Journal of Algebraic Hyperstructures and Logical Algebras

Volume 1, Number 2, (2020), pp. 1-7

On the free S_1^ω-algebras

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Abstract

One and two-generated free MV-algebras are algebraically described in the variety generated by perfect MV-algebras.

Article Information

Corresponding Author:
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Received: May 2020;
Accepted: Invited paper.
Paper type: Original.

Keywords:

MV-algebra, free algebra, generated free, perfect MV-algebra, variety.

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1 Introduction

MV-algebras are the algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. In contrast with what happens for Boolean algebras, there are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals.

Subvarieties of MV-algebras have been studied in [17], [18], [19], [16], [23], [13], [14]. It is known that any such subvariety is generated by finitely many algebras, and explicit axiomatizations have been obtained. Notice that the free algebras over the subvarieties of MV-algebras have been described functionally in [23, 13, 14] using McNaughton functions [21].

As it is well known, MV-algebras form a category that is equivalent to the category of Abelian lattice ordered groups (ℓ-groups, for short) with strong unit [22]. Let us denote by Γ the functor

https://doi.org/10.29252/HATEF.JAHLA.1.2.1

implementing this equivalence. If G is an ℓ-group, then for any element $u \in G$, $u > 0$ we let $[0, u] = \{x \in G : 0 \leq x \leq u\}$ and for each $x, y \in [0, u]$ $x \oplus y = u \wedge (x + y)$ and $\neg x = u - x$. In particular each perfect MV-algebra is associated with an Abelian ℓ-group with a strong unit. Moreover, the category of perfect MV-algebras is equivalent to the category of Abelian ℓ-groups, see [15].

The class of perfect MV-algebras does not form a variety and contains non-simple subdirectly irreducible MV-algebras [15]. It is worth stressing that the variety $\mathcal{V}(S_1^\omega)$, denoted by $\mathbf{MV(C)}$ in [11], generated by all perfect MV-algebras is also generated by a single MV-chain $C(\cong S_1^\omega)$ defined by Chang in [5]. We name by S_1^ω*-algebras* all the algebras from the variety generated by $S_1^\omega(\cong C)$ [11]. Notice that the Lindenbaum algebra of the logic L_P is an S_1^ω-algebra where L_P is the logic corresponding to the variety $\mathcal{V}(S_1^\omega)$. The perfect algebra $C(\cong S_1^\omega)$ has relevant properties. Indeed S_1^ω generates the smallest variety of MV-algebras containing non-boolean non-semisimple algebras. It is also subalgebra of any non-boolean perfect MV-algebra. The variety $\mathcal{V}(S_1^\omega)$ is selected from the variety $\mathbf{MV}$ of all MV-algebras by the identity $2(x^2) = (2x)^2$ [15].

The importance of the class of S_1^ω-algebras and the logic L_P can be perceived by looking further at the role that infinitesimals play in MV-algebras and Łukasiewicz logic. Indeed the pure first order Łukasiewicz predicate logic is not complete with respect to the canonical set of truth values $[0, 1]$, see [25], [2]. The Lindenbaum algebra of the first order Łukasiewicz logic is not semisimple and the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, that is the co-infinitesimals of such algebra. Hence, the valid but unprovable formulas generate the perfect skeleton of the Lindenbaum algebra. So, perfect MV-algebras, the variety generated by them and their logic are intimately related with a crucial phenomenon of the first order Łukasiewicz logic [3].

In this paper we give an algebraic description of one and two-generated free MV-algebras in the variety $\mathcal{V}(S_1^\omega)$ generated by the algebra S_1^ω that was introduced by Komori [19]. Moreover, we give ordered spectral spaces of the free algebras. Notice, that the algebra S_1^ω was firstly defined as algebra C by Chang in [5]. Not exact algebraic description of free m-generated S_1^ω-algebra (or $MV(C)$-algebra) have been given in [11] where the free algebras are represented by subdirect product of infinite family of chains. In this paper we represent free algebras by means of subdirect product of finite family of chains according to Panti's result in [24].

2 Preliminaries

We assume familiarity with MV-algebras; we refer to [5], [6],[22], [7] for all unexplained notions and claims.

