $\neg((x \sim z) \sim (y \sim z)) \cap I \neq \emptyset$ and so

$$\neg((x \sim z) \sim (y \sim z)) \subseteq I. \tag{5.8}$$

Combining (5.7) and (5.8), and by Proposition 3.1(6) and Lemma 3.4(1) we get $\neg t \subseteq I$, whence

$$\neg(\neg(x \sim z) \sim \neg(y \sim z)) \cap I \neq \emptyset.$$

Since I is absorptive, we have $\neg(\neg(x \sim z) \sim \neg(y \sim z)) \subseteq I$. This means that $x \sim z \overline{\overline{\equiv_I}} y \sim z$. By transitivity, similar to the proof of compatibility of $\equiv_I$ with respect to $\wedge$, it is proved that if $x \overline{\overline{\equiv_I}} y$ and $z \overline{\overline{\equiv_I}} w$, then $x \sim z \overline{\overline{\equiv_I}}, y \sim w$. Therefore, $\equiv_I$ is a strong hyper congruence in H. $\square$

We notice that $\equiv_I$ is not regular, in general. But when H is a chain, $\equiv_I$ is regular. To see this, assume that $[x] \leq [y]$ and $a \in [x]$. Since H is a chain, we have $x \leq y$ or $y \leq x$. If $y \leq x$, then $[y] = [x \wedge y] = [x] \overline{\wedge} [y]$ and so $[x] = [y]$, whence $[x] \ll [y]$. If $x \leq y$ and $[x] \not\ll [y]$, then there exists $b \in [x]$ such that $b \not\leq l$, for any $l \in [y]$ and so $l \leq b$, for any $l \in [y]$. $[x] \overline{\wedge} [y] = [b] \overline{\wedge} [l] = [b \wedge l] = [l] = [y]$, means that $[y] \leq [x]$. Hence $[x] = [y]$ and so we must have $[x] \ll [y]$, which is a contradiction. In any case, if $[x] \leq [y]$, then $[x] \ll [y]$. Hence $\equiv_I$ is a regular.

Therefore, considering Theorems 2.4 and 5.4 we get

Theorem 5.5. *Assume that H is a symmetric good involutive hyper equality algebra which is also a chain. If I is an absorptive strong hyper equality ideal of H, then $(H/\equiv_I; \overline{\sim}, \overline{\wedge}, [1])$ is a symmetric good involutive hyper equality algebra which is a chain, as well.*

6 Conclusions

The results of this paper are devoted to introduce (strong) hyper equality ideals in bounded hyper equality algebras which is a generalization of ideals in bounded equality algebras. We presented product of bounded hyper equality algebras and we characterized several important properties (strong) hyper equality ideals in product of hyper equality algebras. Also, we studied relations between strong hyper equality ideals and hyper deductive systems in good bounded hyper equality algebras. Moreover, we constructed a hyper congruence relation via strong hyper equality ideals in good involutive hyper equality algebras, which can be regular under suitable conditions. There's still some open problems, which could be interesting for the future researches.

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