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Multipolar fuzzy hyper BCK-ideals of hyper BCK-algebras

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Abstract

In this paper, we apply m-polar fuzzy set to hyper BCK-algebra. We introduce the notions of k-polar fuzzy hyper BCK-ideal, k-polar fuzzy weak hyper BCK-ideal, k-polar fuzzy s-weak hyper BCK-ideal, k-polar fuzzy strong hyper BCK-ideal and k-polar fuzzy reflexive hyper BCK-ideal, and investigate related properties and their relations. We discuss k-polar fuzzy (weak, s-weak, strong, reflexive) hyper BCK-ideal in relation to k-polar level set.

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1 Introduction

The hyper algebraic structure was introduced by F. Marty [14] in 1934. Bolurian et al. [5] was introduced hyper BCK-algebra as an extension of BCK-algebra. Since then, many scholars have been studying hyper BCK-algebra and its infrastructure and so on. In addition, research using fuzzy and soft set is actively being carried out (see [4], [7], [8], [9], [11]). In 2014, Chen et al. [6] introduced an m-polar fuzzy set which is an extension of bipolar fuzzy set. The m-polar fuzzy set applied to decision making problem (see [1]) and BCK/BCI-algebra (see [2, 3, 15]).

The notion of m-polar fuzzy set is applied to hyper BCK-algebra. The concepts of k-polar fuzzy (weak, s-weak, strong, reflexive) hyper BCK-ideal are introduced, and the relations and properties are investigated in relation to k-polar level set.

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2 Preliminaries

Let $\mathcal{H}$ be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of $\mathcal{H}$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* (see [13]) we mean a nonempty set $\mathcal{H}$ endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ \mathcal{H} \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in \mathcal{H}$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq \mathcal{H}$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in $\mathcal{H}$.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in \mathcal{H})(x \circ y \ll \{x\}). \tag{1}$$

A subset A of a hyper BCK-algebra $\mathcal{H}$ is called

- a *hyper BCK-ideal* of $\mathcal{H}$ (see [13]) if

$$0 \in A, \tag{2}$$
$$(\forall x, y \in \mathcal{H})(x \circ y \ll A,\ y \in A \ \Rightarrow\ x \in A). \tag{3}$$

- a *weak hyper BCK-ideal* of $\mathcal{H}$ (see [13]) if it satisfies (2) and

$$(\forall x, y \in \mathcal{H})(x \circ y \subseteq A,\ y \in A \ \Rightarrow\ x \in A). \tag{4}$$

- a *strong hyper BCK-ideal* of $\mathcal{H}$ (see [12]) if it satisfies (2) and

$$(\forall x, y \in \mathcal{H})((x \circ y) \cap A \neq \emptyset,\ y \in A \ \Rightarrow\ x \in A). \tag{5}$$

- a *reflexive hyper BCK-ideal* of $\mathcal{H}$ (see [12]) if it is a hyper BCK-ideal of $\mathcal{H}$ which satisfies:

$$(\forall x \in \mathcal{H})(x \circ x \subseteq A). \tag{6}$$

Every hyper BCK-algebra $\mathcal{H}$ satisfies the following assertions.

$$(\forall x \in \mathcal{H})(x \circ 0 \ll \{x\},\ 0 \circ x = \{0\},\ 0 \circ 0 = \{0\}), \tag{7}$$
$$(\forall x \in \mathcal{H})(0 \ll x,\ x \ll x,\ x \in x \circ 0), \tag{8}$$
$$(\forall x, y \in \mathcal{H})(x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y), \tag{9}$$
$$(\forall x, y, z \in \mathcal{H})(y \ll z \ \Rightarrow\ x \circ z \ll x \circ y), \tag{10}$$
$$(\forall x, y, z \in \mathcal{H})(x \circ y = \{0\} \ \Rightarrow\ x \circ z \ll y \circ z,\ (x \circ z) \circ (y \circ z) = \{0\}), \tag{11}$$

For any subsets A, B and C of a hyper BCK-algebra $\mathcal{H}$, the following assertions are valid.

