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Fundamental relations in $\mathbf{H_v}$-structures. The 'Judging from the results' proof

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Abstract

The largest class of hyperstructures is the one which satisfy the weak axioms. These are called H_v-structures introduced in 1990 and they proved to have a lot of applications on several sciences. The main tool in the study of H_v-structures is the 'fundamental structure' which is based on the 'fundamental relations'. These relations connect the hyperstructures with the corresponding classical structures. One cannot find the fundamental classes in an analytic way since they depend on the results of hyperoperations used. In this paper we focus on the fact that the fundamental classes depend on the results which gives new proofs and a lot of new important, for applications, large classes of hyperstructures.

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1 Introduction

The $\mathbf{H_v}$-structures is the largest class of hyperstructures, and they satisfy the weak axioms where the non-empty intersection replaces the equality. They were introduced in 1990 by T. Vougiouklis in the 4^{th} AHA congress held in Greece [22], [24].

Some basic definitions:

In a set H equipped with a hyperoperation (abbreviation: **hyperoperation=hope**) $\cdot : H \times H \to P(H) - \{\varnothing\}$, we abbreviate by,

WASS the **weak associativity**: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by

COW the **weak commutativity**: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called $\mathbf{H_v}$- semigroup if it is WASS, it is called $\mathbf{H_v}$-group if it is reproductive H_v-semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

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In a similar way more advanced hyperstructures can be defined:

$(\mathbf{R},+,\cdot)$ is called $\mathbf{H_v}$-ring if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\ (x+y)z\cap(xz+yz)\neq\varnothing,\ \forall x,y,z\in R.$$

Let $(\mathbf{R},+,\cdot)$ be an H_v-ring, $(\mathbf{M},+)$ be a COW H_v-group and there exists an external hope

$$\cdot : R\times M\to P(M):(a,x)\to ax$$

such that $\forall a,b\in\mathbf{R}$ and $\forall x,y\in\mathbf{M}$ we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing,$$

then $\mathbf{M}$ is called an $\mathbf{H_v}$-module over F.

For more definitions and applications on H_v-structures one can see the books and related papers as [1], [2], [3], [4], [5], [8], [16], [17], [20], [23], [24], [25], [26], [27], [29], [30], [31], [42].

In 1970 [12] M. Koskas defined in the classical hypergroup the relation β and its transitive closure β^*. This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v-groups as well. T. Vougiouklis [22],[24] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-modules, respectively. He also named all these relations β^*, γ^* and ϵ^*, *Fundamental Relations* because they play very important role to study hyperstructures.

Definition 1.1. *The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-modules, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively* [9], [22], [24], [25], [27], [29],[34].

Remark 1.2. *In the classical theory the quotient of a group with respect to an invariant subgroup is a group. In 1934, F. Marty* [13] *states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the motivation to introduce the H_v-structures* [9], [22], [24], *is the quotient of a group with respect to any partition (or equivalently, to any equivalence relation) is an H_v-group.*

Specifying the above, we remark that: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, thus, the quotient $(G/R,\cdot)/\beta^*$ is a group, the fundamental. The classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of the R-classes.

Remark that the proof by Koskas and others for the classical hypergroups is very extensive, so is very hard to be applied for general structures. On the other side, the proof by Vougiouklis *[22], [24],* is short and can be applied on hyperstructures with more hopes. The main point is to find the fundamental classes which one cannot find in an analytic way.

The way to find the fundamental classes is given by the following:

Theorem 1.3. *Let $(\mathbf{H},\cdot)$ be an H_v-group and denote by **U** the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\}\subset\mathbf{u}$ where $\mathbf{u}\in\mathbf{U}$. Then β^* is the transitive closure of β.*

We present the proof for an H_v-ring, in order to see how easily can be applied:

Theorem 1.4. *Let $(\mathbf{R},+,\cdot)$ be an H_v-ring. Denote by **U** the set of all finite polynomials of elements of **R**. We define the relation γ in **R** as follows:*

$$x\gamma y \text{ iff } \{x,y\}\subset u \text{ where } u\in\mathbf{U}.$$

Then the relation γ^* is the transitive closure of the relation γ.

Proof. Let $\underline{\gamma}$ be the transitive closure of γ, and denote by $\underline{\gamma}(a)$ the class of the element a. First we prove that the quotient set $\mathbf{M}/\underline{\gamma}$ is a ring.

