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Two dimensional event set and its application in algebraic structures

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Abstract

Two dimensional event set is introduced, and it is applied to algebraic structures. Two dimensional BCK/BCI-eventful algebra, paired B-algebra and paired BCK/BCI-algebra are defined, and several properties are investigated. Conditions for two dimensional eventful algebra to be a B-algebra and a BCK/BCI-algebra are provided. The process of inducing a paired B-algebra using a group is discussed. Using two dimensional BCI-eventful algebra, a commutative group is established.

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1 Introduction and Priliminaries

The notion of neutrosophic set is developed by Smarandache ([9], [10]), and is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Smarandache [11] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in . [1] and [2]. Neutrosophic quadruple BCK/BCI-algebra is studied in [5] and [7]. Using neutrosophic quadruple structures, Jun et al. [4] introduced the notion of events by considering two facts, and applied it to BCK/BCI-algebras. There are many things in our daily lives that we have to choose between two facts. For example, should I read a book or not, go to the movies or not, etc. To consider these two factors, we introduce two-dimensional event sets and try to apply them to algebraic structures. We introduce the notions of two dimensional BCK/BCI-eventful algebra, paired B-algebra and paired BCK/BCI-algebra, and investigate several properties. We provided conditions for two dimensional eventful algebra to be a

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B-algebra and a BCK/BCI-algebra. We discuss the process of inducing a paired B-algebra using a group, and establish a commutative group using two dimensional BCI-eventful algebra.

We describe the basic contents that will be needed in this paper. Let $(X, *, 0)$ be an algebra, i.e., let X be a set with a special element "0" and a binary operation "$*$", and consider the following conditions.

$$(\forall u \in X)\ (u * u = 0). \tag{1}$$
$$(\forall u \in X)\ (u * 0 = u). \tag{2}$$
$$(\forall u \in X)\ (0 * u = 0). \tag{3}$$
$$(\forall u, v, w \in X)\ ((u * v) * w = u * (w * (0 * v))). \tag{4}$$
$$(\forall u, v \in X)\ (u * v = 0,\ v * u = 0\ \Rightarrow\ u = v). \tag{5}$$
$$(\forall u, v \in X)\ ((u * (u * v)) * v = 0). \tag{6}$$
$$(\forall u, v, w \in X)\ (((u * v) * (u * w)) * (w * v) = 0). \tag{7}$$

We say that $X := (X, *, 0)$ is

- a *B-algebra* (see [8]) if it satisfies (1), (2) and (4),
- a *BCI-algebra* (see [6]) if it satisfies (1), (5), (6) and (7),
- a *BCK-algebra* (see [3]) if it is a BCI-algebra satisfying (3).

A BCI-algebra $X := (X, *, 0)$ is said to be *p-semisimple* (see [3]) if it satisfies:

$$(\forall u \in X)(0 * (0 * u) = u). \tag{8}$$

2 Two dimensional event sets

Definition 2.1. *Let $\ell : X \to Q$ be a mapping from a set X to a set Q. For any $a, x \in X$, the ordered pair (x, ℓ_a) is called a two dimensional event on X where ℓ_a is the image of a under ℓ.*

The set of all two dimensional events on X is denoted by (X, ℓ_X), that is,

$$(X, \ell_X) = \{(x, \ell_a) \mid x, a \in X\} \tag{9}$$

and it is called a *two dimensional X-event set.* By a *two dimensional X-eventful algebra* we mean a two dimensional X-event set with a binary operation &, and it is denoted by $\langle (X, \ell_X), \& \rangle$.

Let $\langle (\mathbb{R}, \ell_{\mathbb{R}}), \oplus \rangle$, $\langle (\mathbb{R}, \ell_{\mathbb{R}}), \ominus \rangle$ and $\langle (\mathbb{R}, \ell_{\mathbb{R}}), \odot \rangle$ be two dimensional $\mathbb{R}$-eventful algebras in which "$\oplus$", "$\ominus$" and "$\odot$" are defined as follows:

$$(x, \ell_a) \oplus (y, \ell_b) = (x + y, \ell_{a+b}),$$
$$(x, \ell_a) \ominus (y, \ell_b) = (x - y, \ell_{a-b}),$$
$$(x, \ell_a) \odot (y, \ell_b) = (x \cdot y, \ell_{a \cdot b}),$$

respectively, for any two dimensional events (x, ℓ_a) and (y, ℓ_b) on $\mathbb{R}$. For any $t \in \mathbb{R}$ and a two dimensional event (x, ℓ_a) on $\mathbb{R}$, we define

$$t(x, \ell_a) = (tx, \ell_{ta}). \tag{10}$$

In particular, if $t = -1$, then $-1(x, \ell_a) = (-x, \ell_{-a})$ and $-1(x, \ell_a)$ is simply denoted by $-(x, \ell_a)$.

Proposition 2.2. *Let $\langle (\mathbb{R}, \ell_{\mathbb{R}}), \oplus \rangle$, $\langle (\mathbb{R}, \ell_{\mathbb{R}}), \ominus \rangle$ and $\langle (\mathbb{R}, \ell_{\mathbb{R}}), \odot \rangle$ be two dimensional $\mathbb{R}$-eventful algebras. Then*

(i) $((x, \ell_a) \oplus (y, \ell_b)) \oplus (z, \ell_c) = (x, \ell_a) \oplus ((y, \ell_b) \oplus (z, \ell_c))$,

(ii) $(x, \ell_a) \oplus (y, \ell_b) = (y, \ell_b) \oplus (x, \ell_a)$,

(iii) $(x, \ell_a) \odot (y, \ell_b) = (y, \ell_b) \odot (x, \ell_a)$,

(iv) $((x, \ell_a) \odot (y, \ell_b)) \odot (z, \ell_c) = (x, \ell_a) \odot ((y, \ell_b) \odot (z, \ell_c))$,

(v) $t((x, \ell_a) \oplus (y, \ell_b)) = t(x, \ell_a) \oplus t(y, \ell_b)$ *for all* $t \in \mathbb{R}$,

(vi) $(t + s)(x, \ell_a) = t(x, \ell_a) \oplus s(x, \ell_a)$ *for all* $t, s \in \mathbb{R}$,

(vii) $(x, \ell_a) \oplus (-(x, \ell_a)) = (0, \ell_0)$.

(viii) $(x, \ell_a) \odot (x, \ell_a)^{-1} = (1, \ell_1)$ *where* $(x, \ell_a)^{-1} = (x^{-1}, \ell_{a^{-1}})$

for all two dimensional events (x, ℓ_a), (y, ℓ_b) *and* (z, ℓ_c) *on* $\mathbb{R}$.

Proof. Straightforward. □

By Proposition 2.2, we have the following theorem.