$$A \subseteq B \;\Rightarrow\; A \ll B, \tag{12}$$
$$A \ll \{0\} \;\Rightarrow\; A = \{0\}, \tag{13}$$
$$A \ll A,\ A \circ B \ll A,\ (A \circ B) \circ C = (A \circ C) \circ B, \tag{14}$$
$$A \circ \{0\} = \{0\} \;\Rightarrow\; A = \{0\}. \tag{15}$$

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, then we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

By a *k-polar fuzzy set* on a universe $\mathcal{H}$ (see [6]), we mean a function $\hat{\varphi} : \mathcal{H} \to [0,1]^k$. The membership value of every element $x \in \mathcal{H}$ is denoted by

$$\hat{\varphi}(x) = (\pi_1 \circ \hat{\varphi}(x), \pi_2 \circ \hat{\varphi}(x), \cdots, \pi_k \circ \hat{\varphi}(x)),$$

where $\pi_i : [0,1]^k \to [0,1]$ is the i-th projection for all $i = 1, 2, \cdots, k$.

Given a k-polar fuzzy set on a universe $\mathcal{H}$, we consider the set

$$U(\hat{\varphi}, \hat{t}) := \{x \in \mathcal{H} \mid \hat{\varphi}(x) \geq \hat{t}\}, \tag{16}$$

where $\hat{t} = (t_1, t_2, \cdots, t_k) \in [0,1]^k$, that is,

$$U(\hat{\varphi}, \hat{t}) := \{x \in \mathcal{H} \mid (\pi_i \circ \hat{\varphi})(x) \geq t_i \text{ for all } i = 1, 2, \cdots, k\} \tag{17}$$

which is called a *k-polar level set* of $\hat{\varphi}$. It is clear that $U(\hat{\varphi}, \hat{t}) = \bigcap_{i=1}^{k} U(\hat{\varphi}, \hat{t})^i$ where

$$U(\hat{\varphi}, \hat{t})^i = \{x \in \mathcal{H} \mid (\pi_i \circ \hat{\varphi})(x) \geq t_i\}.$$

3 k-polar fuzzy hyper BCK-ideals

Definition 3.1. *A k-polar fuzzy set $\hat{\varphi}$ on a hyper BCK-algebra $\mathcal{H}$ is called a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$ if it satisfies*

$$(\forall x, y \in \mathcal{H})\,(x \ll y \;\Rightarrow\; \hat{\varphi}(x) \geq \hat{\varphi}(y)), \tag{18}$$
$$(\forall x, y \in \mathcal{H})\left(\hat{\varphi}(x) \geq \left(\bigwedge\{\hat{\varphi}(a) \mid a \in x \circ y\}\right) \wedge \hat{\varphi}(y)\right), \tag{19}$$

that is, $(\pi_i \circ \hat{\varphi})(x) \geq (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in \mathcal{H}$ with $x \ll y$ and

$$(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigwedge\{(\pi_i \circ \hat{\varphi})(a) \mid a \in x \circ y\}\right) \wedge (\pi_i \circ \hat{\varphi})(y) \tag{20}$$

for all $x, y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$.

Example 3.2. *Let $\mathcal{H} = \{0, a, b\}$ be a set with the hyperoperation "$\circ$" in the following Cayley table*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

Then $\mathcal{H}$ is a hyper BCK-algebra (see [13]). Let $\hat{\varphi}$ be a 4-polar fuzzy set on $\mathcal{H}$ defined by

$$\hat{\varphi}:\mathcal{H}\to[0,1]^4,\ x\mapsto\begin{cases}\left(\frac{1}{n},0.9,\frac{1}{m-3},0.7\right) & \text{if } x=0,\\ \left(\frac{1}{2n},0.5,\frac{1}{2m-3},0.7\right) & \text{if } x=a,\\ \left(\frac{1}{3n},0.2,\frac{1}{3m-3},0.4\right) & \text{if } x=b\end{cases}$$

where $m,n\in\mathbb{N}$ and $m\neq 3$. It is toutine to verify that $\hat{\varphi}$ is a 4-polar fuzzy hyper BCK-ideal of $\mathcal{H}$.