In $\mathbf{R}/\underline{\gamma}$ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)\}, \ \forall a, b \in R.$$

Take $a' \in \underline{\gamma}(a)$, $b' \in \underline{\gamma}(b)$. Then we have

$$a'\underline{\gamma}a \text{ iff } \exists x_1, ..., x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, ..., u_m \in U$$

$$\text{such that } \{x_i, x_{i+1}\} \subset u_i, \ i = 1, ..., m, \text{ and}$$

$$b'\underline{\gamma}b \text{ iff } \exists y_1, ..., y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, ..., v_n \in U$$

$$\text{such that } \{y_j, y_{j+1}\} \subset v_j, \ i = 1, ..., n.$$

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, \ i = 1, ..., m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, \ j = 1, ..., n.$$

The sums

$$u_i + v_1 = t_i, \ i = 1, ...m-1 \text{ and } u_m + v_j = t_{im+j-1}, \ j = 1, ..., n$$

are also polynomials, therefore $t_k \in U$ for all $k \in \{1, ..., m+n-1\}$.

Now, pick up elements $z_1, ..., z_{m+n}$ such that

$$z_i \in x_i + y_1, \ i = 1, ..., n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1}, \ j = 1, ..., n,$$

therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k, \ k = 1, ..., m+n-1$.

Thus, every element $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Thus $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c) \text{ for all } c \in \underline{\gamma}(a) + \underline{\gamma}(b)$$

In a similar way we prove that

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d) \text{ for all } d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$$

The WASS and the weak distributivity on $\mathbf{R}$ guarantee that the associativity and the distributivity are valid for the quotient $\mathbf{R}/\underline{\gamma}$. Therefore $\mathbf{R}/\underline{\gamma}$ is a ring.

Now, let σ be an equivalence relation in $\mathbf{R}$ such that $\mathbf{R}/\sigma$ is a ring. Denote $\sigma(a)$ the class of a. Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons for all $a, b \in \mathbf{R}$, i.e.

$$\sigma(a) \oplus \sigma(b) = \sigma(c) \text{ for all } c \in \sigma(a) + \sigma(b), \ \sigma(a) \otimes \sigma(b) = \sigma(d) \text{ for all } d \in \sigma(a) \cdot \sigma(b).$$

Thus we can write, for every $a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B), \ \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B)$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in U$, we have the relation $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently

$$x \in \gamma(a) \text{ implies } x \in \sigma(a) \text{ for every } x \in \mathbf{R}.$$

But σ is transitively closed, so we obtain:

$$x \in \underline{\gamma}(x) \text{ implies } x \in \sigma(a).$$

That means that $\underline{\gamma}$ is the smallest equivalence relation in $\mathbf{R}$ such that $\mathbf{R}/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. □

An element is called **single** if its fundamental class is singleton.

Fundamental relations are used for general definitions. Thus [22], [24].

Definition 1.5. *An H_v-ring $(R, +, \cdot)$ is called $\mathbf{H_v}$-field if R/γ^* is a field. An H_v-module $\boldsymbol{M}$ over an H_v-field $\boldsymbol{F}$, instead of an H_v-ring $\boldsymbol{R}$, it is called $\mathbf{H_v}$-vector space.*

Let ω^* be the kernel of the canonical map from R to R/γ^*; then we call reproductive H_v-field any H_v-field $(R, +, \cdot)$ if the following axiom is valid:

$$x(R - \omega^*) = (R - \omega^*)x = R - \omega^*, \forall x \in R - \omega^*.$$

From the above definition, a new class of hyperstructures introduced [30], [35], [36]:

The h/v-group is a generalization of the H_v-group since the reproductivity is not necessarily valid. Sometimes a kind of reproductivity of classes is valid, i.e. if H is partitioned into equivalence classes $\sigma(x)$, then the quotient is reproductive: $x\sigma(y) = \sigma(xy) = \sigma(x)y, \forall x \in H$. Similarly, the h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces etc, are defined.

An H_v-group is called *cyclic* [18], [19], [25], if there is an element, called generator, which the powers have union the underline set, the minimal power with this property is the period of the generator. If there exist an element and a special power, the minimum one, is the underline set, then the H_v-group is called single-power cyclic.