An MV-algebra $A = (A, 0, \neg, \oplus)$ is an Abelian monoid $(A, 0, \oplus)$ equipped with a unary operation $\neg$ such that $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus y)$. We set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$ [5]. We shall write ab for $a \odot b$ and a^n for $\underbrace{a \odot \cdots \odot a}_{n \text{ times}}$, for given $a, b \in A$.
Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } \neg x \oplus y = 1.$$

$(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

The unit interval of real numbers $[0, 1]$ endowed with the following operations: $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \neg x = 1 - x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0,1], \oplus, \odot, \neg, 0, 1)$ generate the variety $\mathbf{MV}$ of all MV-algebras, i. e. $\mathcal{V}(S) = \mathbf{MV}$.

The subvariety $\mathcal{V}(S_n)$ $(= \mathbf{MV}_n)$ (also named by Grigolia's subvariety [1]) of $\mathbf{MV}$ $(= \mathcal{V}(S))$ is generated by $S_n = (\{0, 1/n, ..., n-1/n, n\}, \oplus, \odot, \neg, 0, 1)$ for $n \geq 2$ have been axiomatized in [17]. Moreover, the free m-generated S_n-algebras $F_{\mathcal{V}(S_n)}(m)$ was described in [17, 12]:

$$F_{\mathcal{V}(S_n)}(m) = S_1^{v_1(m)} \times S_{n_2}^{v_{n_2}(m)} \times ... \times S_{n_{k-1}}^{v_{n_{k-1}}(m)} \times S_n^{v_n(m)}$$

where the function $v_m(x)$ is defined on Z^+ as follows: $v_m(1) = 2^m,\ v_m(2) = 3^m - 2^m, ..., v_m(n) = (n+1)^m - (v_m(n_1) + ... + v_m(n_{k-1}))$, where $n_1 = 1, n_k = n$ and $n_1, n_2, ..., n_{k-1}$ are all the divisors of n except n. In particular,

$$F_{\mathcal{V}(S_1)}(m) = S_1^{v_1(m)} = S_1^{2^m}.$$

The algebra S_1^ω-algebra (or C in Chang's notation), with generator $(0,1)$, is isomorphic to $\Gamma(Z \times_{lex} Z, (1,0))(= S_1^\omega)$. Let $\mathcal{V}(S_1^\omega)$ be the variety generated by perfect algebras. The intersection of all maximal ideals of an MV-algebra A, the radical of A, will be denoted by $Rad(A)$. Notice, that $\mathbf{MV}(\mathbf{C}) = \mathcal{V}(C) = \mathcal{V}(S_1^\omega)$ where $S_1^\omega \cong C$ and $\mathcal{V}(S_1^\omega)$ is the variety generated by S_1^ω-algebras.

Let us introduce some notations.

$$S_1^{\omega(1)} = \Gamma(Z \times_{lex} Z, (1,0)) = C, \quad S_1^{\omega(m)} = \Gamma(\underbrace{Z \times_{lex} ... \times_{lex} Z}_{m+1 \text{ times}}, (1,0,...,0)),$$

where $(1,0,...,0) \in Z^{m+1}$ and $Z \times_{lex} ... \times_{lex} Z$ is the lexicographic product of Z $m+1$ times.

The class of MV-algebras forms a category where the objects of this category are MV-algebras and morphisms between MV-algebras are homomorphisms.

A topological space X is said to be an *MV-space* iff there exists an MV-algebra A such that $Spec(A)$ (= the set of prime filters of the MV-algebra A equipped with spectral topology) and X are homeomorphic. It is well known that $Spec(A)$ with the specialization order R on X (which coincides with the inclusion between prime filters) forms a root system. The class of MV-spaces forms a category where the objects of this category are MV-spaces and morphisms between MV-spaces are the strongly isotone maps, i. e. the continuous maps $\varphi : X \to Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$(for details see [9, 10]).

Let $F_{\mathbb{K}}(m)$ be a free algebra in the variety $\mathbb{K}$ with free generators $g_1, ..., g_m$, i. e. any function $f : \{g_1, ..., g_m\} \to A \in \mathbb{K}$ can be extended to the homomorphism $h : F_{\mathbb{K}}(m) \to A$ where $h(g_i) = f(g_i),\ i = 1, ..., m$.