Proposition 3.3. *If $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$, then*

(1) $(\forall x\in\mathcal{H})(\hat{\varphi}(0)\geq\hat{\varphi}(x))$, *that is,* $(\pi_i\circ\hat{\varphi})(0)\geq(\pi_i\circ\hat{\varphi})(x)$ *for all $x\in\mathcal{H}$ and $i=1,2,\cdots,k$,*

(2) *If $\hat{\varphi}$ satisfies the condition*

$$(\forall T\subseteq\mathcal{H})\left(\exists x_0\in T \text{ s.t. } \hat{\varphi}(x_0)=\bigwedge_{x\in T}\hat{\varphi}(x)\right), \tag{21}$$

then

$$(\forall x,y\in\mathcal{H})\left(\exists a\in x\circ y \text{ s.t. } \hat{\varphi}(x)\geq\hat{\varphi}(a)\wedge\hat{\varphi}(y)\right), \tag{22}$$

that is, for every $x,y\in\mathcal{H}$ there exists $a\in x\circ y$ such that

$$(\pi_i\circ\hat{\varphi})(x)\geq(\pi_i\circ\hat{\varphi})(a)\wedge(\pi_i\circ\hat{\varphi})(y)$$

for $i=1,2,\cdots,k$.

Proof. (1) Since $0\ll x$ for all $x\in\mathcal{H}$, it follows from (18) that $\hat{\varphi}(0)\geq\hat{\varphi}(x)$ for all $x\in\mathcal{H}$.

(2) Assume that $\hat{\varphi}$ satisfies the condition (21). For any $x,y\in\mathcal{H}$, there exists $a_0\in x\circ y$ such that $(\pi_i\circ\hat{\varphi})(a_0)=\bigwedge_{a\in x\circ y}(\pi_i\circ\hat{\varphi})(a)$. It follows from (20) that

$$(\pi_i\circ\hat{\varphi})(x)\geq\left(\bigwedge\{(\pi_i\circ\hat{\varphi})(a)\mid a\in x\circ y\}\right)\wedge(\pi_i\circ\hat{\varphi})(y)=(\pi_i\circ\hat{\varphi})(a_0)\wedge(\pi_i\circ\hat{\varphi})(y)$$

for $i=1,2,\cdots,k$. which proves (2). □

Theorem 3.4. *Let $\hat{\varphi}$ be a k-polar fuzzy set in a hyper BCK-algebra $\mathcal{H}$. If $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$, then the k-polar level set $U(\hat{\varphi},\hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t}\in[0,1]^k$.*

Proof. Assume that $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$ and let $\hat{t}\in[0,1]^k$. It is clear that $0\in U(\hat{\varphi},\hat{t})$. Let $x,y\in\mathcal{H}$ be such that $x\circ y\ll U(\hat{\varphi},\hat{t})$ and $y\in U(\hat{\varphi},\hat{t})$. Then $x\circ y\ll U(\hat{\varphi},\hat{t})^i$ and $y\in U(\hat{\varphi},\hat{t})^i$ for all $i=1,2,\cdots,k$. It follows that

$$(\forall a\in x\circ y)\left(\exists a_0\in U(\hat{\varphi},\hat{t})^i \text{ s.t. } a\ll a_0 \text{ and so } (\pi_i\circ\hat{\varphi})(a)\geq(\pi_i\circ\hat{\varphi})(a_0)\right),$$

which implies that $(\pi_i \circ \hat{\varphi})(a) \geq t_i$ for all $a \in x \circ y$. Hence $\bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \geq t_i$, and so

$$(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigwedge\{(\pi_i \circ \hat{\varphi})(a) \mid a \in x \circ y\}\right) \wedge (\pi_i \circ \hat{\varphi})(y) \geq t_i$$

for all $i = 1, 2, \cdots, k$. Thus $x \in \bigcap_{i=1}^{k} U(\hat{\varphi}, \hat{t})^i = U(\hat{\varphi}, \hat{t})$, and therefore $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. □

In order to consider the converse of Theorem 3.4, we need the following lemma.