Theorem 2.3. *Two dimensional* $\mathbb{R}$*-eventful algebras* $\langle(\mathbb{R}, \ell_{\mathbb{R}}), \oplus\rangle$ *and* $\langle(\mathbb{R}, \ell_{\mathbb{R}}), \odot\rangle$ *are commutative groups with identities* $(0, \ell_0)$ *and* $(1, \ell_1)$*, respectively.*

3 Two dimensional eventful algebras

Let $(X, *, 0)$ be an algebra and $\langle(X, \ell_X), \circledast\rangle$ be a two dimensional X-eventful algebra in which "$\circledast$" is defined by

$$(x, \ell_a) \circledast (y, \ell_b) = (x * y, \ell_{a*b})$$

respectively, for all $x, y, a, b \in X$. In a two dimensional X-eventful algebra $\langle(X, \ell_X), \circledast\rangle$, the order "$\ll$" is defined as follows:

$$(x, \ell_a) \ll (y, \ell_b) \;\Leftrightarrow\; x \leq y \text{ and } a \leq b$$

for all $x, y, a, b \in X$ where $x \leq y$ means $x * y = 0$ and $a \leq b$ means $a * b = 0$.

Theorem 3.1. *If* $(X, *, 0)$ *is a B-algebra, then the two dimensional* X*-eventful algebra* $\langle(X, \ell_X), \circledast\rangle$ *is a B-algebra with the special element* $(0, \ell_0)$.

Proof. For any $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$, we have

$$(x, \ell_a) \circledast (x, \ell_a) = (x * x, \ell_{a*a}) = (0, \ell_0),\ (x, \ell_a) \circledast (0, \ell_0) = (x * 0, \ell_{a*0}) = (x, \ell_a),$$

and

$$\begin{aligned}
&((x, \ell_a) \circledast (y, \ell_b)) \circledast (z, \ell_c) = (x * y, \ell_{a*b}) \circledast (z, \ell_c) \\
&= ((x * y) * z, \ell_{(a*b)*c}) \\
&= (x * (z * (0 * y)), \ell_{a*(c*(0*b))}) \\
&= (x, \ell_a) \circledast (z * (0 * y), \ell_{c*(0*b)}) \\
&= (x, \ell_a) \circledast ((z, \ell_c) \circledast (0 * y, \ell_{0*b})) \\
&= (x, \ell_a) \circledast ((z, \ell_c) \circledast ((0, \ell_0) \circledast (y, \ell_b)))
\end{aligned}$$

by (1), (2) and (4), respectively. Therefore $\langle(X, \ell_X), \circledast\rangle$ is a B-algebra with the special element $(0, \ell_0)$. □

We say that $\langle(X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a *paired B-algebra*.

Table 1: Cayley table for the binary operation “$*$”

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Example 3.2. *Let $X = \{0, a, b\}$ be a set with the binary operation “$*$” as in Table 1.*
For $Q = \{0, \frac{1}{2}, 1\}$, define a mapping ℓ as follows:

$$\ell : X \to Q,\ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = a, \\ 1 & \text{if } x = b. \end{cases}$$

Then

$$\begin{aligned}(X, \ell_X) &= \{(0, \ell_0), (0, \ell_a), (0, \ell_b), (a, \ell_0), (a, \ell_a), (a, \ell_b), (b, \ell_0), (b, \ell_a), (b, \ell_b)\} \\ &= \{(0, 0), (0, \tfrac{1}{2}), (0, 1), (a, 0), (a, \tfrac{1}{2}), (a, 1), (b, 0), (b, \tfrac{1}{2}), (b, 1)\}\end{aligned}$$

and the operation $\circledast$ is given by Table 2. It is routine to verify that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a paired B-algebra.

Table 2: Cayley table for the binary operation “$*$”

$\circledast$	$(0,\ell_0)$	$(0,\ell_a)$	$(0,\ell_b)$	(a,ℓ_0)	(a,ℓ_a)	(a,ℓ_b)	(b,ℓ_0)	(b,ℓ_a)	(b,ℓ_b)
$(0,\ell_0)$	$(0,0)$	$(0,1)$	$(0,\frac{1}{2})$	$(b,0)$	$(b,1)$	$(b,\frac{1}{2})$	$(a,0)$	$(a,1)$	$(a,\frac{1}{2})$
$(0,\ell_a)$	$(0,\frac{1}{2})$	$(0,0)$	$(0,1)$	$(b,\frac{1}{2})$	$(b,0)$	$(b,1)$	$(a,\frac{1}{2})$	$(a,0)$	$(a,1)$
$(0,\ell_b)$	$(0,1)$	$(0,\frac{1}{2})$	$(0,0)$	$(b,1)$	$(b,\frac{1}{2})$	$(b,0)$	$(a,1)$	$(a,\frac{1}{2})$	$(a,0)$
(a,ℓ_0)	$(a,0)$	$(a,1)$	$(a,\frac{1}{2})$	$(0,0)$	$(0,1)$	$(0,\frac{1}{2})$	$(b,0)$	$(b,1)$	$(b,\frac{1}{2})$
(a,ℓ_a)	$(a,\frac{1}{2})$	$(a,0)$	$(a,1)$	$(0,\frac{1}{2})$	$(0,0)$	$(0,1)$	$(b,\frac{1}{2})$	$(b,0)$	$(b,1)$
(a,ℓ_b)	$(a,1)$	$(a,\frac{1}{2})$	$(a,0)$	$(0,1)$	$(0,\frac{1}{2})$	$(0,0)$	$(b,1)$	$(b,\frac{1}{2})$	$(b,0)$
(b,ℓ_0)	$(b,0)$	$(b,1)$	$(b,\frac{1}{2})$	$(a,0)$	$(a,1)$	$(a,\frac{1}{2})$	$(0,0)$	$(0,1)$	$(0,\frac{1}{2})$
(b,ℓ_a)	$(b,\frac{1}{2})$	$(b,0)$	$(b,1)$	$(a,\frac{1}{2})$	$(a,0)$	$(a,1)$	$(0,\frac{1}{2})$	$(0,0)$	$(0,1)$
(b,ℓ_b)	$(b,1)$	$(b,\frac{1}{2})$	$(b,0)$	$(a,1)$	$(a,\frac{1}{2})$	$(a,0)$	$(0,1)$	$(0,\frac{1}{2})$	$(0,0)$

Proposition 3.3. *If $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a paired B-algebra, then*

(i) $(x, \ell_a) \circledast (y, \ell_b) = (x, \ell_a) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (y, \ell_b)))$,

(ii) $((x, \ell_a) \circledast (y, \ell_b)) \circledast ((0, \ell_0) \circledast (y, \ell_b)) = (x, \ell_a)$,

(iii) $(x, \ell_a) \circledast (z, \ell_c) = (y, \ell_b) \circledast (z, \ell_c)$ *implies* $(x, \ell_a) = (y, \ell_b)$,

(iv) $(x, \ell_a) \circledast ((y, \ell_b) \circledast (z, \ell_c)) = ((x, \ell_a) \circledast ((0, \ell_0) \circledast (z, \ell_c))) \circledast (y, \ell_b)$,

(v) $(x, \ell_a) \circledast (y, \ell_b) = (0, \ell_0)$ *implies* $(x, \ell_a) = (y, \ell_b)$,

(vi) $(0, \ell_0) \circledast (x, \ell_a) = (0, \ell_0) \circledast (y, \ell_b)$ *implies* $(x, \ell_a) = (y, \ell_b)$,

(vii) $(0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a)) = (x, \ell_a)$

for all $x, y, z, a, b, c \in X$.