Adapting the Theorem V.1 from [20] for a varieties of algebras we have the following,

Proposition 2.1. *If F is a free algebra in $\mathbb{K}$ with free generators $g_1, ..., g_m \in F$ and satisfy the identity*

(1) $P(g_1, ..., g_m) = Q(g_1, ..., g_m)$ *on the generators $g_1, ..., g_m$, then the identity.*

(2) $P(x_1, ..., x_m) = Q(x_1, ..., x_m)$ *is true in $\mathbb{K}$.*

Conversely, let $g_1, ..., g_m$ generate the algebra F such that the identity (1) holds on the elements $g_1, ..., g_m \in F$, the identity (2) is true in $\mathbb{K}$. Then F is a free algebra in $\mathbb{K}$ with free generators $g_1, ..., g_m \in F$.

3 1-generated free S_1^ω-algebra

In this section we give the descriptions of 1-generated free S_1^ω-algebras in the variety $\mathcal{V}(S_1^\omega)$. It is easy to prove the following

Theorem 3.1. *1)* $S_1^{\omega(m)}$ *is generated by* m *generators:* $(0, ..., 0, 1), ..., (0, 1, 0, ..., 0)$*;*

2) $S_1^{\omega(k)}$ *is a homomorphic image of* $S_1^{\omega(m)}$ *for* $k \le m$*.*

Theorem 3.2. [11] *1-generated free* S_1^ω*-algebra* $F_{\mathcal{V}(S_1^\omega)}(1)$ *is isomorphic to* $(S_1^\omega)^2$ *with free generator* $g = ((0, 1), (1, -1))$*.*

Proof. Firstly, let us show that $(S_1^\omega)^2$ is generated by $g = ((0, 1), (1, -1))$. Indeed,

$$2(((0, 1), (1, -1))^2) = ((0, 0), (1, 0)) \text{ and } 2(\neg((0, 1), (1, -1)))^2 = ((1, 0), (0, 0)),$$

that are atoms of four-element Boolean subalgebra of $(S_1^\omega)^2$. Therefore, since $(0, 1)$ (and $(1, -1)$, as well) generates S_1^ω, we have that g generates $(S_1^\omega)^2$.

Observe that if we have chain perfect S_1^ω-algebra A, then 1-generated subalgebra of A is isomorphic to either S_1^ω or two-element Boolean algebra S_1. Now, suppose that one-variable equation $P = Q$ does not hold in the variety $\mathcal{V}(S_1^\omega)$. It means that this equation does not hold in some 1-generated perfect S_1^ω-algebra A on some element $a \in A$. Then A is isomorphic to S_1^ω. Identify isomorphic elements. Depending on the generator of A, the one belongs to either $RadA$ or $\neg RadA$, we use the projection either $\pi_1 : (S_1^\omega)^2 \to S_1^\omega$ or $\pi_2 : (S_1^\omega)^2 \to S_1^\omega$, sending the generator $((0, 1), (1, -1))$ either to $(0, 1) \in S_1^\omega$ or to $(1, -1) \in S_1^\omega$. From here we conclude that $P = Q$ does not hold in $(S_1^\omega)^2$. Hence, $(S_1^\omega)^2$ is 1-generated free S_1^ω-algebra. □

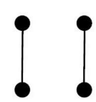

Figure 1: The ordered MV-space of the S_1^ω-algebra $(S_1^\omega)^2$

4 2-generated free S_1^ω-algebra

In this section we give the descriptions of 2-generated free S_1^ω-algebras in the variety $\mathcal{V}(S_1^\omega)$.

For the sake of simplicity let us introduce the following notations for the generating elements of the algebra

$$S_1^{\omega(m)} (m \ge 2) : \; c_1 = (0, 0, ..., 0, 1), c_2 = (0, 0, ..., 1, 0), ..., c_m = (0, 1, ..., 0, 0).$$

Notice, that S_1^ω-algebra $S_1^{\omega(2)}$ is generated by two generators $c_1 = (0, 0, 1)$ and $c_2 = (0, 1, 0)$.

Recall that the radical $Rad(A)$ of an MV-algebra A is the intersection of all its maximal ideals. The algebra A is perfect if $A = Rad^*(A) = Rad(A) \cup \neg Rad(A)$, where $\neg Rad(A) = \{\neg x : x \in Rad(A)\}$ is the intersection of all maximal filters of A.