Lemma 3.5 ([10])**.** *Let A be a subset of a hyper BCK-algebra $\mathcal{H}$. If K is a hyper BCK-ideal of $\mathcal{H}$ such that $A \ll K$, then A is contained in K.*

Theorem 3.6. *Let $\hat{\varphi}$ be a k-polar fuzzy set in a hyper BCK-algebra $\mathcal{H}$ in which the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. Then $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$.*

Proof. Suppose that the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. Let $x, y \in \mathcal{H}$ be such that $x \ll y$ and $\hat{\varphi}(y) = \hat{t}$. Then $y \in U(\hat{\varphi}, \hat{t})$ and so $\{x\} \ll U(\hat{\varphi}, \hat{t})$. It follows from Lemma 3.5 that $\{x\} \subseteq U(\hat{\varphi}, \hat{t})$, i.e., $x \in U(\hat{\varphi}, \hat{t})$. Hence $\hat{\varphi}(x) \geq \hat{t} = \hat{\varphi}(y)$. For any $x, y \in \mathcal{H}$, let $\hat{t} := \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a)\right) \wedge \hat{\varphi}(y)$. Then $y \in U(\hat{\varphi}, \hat{t})$ and

$$\hat{\varphi}(c) \geq \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \geq \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a)\right) \wedge \hat{\varphi}(y) = \hat{t}$$

for all $c \in x \circ y$, i.e., $c \in U(\hat{\varphi}, \hat{t})$. Thus $x \circ y \subseteq U(\hat{\varphi}, \hat{t})$ and so $x \circ y \ll U(\hat{\varphi}, \hat{t})$ by (12). Since $y \in U(\hat{\varphi}, \hat{t})$ and $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$, we have $x \in U(\hat{\varphi}, \hat{t})$ which implies that $\hat{\varphi}(x) \geq \hat{t} = \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a)\right) \wedge \hat{\varphi}(y)$. Therefore $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$. □

Definition 3.7. *A k-polar fuzzy set $\hat{\varphi}$ on a hyper BCK-algebra $\mathcal{H}$ is called a*

- *k-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$ if it satisfies Proposition 3.3(1) and* (19)*.*
- *k-polar fuzzy s-weak hyper BCK-ideal of $\mathcal{H}$ if it satisfies Proposition 3.3(1) and* (22)*.*
- *k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ if it satisfies*

$$(\forall x, y \in \mathcal{H}) \left(\bigwedge_{a \in x \circ x} \hat{\varphi}(a) \geq \hat{\varphi}(x) \geq \left(\bigvee_{b \in x \circ y} \hat{\varphi}(b)\right) \wedge \hat{\varphi}(y)\right), \tag{23}$$

that is, $\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(x) \geq \left(\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b)\right) \wedge (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$.

Example 3.8. *Let $\mathcal{H} = \{0, a, b\}$ be a set with the hyperoperation "$\circ$" in the following Cayley table*

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Then $\mathcal{H}$ is a hyper BCK-algebra (see [13]). Let $\hat{\varphi}$ be a 4-polar fuzzy set on $\mathcal{H}$ defined by

$$\hat{\varphi} : \mathcal{H} \to [0,1]^4, \ x \mapsto \begin{cases} \left(2\pi, \mu(x), \frac{1}{m+3}, 0.8\right) & \text{if } x = 0, \\ \left(\pi, \mu(2x), \frac{1}{m+5}, 0.7\right) & \text{if } x = a, \\ \left(\frac{1}{2}\pi, \mu(3x), \frac{1}{m+7}, 0.4\right) & \text{if } x = b \end{cases}$$

where $m, n \in \mathbb{N}$ and $\mu : [0,1] \to [0,1]$, $x \mapsto \frac{1}{x}$. It is toutine to verify that $\hat{\varphi}$ is a 4-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$.

The following theorem describes the relation between k-polar fuzzy weak hyper BCK-ideal and k-polar fuzzy s-weak hyper BCK-ideal.

Theorem 3.9. *In a hyper BCK-algebra, every k-polar fuzzy s-weak hyper BCK-ideal is a k-polar fuzzy weak hyper BCK-ideal.*

Proof. Let $\hat{\varphi}$ be a k-polar fuzzy s-weak hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ and let $x, y \in \mathcal{H}$. Then there exists $a \in x \circ y$ such that $\hat{\varphi}(x) \geq \hat{\varphi}(a) \wedge \hat{\varphi}(y)$ by (22). Since $\hat{\varphi}(a) \geq \bigwedge_{b \in x \circ y} \hat{\varphi}(b)$, it follows that

$$\hat{\varphi}(x) \geq \left(\bigwedge \{\hat{\varphi}(b) \mid b \in x \circ y\}\right) \wedge \hat{\varphi}(y).$$