Proof. Let $x, y, z, a, b, c \in X$. Then

$$\begin{aligned}(x, \ell_a) \circledast (y, \ell_b) &= (x * y, \ell_{a*b}) = ((x * y) * 0, \ell_{(a*b)*0}) \\ &= (x * (0 * (0 * y)), \ell_{a*(0*(0*b))}) \\ &= (x, \ell_a) \circledast (0 * (0 * y), \ell_{0*(0*b)}) \\ &= (x, \ell_a) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (y, \ell_b)))\end{aligned}$$

which proves (i).

(ii) We have

$$\begin{aligned}&((x, \ell_a) \circledast (y, \ell_b)) \circledast ((0, \ell_0) \circledast (y, \ell_b)) = (x * y, \ell_{a*b}) \circledast (0 * y, \ell_{0*b}) \\ &= ((x * y) * (0 * y), \ell_{(a*b)*(0*b)}) \\ &= (x * ((0 * y) * (0 * y)), \ell_{a*((0*b)*(0*b))}) \\ &= (x * 0, \ell_{a*0}) = (x, \ell_a).\end{aligned}$$

(iii) Assume that $(x, \ell_a) \circledast (z, \ell_c) = (y, \ell_b) \circledast (z, \ell_c)$. It follows from (ii) that

$$\begin{aligned}(x, \ell_a) &= ((x, \ell_a) \circledast (z, \ell_c)) \circledast ((0, \ell_0) \circledast (z, \ell_c)) \\ &= ((y, \ell_b) \circledast (z, \ell_c)) \circledast ((0, \ell_0) \circledast (z, \ell_c)) \\ &= (y, \ell_b).\end{aligned}$$

(iv) Using (i), we have

$$\begin{aligned}&((x, \ell_a) \circledast ((0, \ell_0) \circledast (z, \ell_c))) \circledast (y, \ell_b) \\ &= (x, \ell_a) \circledast ((y, \ell_b) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (z, \ell_c)))) \\ &= (x, \ell_a) \circledast ((y, \ell_b) \circledast (z, \ell_c)).\end{aligned}$$

(v) Suppose that $(x, \ell_a) \circledast (y, \ell_b) = (0, \ell_0)$. Then $(x, \ell_a) \circledast (y, \ell_b) = (y, \ell_b) \circledast (y, \ell_b)$, and so $(x, \ell_a) = (y, \ell_b)$ by (iii).

(vi) Assume that $(0, \ell_0) \circledast (x, \ell_a) = (0, \ell_0) \circledast (y, \ell_b)$. Then

$$\begin{aligned}(0, \ell_0) &= (x, \ell_a) \circledast (x, \ell_a) = (x, \ell_a) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a))) \\ &= (x, \ell_a) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (y, \ell_b))) \\ &= (x, \ell_a) \circledast (y, \ell_b)\end{aligned}$$

and so $(x, \ell_a) = (y, \ell_b)$ by (v).

(vii) We have

$$\begin{aligned}(0, \ell_0) \circledast (x, \ell_a) &= ((0, \ell_0) \circledast (x, \ell_a)) \circledast (0, \ell_0) \\ &= ((0 * x) * 0, \ell_{(0*a)*0}) \\ &= (0 * (0 * (0 * x)), \ell_{0*(0*(0*a))}) \\ &= (0, \ell_0) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a)))\end{aligned}$$

and so $(0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a)) = (x, \ell_a)$ by (vi). □

We provide conditions for two dimensional X-eventful algebra to be a B-algebra.

Theorem 3.4. *For an algebra $(X, *, 0)$, the two dimensional X-eventful algebra $\langle (X, \ell_X), \circledast \rangle$ is a B-algebra with the special element $(0, \ell_0)$ if and only if it satisfies* Proposition 3.3(vii) *and*

$$(x, \ell_a) \circledast (x, \ell_a) = (0, \ell_0), \tag{11}$$

$$((x, \ell_a) \circledast (z, \ell_c)) \circledast ((y, \ell_b) \circledast (z, \ell_c)) = (x, \ell_a) \circledast (y, \ell_b) \tag{12}$$

for all $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$.

Proof. Assume that the two dimensional X-eventful algebra $\langle (X, \ell_X), \circledast \rangle$ is a B-algebra with the special element $(0, \ell_0)$. The condition Proposition 3.3(vii) is by Proposition 3.3. It is clear that (11) is true by the definition of B-algebra. Also, we have

$$\begin{aligned}&((x,\ell_a)\circledast(z,\ell_c))\circledast((y,\ell_b)\circledast(z,\ell_c)) = (x,\ell_a)\circledast(((y,\ell_b)\circledast(z,\ell_c))\circledast((0,\ell_0)\circledast(z,\ell_c)))\\&= (x,\ell_a)\circledast((y,\ell_b)\circledast(((0,\ell_0)\circledast(z,\ell_c))\circledast((0,\ell_0)\circledast(z,\ell_c))))\\&= (x,\ell_a)\circledast((y,\ell_b)\circledast(0,\ell_0)) = (x,\ell_a)\circledast(y,\ell_b)\end{aligned}$$

for all $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$.