Theorem 4.1. *2-generated free* S_1^ω*-algebra* $F_{\mathcal{V}(S_1^\omega)}(2)$ *is isomorphic to* $(Rad^*((S_1^{\omega(2)})^2))^{2^2}$ *with free generators*

$$g_1 = ((c_1, c_2), \neg(c_1, c_2), (c_1, c_2), \neg(c_1, c_2)) \;\; \text{and} \;\; g_2 = ((c_2, c_1), (c_2, c_1), \neg(c_2, c_1), \neg(c_2, c_1)).$$

Proof. The radical $Rad((S_1^{\omega(2)})^2)$ is the intersection of two maximal ideals I_1, I_2 of $(S_1^{\omega(2)})^2$: I_1 is generated by $(1, c_2)$ and I_2 is generated by $(c_2, 1)$. This intersection coincides with the ideal generated by (c_2, c_2), i. e. with $Rad((S_1^{\omega(2)})^2)$. So, $Rad((S_1^{\omega(2)})^2)$ contains the elements (c_2, c_1) and (c_1, c_2).

Notice, that the perfect algebra $Rad^*((S_1^{\omega(2)})^2)$ is isomorphic to a subdirect product of two copies of $S_1^{\omega(2)}$ - $S_{11}^{\omega(2)}$ and $S_{12}^{\omega(2)}$, i. e. it is a subalgebra $Rad^*(S_{11}^{\omega(2)} \times S_{12}^{\omega(2)})$ of direct product $S_{11}^{\omega(2)} \times S_{12}^{\omega(2)}$ with projections $\pi_i : Rad^*(S_{11}^{\omega(2)} \times S_{12}^{\omega(2)}) \to S_{1i}^{\omega(2)}$ $(i = 1, 2)$. The kernel of π_1 is the ideal generated by $(0, c_2)$ and the kernel of π_2 is the ideal generated by $(c_2, 0)$ that are both linearly ordered and prime. It is obvious that $\pi_1^{-1}(0) \cap \pi_2^{-1}(0) = (0, 0)$ and, so, $Rad^*((S_1^{\omega(2)})^2)$ is a subdirect product of $S_{11}^{\omega(2)}$ and $S_{12}^{\omega(2)}$. Moreover, the ideals $\pi_1^{-1}(0), \pi_2^{-1}(0)$ generate the maximal ideal M of $Rad^*((S_1^{\omega(2)})^2)$ generated by (c_2, c_2). In turn, the prime ideals of $Rad^*(S_{11}^{\omega(2)} \times S_{12}^{\omega(2)})$ are generated by $(c_2, c_1), (c_2, 0)$, $(0, c_2), (c_1, c_2)$ and (c_2, c_2) respectively, that are ordered by inclusion, the poset of which is isomorphic to the one component of the poset depicted on Fig. 2. At the same time M is generated by (c_1, c_2) and (c_2, c_1). Therefore, the perfect algebra $Rad^*((S_1^{\omega(2)})^2)$ $(\cong Rad^*(S_{11}^{\omega(2)} \times S_{12}^{\omega(2)}))$ is generated by (c_2, c_1) and (c_1, c_2).

Now, we will show directly that the (c_1, c_2) and (c_2, c_1) generate $Rad^*((S_1^{\omega(2)})^2)$. For this aim it is sufficient to obtain the elements $(c_1, 0), (c_2, 0)$ and $(0, c_1)$, $(0, c_2)$. Indeed,

$$(c_2, c_1) \wedge (c_1, c_2) = (c_1, c_1) \text{ and } (c_2, c_1) \vee (c_1, c_2) = (c_2, c_2).$$

$$(c_1, c_2) \odot \neg(c_1, c_1) = (0, c_2 \odot \neg c_1) \text{ and } (c_2, c_1) \odot \neg(c_1, c_1) = (c_2 \odot \neg c_1, 0).$$

$$(0, c_2 \odot \neg c_1) \wedge (c_1, c_1) = (0, c_1) \text{ and } (c_2 \odot \neg c_1, 0) \wedge (c_1, c_1) = (c_1, 0).$$

$$(0, c_2 \odot \neg c_1) \oplus (0, c_1) = (0, c_2 \vee c_1) = (0, c_2) \text{ and } (c_2 \odot \neg c_1, 0) \oplus (c_1, 0) = (c_1 \vee c_2, 0) = (c_2, 0).$$

So, (c_1, c_2) and (c_2, c_1) generate $Rad^*((S_1^{\omega(2)})^2)$. At the same time $Rad^*((S_1^{\omega(2)})^2)$ are generated by the following couples of generators:

$$\{(c_1, c_2), (c_2, c_1)\}, \{\neg(c_1, c_2), (c_2, c_1)\}, \{(c_1, c_2), \neg(c_2, c_1)\}, \ \{\neg(c_1, c_2), \neg(c_2, c_1)\}.$$