Therefore $\hat{\varphi}$ is a k-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$. □

Theorem 3.10. *Let $\hat{\varphi}$ be a k-polar fuzzy weak hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ which satisfies the condition (21). Then $\hat{\varphi}$ is a k-polar fuzzy s-weak hyper BCK-ideal of $\mathcal{H}$.*

Proof. For any $x, y \in \mathcal{H}$, there exists $a_0 \in x \circ y$ such that $\hat{\varphi}(a_0) = \bigwedge_{a \in x \circ y} \hat{\varphi}(a)$, that is, $(\pi_i \circ \hat{\varphi})(a_0) = \bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a)$ by (21). It follows that

$$(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigwedge \{(\pi_i \circ \hat{\varphi})(a) \mid a \in x \circ y\}\right) \wedge (\pi_i \circ \hat{\varphi})(y) = (\pi_i \circ \hat{\varphi})(a_0) \wedge (\pi_i \circ \hat{\varphi})(y).$$

Therefore $\hat{\varphi}$ is a k-polar fuzzy s-weak hyper BCK-ideal of $\mathcal{H}$. □

Proposition 3.11. *Every k-polar fuzzy strong hyper BCK-ideal $\hat{\varphi}$ of a hyper BCK-algebra $\mathcal{H}$ satisfies the following assertions.*

(1) $(\forall x \in \mathcal{H})(\hat{\varphi}(0) \geq \hat{\varphi}(x))$, *that is,* $(\pi_i \circ \hat{\varphi})(0) \geq (\pi_i \circ \hat{\varphi})(x)$ *for all* $x \in \mathcal{H}$ *and* $i = 1, 2, \cdots, k$,

(2) $(\forall x, y \in \mathcal{H})(x \ll y \ \Rightarrow \ \hat{\varphi}(x) \geq \hat{\varphi}(y))$, *that is,* $(\pi_i \circ \hat{\varphi})(x) \geq (\pi_i \circ \hat{\varphi})(y)$ *for all* $x, y \in \mathcal{H}$ *with* $x \ll y$ *and* $i = 1, 2, \cdots, k$.

(3) $(\forall a, x, y \in \mathcal{H})\,(a \in x \circ y \ \Rightarrow \ \hat{\varphi}(x) \geq \hat{\varphi}(a) \wedge \hat{\varphi}(y))$.

Proof. (1) Since $0 \in x \circ x$ for all $x \in \mathcal{H}$, we get

$$\hat{\varphi}(0) \geq \bigwedge_{a \in x \circ x} \hat{\varphi}(a) \geq \hat{\varphi}(x)$$

for all $x \in \mathcal{H}$.

(2) Let $x, y \in \mathcal{H}$ be such that $x \ll y$. Then $0 \in x \circ y$ and thus $\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \geq (\pi_i \circ \hat{\varphi})(0)$ for $i = 1, 2, \cdots, k$. It follows from (1) that

$$(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \right) \wedge (\pi_i \circ \hat{\varphi})(y) \geq (\pi_i \circ \hat{\varphi})(0) \wedge (\pi_i \circ \hat{\varphi})(y) = (\pi_i \circ \hat{\varphi})(y),$$

for $i = 1, 2, \cdots, k$, that is, $\hat{\varphi}(x) \geq \hat{\varphi}(y)$ for all $x, y \in \mathcal{H}$ with $x \ll y$.

(3) Let $a, x, y \in \mathcal{H}$ be such that $a \in x \circ y$. Then

$$(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \right) \wedge (\pi_i \circ \hat{\varphi})(y) \geq (\pi_i \circ \hat{\varphi})(a) \wedge (\pi_i \circ \hat{\varphi})(y),$$

for $i = 1, 2, \cdots, k$. Hence $\hat{\varphi}(x) \geq \hat{\varphi}(a) \wedge \hat{\varphi}(y)$ for all $a, x, y \in \mathcal{H}$ with $a \in x \circ y$. □

Corollary 3.12. *If $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$, then*

$$(\forall x, y \in \mathcal{H}) \left(\hat{\varphi}(x) \geq \hat{\varphi}(y) \wedge \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a) \right) \right).$$