Let $(H, \cdot), (H, *)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called smaller than $(*)$, and $(*)$ greater than $(\cdot)$, iff there exists an

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y),\ \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ **contains** $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called basic structure and $(H, *)$ is called $\mathbf{H_b} - \mathbf{structure}$.

Theorem 1.6. *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on H_v-structures and mainly to a correspondence between hyperstructures and posets. Using the partial ordering with the fundamental relations one can give several definitions to obtain constructions used in several applications [28], [30], [33]:

Let $(H, \cdot)$ be hypergroupoid. We remove $h \in H$, if we take the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ absorbs $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ **merges** with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

2 Some large classes and applications of H_v-structures

A class of H_v-structures, introduced in [21], [33] is the following:

Definition 2.1. *An H_v-structure is called* ***very thin*** *iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Thus, in a very thin H_v-structure in H there exists a hope $(\cdot)$ and a pair $(a,b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, with respect to any other hopes (so, operations), are singletons.*

Another large class of H_v-structures is the following [31], [34]:

Definition 2.2. *Let $(G,\cdot)$ be a groupoid (resp. hypergroupoid) and $f : G \to G$ be a map. We define a hope (∂), called theta-hope, we write* ***∂-hope****, on G as follows*

$$x\partial y = \{f(x)\cdot y, x\cdot f(y)\}, \quad \forall x,y \in G. \textit{ (resp. } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)), \quad \forall x,y \in G)$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

Let $(G,\cdot)$ be a groupoid (or hypergroupoid) and $f : G \to P(G) - \{\varnothing\}$ be any multivalued map. We define the (∂), on G as follows

$$x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)), \quad \forall x,y \in G$$

Let $(G,\cdot)$ be a groupoid, $f_i : G \to G, i \in I$, be a set of maps on G. The

$$f_\cup : G \to \mathbf{P}(G) : f_\cup(x) = \{f_i(x) | i \in I,\}$$

is the union of $f_i(x)$. We have the union ∂-hope (∂), on G if we take $f_\cup(x)$. If $\underline{f} \equiv f \cup (id)$, then we have the b-$\partial$-hope.

Motivation for the definition of the theta-hope is the map derivative where we can use only the product. The basic property is that if $(G,\cdot)$ is a semigroup then ∀f, the (∂) is WASS.

Consider the group of integers $(\mathbf{Z},+)$ and $n \neq 0$ be a natural number. Take the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{0\}$. Then

$$(\mathbf{Z},\partial)/\beta^* \cong (\mathbf{Z}_n,+)$$

Theorem 2.3. *(a) Take the ring of integers $(\mathbf{Z},+,\cdot)$ and fix $n \neq 0$ a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{0\}$. Then $(Z,\partial_+,\partial.)$, where ∂_+ and $\partial.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with*

$$(\mathbf{Z},\partial_+,\partial.)/\gamma^* \cong \mathbf{Z}_n.$$

(b) Consider the $(\mathbf{Z},+,\cdot)$ and $n \neq 0$ a natural. Take the map f such that $f(n) = 0$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z},\partial_+,\partial.)$ is an H_v-ring, moreover,

$$(\mathbf{Z},\partial_+,\partial.)/\gamma^* \cong \mathbf{Z}_n.$$

Special case of the above is for $n = p$, prime, then $(Z,\partial_+,\partial.)$ is an H_v-field.

In classical hypergoups, were introduced and studied the ***P-hopes****. In H_v-structures they gave very interesting results and applications* [2], [9], [18], [19], [20], [24], [35], [36], [39].

Definition 2.4. *Let $(G, \cdot)$ be a groupoid, then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-hopes: $\forall x, y \in G$,*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \quad \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-hyperstructures. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

During last decades hyperstructures there is a variety of applications in other branches of mathematics and other sciences. These applications are on biomathematics -conchology, inheritance- and hadronic physics to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too. In several books and extensive papers [1], [2], [6], [9], [24], [35], one can find numerous applications.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed in this theory correspond to the hyperstructures were introduced by Santilli & Vougiouklis in 1996[14] and they are called e-hyperfields. The H_v-fields can give e-hyperfields which can be used in the isotopy theory. We present the main definitions and results restricted in the H_v-structures [7], [14], [15], [34].