It is easy to check that

$$g_1 = ((c_1, c_2), \neg(c_1, c_2), (c_1, c_2), \neg(c_1, c_2)) \text{ and } g_2 = ((c_2, c_1), (c_2, c_1), \neg(c_2, c_1), \neg(c_2, c_1))$$

generate $(Rad^*((S_1^{\omega(2)})^2))^{2^2}$ (notice, that g_1 contains as the first component c_1 and as the second component c_2; and g_2 contains as the first component c_2 and as the second component c_1). Indeed, the elements $2(g_1^2) = (0, 1, 0, 1)$ and $2(g_2^2) = (0, 0, 1, 1)$ generate all (four-element) Boolean elements of $(Rad^*((S_1^{\omega(2)}))^2)^{2^2}$. Therefore, $(Rad^*((S_1^{\omega(2))})^2)^{2^2}$ is generated by g_1, g_2.

Observe that if we have chain perfect S_1^ω-algebra A, then 2-generated subalgebra of A is isomorphic to either S_1^ω, $S_1^{\omega(2)}$ or two-element Boolean algebra S_1. Notice that all these algebras are homomorphic image of the algebra $Rad^*((S_1^{\omega(2)})^2)$ which is also 2-generated. Moreover, all 2-generated S_1^ω-algebra is a homomorphic image of S_1^ω-algebra $Rad^*((S_1^{\omega(2)})^2)$. Now, suppose that two-variable equation $P(x_1, x_2) = Q(x_1, x_2)$ does not hold in the variety $\mathcal{V}(S_1^\omega)$. It means that this equation does not hold in some 2-generated perfect chain S_1^ω-algebra $S_1^{\omega(2)}$ on some elements $a_1, a_2 \in S_1^{\omega(2)}$. Then $S_1^{\omega(2)}$ is a homomorphic image of S_1^ω-algebra $(Rad^*(S_1^{\omega(2)})^2)^{2^2}$ such that the generators g_1, g_2 send to the generators $a_1, a_2 \in S_1^{\omega(2)}$, respectively. From here we conclude that $P(x_1, x_2) = Q(x_1, x_2)$ does not hold in $(Rad^*(S_1^{\omega(2)})^2)^{2^2}$.

Now suppose that $P(x_1, x_2) = Q(x_1, x_2)$ does not hold in S_1^ω on some two generators, say a_1, a_2. But as we know S_1^ω is one-generated, say by a_0, that is equal to either c_1 or $\neg c_1$. Then there exist one-variable polinomials $P_1(x)$ and $P_2(x)$ such that $P_1(a_0) = a_1$ and $P_2(a_0) = a_2$. Hence $P(P_1(x), P_2(x)) = Q(P_1(x), P_2(x))$ does not hold on the generator a_0 of S_1^ω. There exists a homomorphism from $h : \pi_i((Rad^*(S_1^{\omega(2)} \times S_1^{\omega(2)}))^{2^2}) \to S_1^\omega$ for corresponding $i \in \{1, 2, 3, 4\}$. Hence $P(x_1, x_2) = Q(x_1, x_2)$ does not hold in $(Rad^*(S_1^{\omega(2)} \times S_1^{\omega(2)}))^{2^2}$.

By the same argument we can prove for two-element Boolean algebra S_1.

Therefore, $(Rad^*(S_1^{\omega(2)})^2)^{2^2}$ is 2-generated free S_1^ω-algebra. □

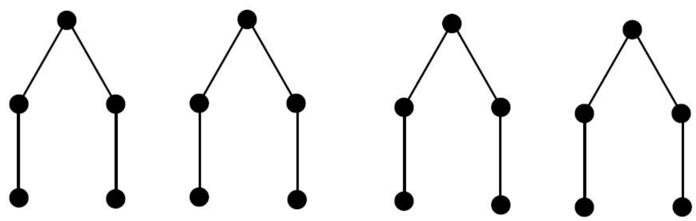

Figure 2: The ordered MV-space of the S_1^ω-algebra $(Rad^*((S_1^{\omega(2)}))^2)^{2^2}$

5 Conclusions

In this paper, one and two-generated free MV-algebras are algebraically described in the variety generated by perfect MV-algebras.

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