Proof. It is straightforward by Proposition 3.11(3). □

Corollary 3.13. *Every k-polar fuzzy strong hyper BCK-ideal is a k-polar fuzzy hyper BCK-ideal and a k-polar fuzzy s-weak hyper BCK-ideal (and hence a k-polar fuzzy weak hyper BCK-ideal).*

In general, a k-polar fuzzy (weak) hyper BCK-ideal may not be a k-polar fuzzy strong hyper BCK-ideal. In fact, the 4-polar fuzzy hyper BCK-ideal $\hat{\varphi}$ of $\mathcal{H}$ in Example 3.2 is not a 4-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ since

$$(\pi_3 \circ \hat{\varphi})(b) = \tfrac{1}{3m-3} < \tfrac{1}{2m-3} = (\pi_3 \circ \hat{\varphi})(a) = (\pi_3 \circ \hat{\varphi})(a) \wedge \bigvee_{x \in b \circ a} (\pi_3 \circ \hat{\varphi})(x).$$

It is clear that every k-polar fuzzy hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ is a k-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$. But the converse is not true in general as seen in the following example.

Example 3.14. *Let $\mathcal{H} = \{0, a, b\}$ be a hyper BCK-algebra as in Example 3.2. Let $\hat{\varphi}$ be a 3-polar fuzzy set on $\mathcal{H}$ defined by*

$$\hat{\varphi} : \mathcal{H} \to [0,1]^3, \ x \mapsto \begin{cases} \left(5n, \frac{1}{m-3}, 0.7\right) & \text{if } x = 0, \\ \left(n, \frac{1}{3m-3}, 0.1\right) & \text{if } x = a, \\ \left(3n, \frac{1}{2m-3}, 0.5\right) & \text{if } x = b \end{cases}$$

where $m, n \in \mathbb{N}$ and $m \neq 3$. Then $\hat{\varphi}$ is a 3-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$. But it is not a 3-polar fuzzy hyper BCK-ideal of $\mathcal{H}$ since $a \ll b$ and $\hat{\varphi}(a) \not\geq \hat{\varphi}(b)$.

By the simiar way to the proofs of Theorems 3.4 and 3.6, we have a characterization of a k-polar fuzzy weak hyper BCK-ideal.

Theorem 3.15. *Given a k-polar fuzzy set $\hat{\varphi}$ in a hyper BCK-algebra $\mathcal{H}$, the following are equivalent.*

(1) *$\hat{\varphi}$ is a k-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$.*

(2) *The k-polar level set $U(\hat{\varphi}, \hat{t})$ is a weak hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0,1]^k$.*

Theorem 3.16. *Let $\hat{\varphi}$ be a k-polar fuzzy set in a hyper BCK-algebra $\mathcal{H}$. If $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$, then the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0,1]^k$.*

Proof. Assume that $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ and let $\hat{t} \in [0,1]^k$ be such that the k-polar level set $U(\hat{\varphi}, \hat{t})$ is nonempty. Then there exists $a \in U(\hat{\varphi}, \hat{t})$ and so $\hat{\varphi}(a) \geq \hat{t}$, that is, $(\pi_i \circ \hat{\varphi})(a) \geq t_i$ for all $i = 1, 2, \ldots, k$. It is clear that $0 \in U(\hat{\varphi}, \hat{t})$ by Proposition 3.11(1). Let $x, y \in \mathcal{H}$ be such that $y \in U(\hat{\varphi}, \hat{t})$ and $(x \circ y) \cap U(\hat{\varphi}, \hat{t}) \neq \emptyset$. Then there exists $a_0 \in (x \circ y) \cap U(\hat{\varphi}, \hat{t})$ and so $\hat{\varphi}(a_0) \geq \hat{t}$, i.e., $(\pi_i \circ \hat{\varphi})(a_0) \geq t_i$ for $i = 1, 2, \cdots, k$. It follows that

$$(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y) \geq (\pi_i \circ \hat{\varphi})(a_0) \wedge (\pi_i \circ \hat{\varphi})(y) \geq t_i$$

for all $i = 1, 2, \cdots, k$. Hence $x \in \bigcap_{i=1}^{k} U(\hat{\varphi}, \hat{t})^i = U(\hat{\varphi}, \hat{t})$. Therefore $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$. □

Theorem 3.17. *Let $\hat{\varphi}$ be a k-polar fuzzy set on a hyper BCK-algebra $\mathcal{H}$ which satisfies the condition*

$$(\forall T \subseteq \mathcal{H}) \left(\exists x_0 \in T \text{ s.t. } \hat{\varphi}(x_0) = \bigvee_{x \in T} \hat{\varphi}(x) \right). \tag{24}$$

If the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0,1]^k$, then $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$.