Definition 2.5. $(\mathbf{H}, \cdot)$ *is called e-hyperstructure, if it contains a unique scalar unit e. We assume that $\forall x$, there exists an, not necessarily unique, inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

Definition 2.6. *A hyperstructure $(\mathbf{F}, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid:*

1. $(\mathbf{F}, +)$ *is an abelian group with the additive unit 0,*
2. $(\cdot)$ *is WASS,*
3. $(\cdot)$ *is weak distributive with respect to $(+)$,*
4. *0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,*
5. *exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$,*
6. $\forall x \in F$ *there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

The elements of an e-hyperfield are called e-hypernumbers. If $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we have a strong *e-hyperfield.*

Now we present a general construction based on the partial ordering of the H_v-structures and on the Little Theorem.

Definition 2.7. *The Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$g_1, g_2,...$ are not necessarily the same for each pair (x,y). Then $(G, \otimes)$ is an H_b-group, an e-hypergroup which contains the $(G, \cdot)$. If $\forall x, y$ with $xy = e$, so $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup.

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. The unit e is unique scalar and $\forall x \in G$, there is a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. Thus, the hyperstructure $(G, \otimes)$ is strong e-hypergroup.

Example 2.8. *Take the quaternion group $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$ whose product is given by $i^2 = j^2 = -1, ij = -ji = k$. On this we define hopes and e-groups: For example, denoting $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$ we define the $(*)$ by $(-1) * k = k * (-1) = \underline{k}$, and in the rest cases $(*)$ coincides to the original operation. $(\mathbf{Q}, *)$ is strong e-hypergroup because 1 is scalar unit and the -1, i, -i, j, -j, k and -k have unique inverses the elements -1, -i, i, -j, j, -k and k, respectively, which are the inverses in the basic group.*

A generalization of P-hopes is the following needed in Santilli's theory is the following:

Construction 2.9. *Let $(G, \cdot)$ be an abelian group and P a subset of G. We define the hope $\times_P$ by:*

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is an abelian H_v-group.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [11], see also [10], [37], [38]. The suggestion is the following:

Definition 2.10. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ____________________ *1*

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feels expresses their answer to the specific question.

The use of the bar of Vougiouklis & Vougiouklis bar instead of a scale has several advantages during both the filling-in and the research processing [19]. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

3 Representation theory of hyperstructures

We abbreviate representations by rep. Reps of H_v-groups can be considered either by generalized permutations [23], [24] or by H_v-matrices [19], [20], [24], [27], [29], [34], [35]. We present here the hypermatrix rep in H_v-structures and the same is for the h/v-structures.

An introduction on the rep problem is by Generalized Permutations (we write gp):

Definition 3.1. *Let X be a set, then a map $f : X \to P(X) - \{\varnothing\}$, is a gp of X if*

$$\bigcup_{x \in X} f(x) = f(X) = X,$$

i.e. it is reproductive. Denote by M_X the set of all gps on X. For an H_v-group $(X, \cdot)$ and $a \in X$, the gp f_a defined by $f_a(x) = ax$ is an inner gp. Arrow of f is any $(x, y) \in X^2$ with

$y \in f(x)$. $f_2 \in M_X$ contains $f_1 \in M_X$ or f_1 is a sub-gp of f_2, if $f_1(x) \subset f_2(x), \forall x \in X$, then we write $f_1 \subset f_2$. If, moreover, $f_1 \neq f_2$, then f_1 is a proper sub-gp of f_2. An $f \in M_X$ is called minimal if it has no proper sub-gp. Denote $\underline{M}_X$ the set of all minimal gps of M_X. The gp t with $t(x) = X, \forall x \in X$, is called universal and contains all elements of M_X. The converse of a gp f is the gp $\underline{f}$ defined by $\underline{f}(x) = \{z \in X : f(z) \ni x\}$, thus $\underline{f}$ is obtained by reversing arrows. We call associated to $f \in M_X$ the gp $f \circ \underline{f}$, where $(\circ)$ is the map composition. The union $f = \bigcup_{i \in I} f_i$ of a family of gps $\{f_i : i \in I\}$, is defined by $f(x) = \bigcup_{i \in I} f_i(x), \forall x \in X$.

For finite X, we reach a minimal gp, by the deleting arrows method.