Conversely, suppose that $\langle (X, \ell_X), \circledast \rangle$ satisfies three conditions (11), (12) and Proposition 3.3(vii). Then

$$\begin{aligned}(x,\ell_a) &= (0,\ell_0)\circledast((0,\ell_0)\circledast(x,\ell_a))\\&= ((x,\ell_a)\circledast(x,\ell_a))\circledast((0,\ell_0)\circledast(x,\ell_a))\\&= (x,\ell_a)\circledast(0,\ell_0)\end{aligned}$$

and

$$((x,\ell_a)\circledast(y,\ell_b))\circledast((0,\ell_0)\circledast(y,\ell_b)) = (x,\ell_a). \tag{13}$$

Combining (12) with (13) induces

$$\begin{aligned}&(x,\ell_a)\circledast((z,\ell_c)\circledast((0,\ell_0)\circledast(y,\ell_b)))\\&= (((x,\ell_a)\circledast(y,\ell_b)) * ((0,\ell_0)\circledast(y,\ell_b)))\circledast((z,\ell_c)\circledast((0,\ell_0)\circledast(y,\ell_b)))\\&= ((x,\ell_a)\circledast(y,\ell_b))\circledast(z,\ell_c)\end{aligned}$$

for all $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$. Therefore $\langle (X, \ell_X), \circledast \rangle$ is a B-algebra with the special element $(0, \ell_0)$. □

Theorem 3.5. *For an algebra $(X, *, 0)$, the two dimensional X-eventful algebra $\langle (X, \ell_X), \circledast \rangle$ is a B-algebra with the special element $(0, \ell_0)$ if and only if it satisfies (11) and*

$$(x,\ell_a)\circledast((((0,\ell_0)\circledast(y,\ell_b))\circledast(z,\ell_c))\circledast(((0,\ell_0)\circledast(x,\ell_a))\circledast(z,\ell_c))) = (y,\ell_b) \tag{14}$$

for all $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$.

Proof. Assume that the two dimensional X-eventful algebra $\langle (X, \ell_X), \circledast \rangle$ is a B-algebra with the special element $(0, \ell_0)$. Then (11) is valid in Theorem 3.4. Using (12), we get

$$\begin{aligned}&(((0,\ell_0)\circledast(y,\ell_b))\circledast(z,\ell_c))\circledast(((0,\ell_0)\circledast(x,\ell_a))\circledast(z,\ell_c))\\&= ((0,\ell_0)\circledast(y,\ell_b))\circledast((0,\ell_0)\circledast(x,\ell_a)).\end{aligned} \tag{15}$$

It follows that

$$\begin{aligned}&(x,\ell_a)\circledast((((0,\ell_0)\circledast(y,\ell_b))\circledast(z,\ell_c))\circledast(((0,\ell_0)\circledast(x,\ell_a))\circledast(z,\ell_c)))\\&= (x,\ell_a)\circledast(((0,\ell_0)\circledast(y,\ell_b))\circledast((0,\ell_0)\circledast(x,\ell_a)))\\&= ((x,\ell_a)\circledast(x,\ell_a))\circledast((0,\ell_0)\circledast(y,\ell_b))\\&= (0,\ell_0)\circledast((0,\ell_0)\circledast(y,\ell_b))\\&= (y,\ell_b)\end{aligned}$$

which proves (14).

Conversely, suppose that $\langle (X, \ell_X), \circledast \rangle$ satisfies (11) and (14). If we substitute (y, ℓ_b) for (x, ℓ_a) in (14) and use (11), then

$$\begin{aligned}(x,\ell_a) &= (x,\ell_a)\circledast((((0,\ell_0)\circledast(x,\ell_a))\circledast(z,\ell_c))\circledast(((0,\ell_0)\circledast(x,\ell_a))\circledast(z,\ell_c)))\\&= (x,\ell_a)\circledast(0,\ell_0)\end{aligned} \tag{16}$$

If we put $(x, \ell_a) = (0, \ell_0) = (z, \ell_c)$ and $(y, \ell_a) = (x, \ell_a)$ in (14), then

$$\begin{aligned}(x, \ell_a) &= (0, \ell_0) \circledast ((((0, \ell_0) \circledast (x, \ell_a)) \circledast (0, \ell_0)) \circledast (((0, \ell_0) \circledast (0, \ell_0)) \circledast (0, \ell_0))) \\ &= (0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a))\end{aligned} \tag{17}$$

by (16). Assume that $(0, \ell_0) \circledast (x, \ell_a) = (0, \ell_0) \circledast (y, \ell_b)$. Then

$$(x, \ell_a) = (0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a)) = (0, \ell_0) \circledast ((0, \ell_0) \circledast (y, \ell_b)) = (y, \ell_b)$$

by (17) which proves

$$(0, \ell_0) \circledast (x, \ell_a) = (0, \ell_0) \circledast (y, \ell_b) \;\Rightarrow\; (x, \ell_a) = (y, \ell_b). \tag{18}$$

Putting $(x, \ell_a) = (0, \ell_0)$, $(y, \ell_b) = (0, \ell_0) \circledast (y', \ell_{b'})$ and $(z, \ell_c) = (z', \ell_{c'})$ in (14) induces

$$\begin{aligned}(0, \ell_0) \circledast (y', \ell_{b'}) &= (0, \ell_0) \circledast ((((0, \ell_0) \circledast ((0, \ell_0) \circledast (y', \ell_{b'})) \circledast (z', \ell_{c'}))\circledast \\ &\quad (((0, \ell_0) \circledast (0, \ell_0)) \circledast (z', \ell_{c'}))) \\ &= (0, \ell_0) \circledast (((y', \ell_{b'}) \circledast (z', \ell_{c'})) \circledast ((0, \ell_0) \circledast (z', \ell_{c'}))).\end{aligned}$$

It follows from (18) that

$$(y', \ell_{b'}) = (((y', \ell_{b'}) \circledast (z', \ell_{c'})) \circledast ((0, \ell_0) \circledast (z', \ell_{c'}))). \tag{19}$$

If we substitute (x, ℓ_a), (y, ℓ_b) and (z, ℓ_c) for $(x', \ell_{a'})$, $(0, \ell_0) \circledast ((z', \ell_{c'}) \circledast (x', \ell_{a'}))$ and $(0, \ell_0)$, respectively, in (14), then

$$\begin{aligned}&(0, \ell_0) \circledast ((z', \ell_{c'}) \circledast (x', \ell_{a'})) \\ &= (x', \ell_{a'}) \circledast ((((0, \ell_0) \circledast ((0, \ell_0) \circledast ((z', \ell_{c'}) \circledast (x', \ell_{a'})))) \circledast (0, \ell_0)) \\ &\quad \circledast (((0, \ell_0) \circledast (x', \ell_{a'})) \circledast (0, \ell_0))) \\ &= (x', \ell_{a'}) \circledast (((z', \ell_{c'}) \circledast (x', \ell_{a'})) \circledast ((0, \ell_0) \circledast (x', \ell_{a'}))) \\ &= ((x', \ell_{a'}) \circledast (z', \ell_{c'}))\end{aligned} \tag{20}$$