Proof. Assume that the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0,1]^k$. Then $x \in U(\hat{\varphi}, \hat{t})$ for some $x \in \mathcal{H}$, and so $x \circ x \ll \{x\} \subseteq U(\hat{\varphi}, \hat{t})$. This implies from Lemma 3.5 that $x \circ x \subseteq U(\hat{\varphi}, \hat{t})$. Hence for every $a \in x \circ x$, we get $a \in U(\hat{\varphi}, \hat{t})$ and so $(\pi_i \circ \hat{\varphi})(a) \geq t_i$ for all $i = 1, 2, \cdots, k$. It follows that

$$\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq t_i = (\pi_i \circ \hat{\varphi})(x)$$

for $i = 1, 2, \cdots, k$. For any $x, y \in \mathcal{H}$, put $\hat{d} := \left(\bigvee_{a \in x \circ y} \hat{\varphi}(a) \right) \wedge \hat{\varphi}(y)$, that is, $d_i := \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y)$ for $i = 1, 2, \cdots, k$. Then $U(\hat{\varphi}, \hat{d})$ is a strong hyper BCK-ideal of $\mathcal{H}$ by hypothesis. The condition (24) implies that there exists $a_0 \in x \circ y$ such that $\hat{\varphi}(a_0) = \bigvee_{a \in x \circ y} \hat{\varphi}(a)$, i.e., $(\pi_i \circ \hat{\varphi})(a_0) = \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a)$ for $i = 1, 2, \cdots, k$. Hence

$$(\pi_i \circ \hat{\varphi})(a_0) = \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \geq \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y) = d_i$$

for $i = 1, 2, \cdots, k$, which implies that $a_0 \in \bigcap_{i=1}^k U(\hat{\varphi}, \hat{d})^i = U(\hat{\varphi}, \hat{d})$. Hence $(x \circ y) \cap U(\hat{\varphi}, \hat{d}) \neq \emptyset$, and thus $x \in U(\hat{\varphi}, \hat{d})$. It follows that

$$(\pi_i \circ \hat{\varphi})(x) \geq d_i = \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y)$$

for $i = 1, 2, \cdots, k$. Therefore $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$. □

Definition 3.18. *A k-polar fuzzy set $\hat{\varphi}$ on a hyper BCK-algebra $\mathcal{H}$ is called a k-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ if it satisfies:*

$$(\forall x, y \in \mathcal{H}) \left(\hat{\varphi}(y) \leq \bigwedge_{a \in x \circ x} \hat{\varphi}(a) \right), \tag{25}$$

$$(\forall x, y \in \mathcal{H}) \left(\hat{\varphi}(x) \geq \left(\bigvee_{a \in x \circ y} \hat{\varphi}(a) \right) \wedge \hat{\varphi}(y) \right), \tag{26}$$

that is, $(\pi_i \circ \hat{\varphi})(y) \leq \bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a)$ and $(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$.

The following theorem is straightforward.

Theorem 3.19. *Every k-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$.*

Theorem 3.20. *If $\hat{\varphi}$ is a k-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$, then the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$.*

Proof. Assume that $\hat{\varphi}$ is a k-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$. Then $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ by Theorem 3.19, and so $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$. It follows from Theorem 3.4 that the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. Let $\hat{t} \in [0, 1]^k$ be such that $U(\hat{\varphi}, \hat{t})$ is nonempty. Then $\hat{\varphi}(c) \geq \hat{t}$ for some $c \in \mathcal{H}$. For any $x \in \mathcal{H}$, let $b \in x \circ x$. The condition (25) implies that $\hat{\varphi}(b) \geq \bigwedge_{a \in x \circ x} \hat{\varphi}(a) \geq \hat{\varphi}(c) \geq \hat{t}$, that is, $b \in U(\hat{\varphi}, \hat{t})$. Thus $x \circ x \subseteq U(\hat{\varphi}, \hat{t})$ for all $x \in \mathcal{H}$, and therefore $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. □

Lemma 3.21 ([12])**.** *Every reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ is a strong hyper BCK-ideal of $\mathcal{H}$.*

In order to consider the converse of Theorem 3.20, we need an additional condition.