Theorem 3.2. *Let $f \in M_X$, then $f \in \underline{M}_X$ if and only if, the following condition is valid: if $a \neq b$ and $f(a) \cap f(b) \neq \varnothing$, then $f(a) = f(b)$ and $f(a)$ is a singleton.*

Corollary 3.3. *If $f \in \underline{M}_X$ then $\underline{f} \in \underline{M}_X$. Explicit description of $\underline{M}_X$:*

$$(f \circ \underline{f})(x) = f\{u : f(u) \ni x\} = \bigcup_{f(u) \ni x} f(u), \forall x \in X$$

So $(f \circ \underline{f})(x) = \{y : \exists u \in X, \{x, y\} \subset f(u)\}$ and if I is the identity permutation, then $I \subset f \circ \underline{f}, \forall f \in M_X$.

Remark that there is a direct relation of β^* and the associated gp $f \circ \underline{f}$. We see this relation, for finite X, in the following theorem:

Theorem 3.4. *If $f \in \underline{M}_X$ then $(f \circ \underline{f})(x) = \{y \in X : f(y) = f(x)\}$.*

In the classical theory of reps, we have the following definitions.

Definition 3.5. *Let $\boldsymbol{G}$ be a group and $\mathfrak{V}$ be a finite dimensional vector space over the field $\boldsymbol{F}$. A representation of $\boldsymbol{G}$ is a homomorphism $\rho : \mathbf{G} \to Aut(\mathfrak{V})$ of $\boldsymbol{G}$ into the set of automorphisms of $\mathfrak{V}$*

Analogous definitions are given for complicate structures: Let $\mathfrak{L}$ be a Lie algebra then a rep of $\mathfrak{L}$ is a homomorphism $\rho : \mathfrak{L} \to \mathfrak{gl}(\mathfrak{V})$, from $\mathfrak{L}$ into linear transformations on $\mathfrak{V}$ over $\mathbf{F}$.

The problem of the $\mathbf{H_v}$-matrix representations is the following [19], [20], [24]:

Definition 3.6. $\mathbf{H_v}$*-matrix* *is called a matrix with entries from an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner, and it is a set of $m \times r$ H_v-matrices.*

The sum of products of elements of the H_v-ring is the union of the sets obtained with all possible parentheses put on them, called n-ary circle hope on the hyperaddition [36]. The hyperproduct of H_v-matrices is not nessesarily WASS.

Let $(H, \cdot)$ be an H_v-group. Find an H_v-ring $(R, +, \cdot)$, a set $M_R = \{(a_{ij}) | a_{ij} \in R\}$ and a map

$$T : H \to M_R : h \mapsto T(h) \text{ such that } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is called $\mathbf{H_v}$-matrix rep. If the $T(h_1 h_2) \subset T(h_1)(h_2), \forall h_1, h_2 \in H$, then T is called inclusion rep. If $T(h_1 h_2) = T(h_1)(h_2) = \{T(h) | h \in h_1 h_2\}$, $\forall h_1, h_2 \in H$, then T is called good rep and then an induced rep T^* for the hypergroup algebra is obtained. If T is one to one and good then it is a faithful rep.

The problem of reps is complicated because the cardinality of the product of H_v-matrices is very big. It can be simplified in special cases such as:

1. The H_v-matrices are over H_v-rings with 0 and 1 and if these are scalars.
2. The H_v-matrices are over very thin H_v-rings.
3. The case of $2{\times}2$ H_v-matrices, since the circle hope coincides with the hyperaddition.

The main theorem of reps is the following:

Theorem 3.7. *A necessary condition in order to have an inclusion rep T of an H_v-group $(H,\cdot)$ by $n \times n$ H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following:*
For all classes $\beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H, i,j \in \{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) | a'_{ij} \in \gamma^*(a_{ij}), i,j \in \{1,...,n\}\}$$

So every inclusion rep $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces an homomorphic T^* of the group H/β^* over the ring R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \forall \beta^*(a) \in H/\beta^*$, where the $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$. Then T^* is called fundamental induced representation of T.

Denote $tr_\phi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T : H \to R/\gamma^* : x \mapsto X_{T(x)} = tr_\phi(T(x)) = trT^*(x)$$

is called fundamental character. There are several types of traces.