In (14), taking $(x, \ell_a) = (y', \ell_{b'}) \circledast (w, \ell_d)$, $(y, \ell_b) = (y', \ell_{b'}) \circledast (z', \ell_{c'})$ and $(z, \ell_c) = (0, \ell_0) \circledast (y', \ell_{b'})$ imply that

$$\begin{aligned}(y', \ell_{b'}) \circledast (z', \ell_{c'}) &= ((y', \ell_{b'}) \circledast (w, \ell_d)) \circledast [(((0, \ell_0) \circledast ((y', \ell_{b'}) \circledast (z', \ell_{c'}))) \\ &\quad \circledast ((0, \ell_0) \circledast (y', \ell_{b'}))) \circledast (((0, \ell_0) \circledast ((y', \ell_{b'}) \circledast (w, \ell_d))) \\ &\quad \circledast ((0, \ell_0) \circledast (y', \ell_{b'})))]\end{aligned} \tag{21}$$

Using (19) and (20), we get

$$\begin{aligned}&((0, \ell_0) \circledast ((y', \ell_{b'}) \circledast (z', \ell_{c'}))) \circledast ((0, \ell_0) \circledast (y', \ell_{b'})) \\ &= ((z', \ell_{c'}) \circledast (y', \ell_{b'})) \circledast ((0, \ell_0) \circledast (y', \ell_{b'})) = (z', \ell_{c'}).\end{aligned} \tag{22}$$

Similarly, we have

$$((0, \ell_0) \circledast ((y', \ell_{b'}) \circledast (w, \ell_d))) \circledast ((0, \ell_0) \circledast (y', \ell_{b'})) = (w, \ell_d). \tag{23}$$

Combining (21), (22) and (23) induces

$$(y', \ell_{b'}) \circledast (z', \ell_{c'}) = ((y', \ell_{b'}) \circledast (w, \ell_d)) \circledast ((z', \ell_{c'}) \circledast (w, \ell_d)).$$

Therefore $\langle (X, \ell_X), \circledast \rangle$ is a B-algebra with the special element $(0, \ell_0)$ by Theorem 3.4. $\square$

The following theorem shows the process of inducing a paired B-algebra using a group.

Theorem 3.6. *If $(X, \circ, 0)$ is a group, then the two dimensional X-eventful algebra $\langle (X,\, \ell_X),\, \circledast,\, (0, \ell_0)\rangle$ is a paired B-algebra where*

$$\circledast : (X, \ell_X) \times (X, \ell_X) \to (X, \ell_X),\ \ ((x, \ell_a), (y, \ell_b)) \mapsto \left(x \circ y^{-1}, \ell_{a \circ b^{-1}}\right) = (x * y, \ell_{a*b}).$$

Proof. Let $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$. Then $(x, \ell_a) \circledast (x, \ell_a) = \left(x \circ x^{-1}, \ell_{a\circ a^{-1}}\right) = (0, \ell_0)$ and $(x, \ell_a) \circledast (0, \ell_0) = \left(x \circ 0^{-1}, \ell_{a \circ 0^{-1}}\right) = (x \circ 0, \ell_{a\circ 0}) = (x, \ell_a)$. Also

$$\begin{aligned} ((x, \ell_a) \circledast (y, \ell_b)) \circledast (z, \ell_c) &= \left(x \circ y^{-1}, \ell_{a\circ b^{-1}}\right) \circledast (z, \ell_c) \\ &= \left((x \circ y^{-1}) \circ z^{-1}, \ell_{(a\circ b^{-1})\circ c^{-1}}\right) \\ &= \left(x \circ (z \circ y)^{-1}, \ell_{a\circ (c\circ b)^{-1}}\right) \end{aligned}$$

and

$$\begin{aligned} &(x, \ell_a) \circledast ((z, \ell_c) \circledast ((0, \ell_0) \circledast (y, \ell_b))) = (x, \ell_a) \circledast \left((z, \ell_c) \circledast (0 \circ y^{-1}, \ell_{0\circ b^{-1}})\right) \\ &= (x, \ell_a) \circledast \left((z, \ell_c) \circledast (y^{-1}, \ell_{b^{-1}})\right) = (x, \ell_a) \circledast \left(z \circ (y^{-1})^{-1}, \ell_{c\circ (b^{-1})^{-1}}\right) \\ &= (x, \ell_a) \circledast (z \circ y, \ell_{c\circ b}) = \left(x \circ (z \circ y)^{-1}, \ell_{a\circ(c\circ b)^{-1}}\right). \end{aligned}$$

Hence $((x, \ell_a) \circledast (y, \ell_b)) \circledast (z, \ell_c) = (x, \ell_a) \circledast ((z, \ell_c) \circledast ((0, \ell_0) \circledast (y, \ell_b)))$. Therefore $\langle (X,\, \ell_X),\, \circledast,\, (0, \ell_0)\rangle$ is a paired B-algebra. □

Let $X := (X, *, 0)$ be an algebra. In a two dimensional X-eventful algebra $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$, we consider the following assertions.

$$(((x, \ell_a) \circledast (y, \ell_b)) \circledast ((x, \ell_a) \circledast (z, \ell_c))) \circledast ((z, \ell_c) \circledast (y, \ell_b)) = (0, \ell_0), \tag{24}$$

$$((x, \ell_a) \circledast ((x, \ell_a) \circledast (y, \ell_b))) \circledast (y, \ell_b) = (0, \ell_0), \tag{25}$$

$$(0, \ell_0) \circledast (x, \ell_a) = (0, \ell_0), \tag{26}$$

$$(x, \ell_a) \circledast (y, \ell_b) = (0, \ell_0),\ (y, \ell_b) \circledast (x, \ell_a) = (0, \ell_0)\ \Rightarrow\ (x, \ell_a) = (y, \ell_b) \tag{27}$$

for all $x, y, z, a, b, c \in X$.

Definition 3.7. *Given an algebra $X := (X, *, 0)$, let $\langle (X, \ell_X), \circledast\rangle$ be a two dimensional X-eventful algebra with a special element $(0, \ell_0)$. If it satisfies (11), (24) and (25), we say that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a two dimensional BCI-eventful algebra. If a two dimensional BCI-eventful algebra $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ satisfies the condition (26), it is called a two dimensional BCK-eventful algebra.*

Example 3.8. *(1) Consider an algebra $X := (X, *, 0)$ where $X = \{0, a\}$ and the binary operation "$*$" is given by Table 3. Given a set $Q = \{\alpha, \beta\}$, define a mapping ℓ as follows:*

Table 3: Cayley table for the binary operation "$*$"

$*$	0	a
0	0	0
a	a	0

$$\ell : X \to Q,\ x \mapsto \begin{cases} \alpha & \text{if } x = 0, \\ \beta & \text{if } x = a. \end{cases}$$

Then $(X, \ell_X) = \{(0, \ell_0), (0, \ell_a), (a, \ell_0), (a, \ell_a)\} = \{(0, \alpha), (0, \beta), (a, \alpha), (a, \beta)\}$ and the operation $\circledast$ is given by Table 4. It is routine to verify that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a two dimensional BCK-eventful algebra.