Theorem 3.22. *Let $\hat{\varphi}$ be a k-polar fuzzy set on a hyper BCK-algebra $\mathcal{H}$ which satisfies the condition* (24)*. If the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$, then $\hat{\varphi}$ is a k-polar fuzzy reflexive hyper BCK-ideal of $\mathcal{H}$.*

Proof. Assume that the k-polar level set $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. Then $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$ by Lemma 3.21. Using Theorem 3.17, we know that $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ and so (26) is valid. Let $x, y \in \mathcal{H}$

and $(\pi_i \circ \hat{\varphi})(y) = t_i$ for $i = 1, 2, \cdots, k$. Since $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$, we get $x \circ x \subseteq U(\hat{\varphi}, \hat{t})$ and thus $c \in U(\hat{\varphi}, \hat{t})$ for all $c \in x \circ x$. Hence $(\pi_i \circ \hat{\varphi})(c) \geq t_i$ which implies that

$$\bigwedge_{c \in x \circ x} (\pi_i \circ \hat{\varphi})(c) \geq t_i = (\pi_i \circ \hat{\varphi})(y)$$

for all $i = 1, 2, \cdots, k$. Therefore $\hat{\varphi}$ is a k-polar fuzzy reflexive hyper BCK-ideal of $\mathcal{H}$. □

Theorem 3.23. *Let $\hat{\varphi}$ be a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ which satisfies the condition (24). Then $\hat{\varphi}$ is a k-polar fuzzy reflexive hyper BCK-ideal of $\mathcal{H}$ if and only if $\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(0)$ for all $x \in \mathcal{H}$ and $i = 1, 2, \cdots, k$.*

Proof. Assume that $\hat{\varphi}$ is a k-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ which satisfies the condition (24). The necessity is clear. Assume that $\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(0)$ for all $x \in \mathcal{H}$ and $i = 1, 2, \cdots, k$. Since $\hat{\varphi}$ is a k-polar fuzzy hyper BCK-ideal of $\mathcal{H}$ by Corollary 3.13, we have $(\pi_i \circ \hat{\varphi})(0) \geq (\pi_i \circ \hat{\varphi})(y)$ for all $y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$. It follows that $\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$. For any $x, y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$, let $t_i := (\pi_i \circ \hat{\varphi})(y) \wedge \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right)$. The condition (24) implies that there exists $a_0 \in x \circ y$ such that $\hat{\varphi}(a_0) = \bigvee_{a \in x \circ y} \hat{\varphi}(a)$ and so $\hat{\varphi}(a_0) \geq \hat{t}$, i.e., $a_0 \in U(\hat{\varphi}, \hat{t})$. Hence $(x \circ y) \cap U(\hat{\varphi}, \hat{t}) \neq \emptyset$. Since $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$ by Theorem 3.16, it follows that $x \in U(\hat{\varphi}, \hat{t})$. Hence $(\pi_i \circ \hat{\varphi})(x) \geq t_i = (\pi_i \circ \hat{\varphi})(y) \wedge \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right)$. Therefore $\hat{\varphi}$ is a k-polar fuzzy reflexive hyper BCK-ideal of $\mathcal{H}$. □

4 Conclusion

We have applied the m-polar fuzzy set to hyper BCK-algebra. We have introduced the notions of k-polar fuzzy hyper BCK-ideal, k-polar fuzzy weak hyper BCK-ideal, k-polar fuzzy s-weak hyper BCK-ideal, k-polar fuzzy strong hyper BCK-ideal and k-polar fuzzy reflexive hyper BCK-ideal, and have investigated related properties and their relations. We have discussd k-polar fuzzy (weak, s-weak, strong, reflexive) hyper BCK-ideal in relation to k-polar level set. In the future work, we will use the idea and results in this paper to study hyper MV-algebra, hyper hoop, hyper equality algebra, hyper BCI-algebra etc.

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