(2) The paired B-algebra $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ in Example 3.2 is two dimensional BCI-eventful algebra.

Table 4: Cayley table for the binary operation "$*$"

$\circledast$	$(0,\ell_0)$	$(0,\ell_a)$	(a,ℓ_0)	(a,ℓ_a)
$(0,\ell_0)$	$(0,\alpha)$	$(0,\alpha)$	$(0,\alpha)$	$(0,\alpha)$
$(0,\ell_a)$	$(0,\beta)$	$(0,\alpha)$	$(0,\beta)$	$(0,\alpha)$
(a,ℓ_0)	(a,α)	(a,α)	$(0,\alpha)$	$(0,\alpha)$
(a,ℓ_a)	(a,β)	(a,α)	$(0,\beta)$	$(0,\alpha)$

Table 5: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

In general, two dimensional BCK/BCI-eventful algebra $\langle (X,\ell_X),\circledast,(0,\ell_0)\rangle$ does not satisfy the condition (27) as seen in the following examples.

Example 3.9. *Consider an algebra $X = (X,*,0)$ where $X = \{0,a,b,c\}$ and the binary operation "$*$" is given by Table 5.*
Given a set $Q = \{0.2, 0.5, 0.7\}$, define a mapping ℓ as follows:

$$\ell : X \to Q,\ x \mapsto \begin{cases} 0.2 & \text{if } x \in \{0,a\} \\ 0.7 & \text{if } x \in \{b,c\}. \end{cases}$$

Then the two dimensional X-event set is given as follows:

$$(X,\ell_X) = \{(0,0.2),(0,0.7),(a,0.2),(a,0.7),(b,0.2),(b,0.7),(c,0.2),(c,0.7)\}$$

and it is routine to check that $\langle (X,\ell_X),\circledast,(0,\ell_0)\rangle$ is a two dimensional BCI-eventful algebra. But it is not a two dimensional BCK-eventful algebra since $(0,\ell_0)\circledast(b,\ell_c) = (c,\ell_b) \neq (0,\ell_0)$. Note that $(0,\ell_0)\circledast(0,\ell_b) = (0,\ell_c) = (0,0.7)$ and $(0,\ell_b)\circledast(0,\ell_0) = (0,\ell_b) = (0,0.7)$, but $(0,\ell_0) \neq (0,\ell_b)$. Hence $\langle (X,\ell_X),\circledast,(0,\ell_0)\rangle$ does not satisfy the condition (27).

Lemma 3.10. *If $X := (X,*,0)$ is a BCK/BCI-algebra, then $\langle (X,\ell_X),\circledast,(0,\ell_0)\rangle$ is a two dimensional BCK/BCI-eventful algebra.*

Proof. Straightforward. □

By a *paired BCK/BCI-algebra* we mean a two dimensional BCK/BCI-eventful algebra $\langle (X,\ell_X),\ \circledast,\ (0,\ell_0)\rangle$ which satisfies the condition (27).

Example 3.11. *(1) Consider a BCK-algebra $X = (X,*,0)$ where $X = \{0,a,b,c\}$ and the binary operation "$*$" is given by Table 6.*
Given a nonempty set Q, define a mapping ℓ as follows:

$$\ell : X \to Q,\ x \mapsto u.$$

Then $(X,\ell_X) = \{(0,u),(a,u),(b,u),(c,u)\}$, and it is clear that $\langle (X,\ell_X),\ \circledast,\ (0,u)\rangle$ is a two dimensional BCK-eventful algebra by Lemma 3.10. It is routine to verify that $\langle (X,\ell_X),\ \circledast,\ (0,u)\rangle$ satisfies the condition (27)*. Hence it is a paired BCK-algebra.*

Table 6: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

(2) Consider the algebra $X = (X, *, 0)$ in Example 3.9 and let ℓ be a mapping from X to a nonempty set Q given by $\ell(x) = v \in Q$ for all $x \in X$. Then $(X, \ell_X) = \{(0, v), (a, v), (b, v), (c, v)\}$, we recall that $X = (X, *, 0)$ is a BCI-algebra, and thus $\langle (X, \ell_X), \circledast, (0, v)\rangle$ is a two dimensional BCI-eventful algebra by Lemma 3.10. It is routine to verify that $\langle (X, \ell_X), \circledast, (0, u)\rangle$ satisfies the condition (27). *Hence it is a paired BCI-algebra.*

We consider a generalization of Example 3.11.

Theorem 3.12. *Let $X = (X, *, 0)$ be a BCK/BCI-algebra. Given a nonempty set Q, let $\ell : X \to Q$ be a constant mapping, say $\ell(x) = q$ for all $x \in X$. Then $\langle (X, \ell_X), \circledast, (0, q)\rangle$ is a paired BCK/BCI-algebra.*

Proof. Using Lemma 3.10, we know that $\langle (X, \ell_X), \circledast, (0, q)\rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $x, y, a, b \in X$ be such that $(x, \ell_a) \circledast (y, \ell_b) = (0, q)$ and $(y, \ell_b) \circledast (x, \ell_a) = (0, q)$. Then $(0, q) = (x, \ell_a) \circledast (y, \ell_b) = (x * y, \ell_{a*b}) = (x * y, q)$ and $(0, q) = (y, \ell_b) \circledast (x, \ell_a) = (y * x, \ell_{b*a}) = (y * x, q)$. It follows that $x * y = 0$ and $y * x = 0$. Hence $x = y$, and so $(x, \ell_a) = (x, q) = (y, q) = (y, \ell_b)$. This shows that $\langle (X, \ell_X), \circledast, (0, u)\rangle$ satisfies the condition (27). Therefore it is a paired BCK/BCI-algebra. □

Theorem 3.12 shows that it can induce many other BCK/BCI-algebras from given a BCK/BCI-algebra. These induced BCK/BCI-algebras are isomorphic each other. Thus a BCK/BCI-algebra induce a unique paired BCK/BCI-algebra up to isomorphism.

Theorem 3.13. *Let $X = (X, *, 0)$ be a BCK/BCI-algebra. Given a nonempty set Q, if a mapping $\ell : X \to Q$ is ono-to-one, then $\langle (X, \ell_X), \circledast, (0, \ell_0))\rangle$ is a paired BCK/BCI-algebra.*

Proof. Using Lemma 3.10, we know that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $x, y, a, b \in X$ be such that $(x, \ell_a) \circledast (y, \ell_b) = (0, \ell_0)$ and $(y, \ell_b) \circledast (x, \ell_a) = (0, \ell_0)$. Then $(0, \ell_0) = (x, \ell_a) \circledast (y, \ell_b) = (x * y, \ell_{a*b})$ and $(0, \ell_0) = (y, \ell_b) \circledast (x, \ell_a) = (y * x, \ell_{b*a})$. It follows that $x * y = 0$, $y * x = 0$, $\ell_{a*b} = \ell_0$ and $\ell_{b*a} = \ell_0$. Since ℓ is one-to-one, we have $a * b = 0$ and $b * a = 0$. It follows that $x = y$ and $a = b$. Hence $(x, \ell_a) = (y, \ell_b)$. This shows that $\langle (X, \ell_X), \circledast, (0, \ell_0\rangle$ satisfies the condition (27). Therefore it is a paired BCK/BCI-algebra. □

Theorem 3.14. *Let $X = (X, *, 0)$ be a BCK/BCI-algebra. Given a nonempty set Q, if a mapping $\ell : X \to Q$ satisfies $\ell^{-1}(\ell_0) = \{0\}$, then $\langle (X, \ell_X), \circledast, (0, \ell_0))\rangle$ is a paired BCK/BCI-algebra.*

Proof. Using Lemma 3.10, we know that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $x, y, a, b \in X$ be such that $(x, \ell_a) \circledast (y, \ell_b) = (0, \ell_0)$ and $(y, \ell_b) \circledast (x, \ell_a) = (0, \ell_0)$. Then $(0, \ell_0) = (x, \ell_a) \circledast (y, \ell_b) = (x * y, \ell_{a*b})$ and $(0, \ell_0) = (y, \ell_b) \circledast (x, \ell_a) = (y * x, \ell_{b*a})$. It follows that $x * y = 0$, $y * x = 0$, $\ell_{a*b} = \ell_0$ and $\ell_{b*a} = \ell_0$. Hence $a * b \in \ell^{-1}(\ell_0) = \{0\}$ and $b * a \in \ell^{-1}(\ell_0) = \{0\}$ which shows that $a * b = 0$ and $b * a = 0$. It follows that $x = y$ and $a = b$. Hence $(x, \ell_a) = (y, \ell_b)$. This shows that $\langle (X, \ell_X), \circledast, (0, \ell_0\rangle$ satisfies the condition (27). Therefore it is a paired BCK/BCI-algebra. □

Lemma 3.15 ([3]). *Given a BCI-algebra $X = (X, *, 0)$, the following are equivalent,*

(i) *X is p-semisimple.*

(ii) *$x * (0 * y) = y * (0 * x)$ for all $x, y \in X$.*

Theorem 3.16. *The two dimensional BCI-eventful algebra* $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ *induced by a p-semisimple BCI-algebra* $X = (X, *, 0)$ *is a commutative group under the operation* $\odot$ *which is given by*

$$\odot : (X, \ell_X) \times (X, \ell_X) \to (X, \ell_X),\ ((x, \ell_a), (y, \ell_b)) \mapsto (x, \ell_a) \circledast ((0, \ell_0) \circledast (y, \ell_b)).$$

Proof. By Lemma 3.10, we know that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a two dimensional BCI-eventful algebra. Let $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$. Then

$$\begin{aligned}(x, \ell_a) \odot (y, \ell_b) &= (x, \ell_a) \circledast ((0, \ell_0) \circledast (y, \ell_b)) = (x, \ell_a) \circledast (0 * y, \ell_{0*b}) \\ &= (x * (0 * y), \ell_{a*(0*b)}) = (y * (0 * x), \ell_{b*(0*a)}) \\ &= (y, \ell_b) \circledast (0 * x, \ell_{0*a}) = (y, \ell_b) \circledast ((0, \ell_0) \circledast (x, \ell_a)) \\ &= (y, \ell_b) \odot (x, \ell_a)\end{aligned} \tag{28}$$

by Lemma 3.15, and

$$\begin{aligned}&((y, \ell_b) \odot (z, \ell_c)) \odot (x, \ell_a) \\ &= ((y, \ell_b) \circledast ((0, \ell_0) \circledast (z, \ell_c))) \circledast ((0, \ell_0) \circledast (x, \ell_a)) \\ &= \big((y * (0 * z)) * (0 * x), \ell_{(b*(0*c))*(0*a)}\big) \\ &= \big((y * (0 * x)) * (0 * z), \ell_{(b*(0*a))*(0*c)}\big) \\ &= ((y, \ell_b) \circledast ((0, \ell_0) \circledast (x, \ell_a))) \circledast ((0, \ell_0) \circledast (z, \ell_c)) \\ &= ((y, \ell_b) \odot (x, \ell_a)) \odot (z, \ell_c).\end{aligned} \tag{29}$$

Using (28) and (29), we get

$$(x, \ell_a) \odot ((y, \ell_b) \odot (z, \ell_c)) = ((y, \ell_b) \odot (z, \ell_c)) \odot (x, \ell_a) = ((y, \ell_b) \odot (x, \ell_a)) \odot (z, \ell_c) = ((x, \ell_a) \odot (y, \ell_b)) \odot (z, \ell_c).$$

Now,

$$\begin{aligned}(0, \ell_0) \odot (x, \ell_a) &= (0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a)) \\ &= (0, \ell_0) \circledast (0 * x, \ell_{0*a}) \\ &= (0 * (0 * x), \ell_{0*(0*a)}) = (x, \ell_a),\end{aligned}$$

which shows that $(0, \ell_0)$ is the identity element of (X, ℓ_X). Finally, we show that $(0, \ell_0) \circledast (x, \ell_a)$ is the inverse of any element (x, ℓ_a). In fact,

$$\begin{aligned}&(x, \ell_a) \odot ((0, \ell_0) \circledast (x, \ell_a)) = (x, \ell_a) \circledast ((0, \ell_0) \circledast ((0, \ell_0) \circledast (x, \ell_a))) \\ &= \big(x * (0 * (0 * x)), \ell_{a*(0*(0*a))}\big) \\ &= (x * x, \ell_{a*a}) = (0, \ell_0).\end{aligned}$$

Therefore $\langle (X, \ell_X), \odot, (0, \ell_0)\rangle$ is a commutative group. □

Corollary 3.17. *Let* $X = (X, *, 0)$ *be a BCI-algebra which satisfies any one of the following assertions.*

$$(\forall x \in X)(0 * x = 0 \ \Rightarrow\ x = 0), \tag{30}$$
$$(\forall a \in X)(X = \{a * x \mid x \in X\}), \tag{31}$$
$$(\forall a, x \in X)(a * (a * x) = x), \tag{32}$$
$$(\forall a, x, y, z \in X)((x * y) * (z * a) = (x * z) * (y * a)), \tag{33}$$
$$(\forall x, y \in X)(0 * (y * x) = x * y), \tag{34}$$
$$(\forall x, y, z \in X)((x * y) * (x * z) = z * y). \tag{35}$$

Then the two dimensional BCI-eventful algebra $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ *is a commutative group under the operation* $\odot$.

Theorem 3.18. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ satisfies the condition (27), then $\langle (Y, \zeta_Y), \circledast, (0, \zeta_0)\rangle$ is a paired BCK/BCI-algebra where ζ is a mapping from Y to Q.*

Proof. By Lemma 3.10, we know that $\langle (X, \ell_X), \circledast, (0, \ell_0)\rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $(x', \zeta_{a'}), (y', \zeta_{b'}), (z', \zeta_{c'}) \in (Y, \zeta_Y)$. Then there exist $x, y, z, a, b, c \in X$ such that $f(x) = x'$, $f(y) = y'$, $f(z) = z'$, $f(a) = a'$, $f(b) = b'$ and $f(c) = c'$. Hence

$$\begin{aligned}
&(((x', \zeta_{a'}) \circledast (y', \zeta_{b'})) \circledast ((x', \zeta_{a'}) \circledast (z', \zeta_{c'}))) \circledast ((z', \zeta_{c'}) \circledast (y', \zeta_{b'}))\\
&= (((f(x), \zeta_{f(a)}) \circledast (f(y), \zeta_{f(b)})) \circledast ((f(x), \zeta_{f(a)}) \circledast (f(z), \zeta_{f(c)}))) \circledast ((f(z), \zeta_{f(c)}) \circledast (f(y), \zeta_{f(b)}))\\
&= ((f(x) * f(y), \zeta_{f(a)*f(b)}) \circledast (f(x) * f(z), \zeta_{f(a)*f(c)})) \circledast (f(z) * f(y), \zeta_{f(c)*f(b)})\\
&= ((f(x * y), \zeta_{f(a*b)}) \circledast (f(x * z), \zeta_{f(a*c)})) \circledast (f(z * y), \zeta_{f(c*b)})\\
&= (f(x * y) * f(x * z), \zeta_{f(a*b)*f(a*c)}) \circledast (f(z * y), \zeta_{f(c*b)})\\
&= (f(((x * y) * (x * z)) * (z * y)), \zeta_{f(((a*b)*(a*c))*(c*b))})\\
&= (f(0), \zeta_{f(0)}) = (0, \zeta_0),
\end{aligned}$$

$$\begin{aligned}
&(((x', \zeta_{a'}) \circledast ((x', \zeta_{a'}) \circledast (y', \zeta_{b'}))) \circledast (y', \zeta_{b'}))\\
&= (((f(x), \zeta_{f(a)}) \circledast ((f(x), \zeta_{f(a)}) \circledast (f(y), \zeta_{f(b)}))) \circledast (f(y), \zeta_{f(b)}))\\
&= ((f(x), \zeta_{f(a)}) \circledast (f(x) * f(y), \zeta_{f(a)*f(b)})) \circledast (f(x), \zeta_{f(a)})\\
&= ((f(x), \zeta_{f(a)}) \circledast (f(x * y), \zeta_{f(a*b)})) \circledast (f(x), \zeta_{f(a)})\\
&= (f(x * (x * y)), \zeta_{f(a*(a*b))}) \circledast (f(x), \zeta_{f(a)})\\
&= (f((x * (x * y)) * y), \zeta_{f((a*(a*b)))*b})\\
&= (f(0), \zeta_{f(0)}) = (0, \zeta_0),
\end{aligned}$$

and $(x', \zeta_{a'}) \circledast (x', \zeta_{a'}) = (f(x), \zeta_{f(a)}) \circledast (f(x), \zeta_{f(a)}) = (f(x * x), \zeta_{f(a*a)}) = (f(0), \zeta_{f(0)}) = (0, \zeta_0)$. Hence $\langle (Y, \zeta_Y), \circledast, (0, \zeta_0)\rangle$ is a two dimensional BCI-eventful algebra. Since $(0, \zeta_0) \circledast (x', \zeta_{a'}) = (f(0), \zeta_{f(0)}) \circledast (f(x), \zeta_{f(a)}) = (f(0 * x), \zeta_{f(0*a)}) = (f(0), \zeta_{f(0)}) = (0, \zeta_0)$, we know that $\langle (Y, \zeta_Y), \circledast, (0, \zeta_0)\rangle$ is a two dimensional BCK-eventful algebra. Assume that $(x', \zeta_{a'}) \circledast (y', \zeta_{b'}) = (0, \zeta_0)$ and $(y', \zeta_{b'}) \circledast (x', \zeta_{a'}) = (0, \zeta_0)$. Then

$$(0, \zeta_0) = (x', \zeta_{a'}) \circledast (y', \zeta_{b'}) = (f(x), \zeta_{f(a)}) \circledast (f(y), \zeta_{f(b)}) = (f(x) * f(y), \zeta_{f(a)*f(b)})$$

and

$$(0, \zeta_0) = (y', \zeta_{b'}) \circledast (x', \zeta_{a'}) = (f(y), \zeta_{f(b)}) \circledast (f(x), \zeta_{f(a)}) = (f(y) * f(x), \zeta_{f(b)*f(a)}),$$

which imply that $f(x) * f(y) = 0$, $f(y) * f(x) = 0$, $f(a) * f(b) = 0$ and $f(b) * f(a) = 0$. Hence $x' = f(x) = f(y) = y'$ and $a' = f(a) = f(b) = b'$. Therefore $(x', \zeta_{a'}) = (y', \zeta_{b'})$. Consequently, $\langle (Y, \zeta_Y), \circledast, (0, \zeta_0)\rangle$ is a paired BCK/BCI-algebra. □

4 Conclusions

We have introduced two-dimensional event sets and have applied it to algebraic structures. We have introduced the notions of two dimensional BCK/BCI-eventful algebra, paired B-algebra and paired BCK/BCI-algebra, and have investigated several properties. We have considered conditions for two dimensional eventful algebra to be a B-algebra and a BCK/BCI-algebra. We have discussed the process of inducing a paired B-algebra using a group, and have established a commutative group using two dimensional BCI-eventful algebra. We have presented examples to show that a two dimensional eventful BCK/BCI-algebra is not a BCK/BCI-algebra, and then we have considered conditions for a two dimensional eventful BCK/BCI-algebra to be a BCK/BCI-algebra. We have studied a paired BCK/BCI-algebra in relation to the BCK/BCI-homomorphism.

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