Reps H_v-structures can be faced in several ways [29], [34], [15]:

Definition 3.8. *Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over a ring $\mathbf{R}$ and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a P-hope $\underline{P}$ on $\mathbf{M}$ as follows*

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A,B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P.

The hope $\underline{P}$, is a generalization of Rees operation where, instead of one sandwich matrix, a set of sandwich matrices is used. $\underline{P}$ is strong associative and the inclusion distributivity is valid:

$$A\underline{P}(B+C) \subseteq A\underline{P}B + A\underline{P}C \text{ for all } A,B,C \text{ in } \mathbf{M}$$

Thus, $(\mathbf{M},+,\underline{P})$ defines a multiplicative hyperring, only the product is hope, on non-square matrices.

Definition 3.9. *Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over R and let us take sets $\mathbf{S} = \{s_k : k \in K\} \subseteq R$, $\mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}$, $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. Define three hopes as follows*

$$\underline{S} : R \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (r,A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A,B) \to A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A,B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over $\boldsymbol{R}$ called general matrix P-hyperalgebra.

In the rep theory can be used hopes on any type of ordinary matrices. The new hopes can be defined which are called, helix hopes [8], [17], [40], [41].

4 The H_v-Lie algebra

Since the algebras are defined on vector spaces, we now present a proof for the fundamental relation analogous to Theorem 1.3, in the case of an H_v-module [9], [32], [34]:

Theorem 4.1. *Let $(\boldsymbol{M},+)$ be an H_v-module over the H_v-ring $\boldsymbol{R}$. Denote by $\boldsymbol{U}$ the set of all expressions consisting of finite hopes either on $\boldsymbol{R}$ and $\boldsymbol{M}$ or the external hope applied on finite sets of elements $\boldsymbol{R}$ and $\boldsymbol{M}$. We define the relation ϵ in $\boldsymbol{M}$ as follows:*

$$x\epsilon y \text{ iff } \{x,y\} \subset u \text{ where } u \in \boldsymbol{U}$$

Then the relation ϵ^* is the transitive closuer of the relation ϵ

Proof. Let $\underline{\epsilon}$ be the transitive closure of ϵ, and denote by $\underline{\epsilon}(x)$ the class of the element x. First we prove that the quotient set $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$.

In $\mathbf{M}/\underline{\epsilon}$ the sum ($\oplus$) and the external product ($\otimes$), using the γ^* classes in $\mathbf{R}$, are defined in the usual manner:

$$\underline{\epsilon}(x) \oplus \underline{\epsilon}(y) = \{\underline{\epsilon}(z) : z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)\},$$

$$\gamma^*(a) \otimes \underline{\epsilon}(x) = \{\underline{\epsilon}(z) : z \in \gamma^*(a) \cdot \underline{\epsilon}(x)\},\ \ \forall a \in \mathbf{R},\ x, y \in \mathbf{M}$$

Take $x' \in \underline{\epsilon}(x)$, $y' \in \underline{\epsilon}(y)$. Then we have $x'\underline{\epsilon}x$ iff $\exists x_1, ..., x_{m+1}$ with $x_1 = x', x_{m+1} = x$ and $u_1, ..., u_m \in \mathbf{U}$ such that $\{x_i, x_{i+1}\} \subset u_i, i = 1, ..., m$, and $y'\underline{\epsilon}y$ iff $\exists y_1, ..., y_{n+1}$ with $y_1 = y', y_{n+1} = y$ and $v_1, ..., v_n \in \mathbf{U}$ such that $\{y_j, y_{j+1}\} \subset v_j,\ i = 1, ..., n$. From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1,\ \ i = 1, ..., m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j,\ \ j = 1, ..., n.$$

The sums

$$u_i + v_1 = t_i,\ \ i = 1, .., .m-1 \text{ and } u_m + v_j = t_{m+j-1},\ \ j = 1, ..., n$$

are also elements of $\mathbf{U}$, thus, $t_k \in \mathbf{U}$ for all $k \in \{1, ..., m+n-1\}$. Now, take elements $z_1, ..., z_{m+n}$ such that

$$z_i \in x_i + y_1,\ \ i = 1, ..., n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1},\ \ j = 1, ..., n,$$

therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k,\ \ k = 1, ..., m+n-1$. Thus, every element $z_1 \in x_1 + y_1 = x' + y'$ is $\underline{\epsilon}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = x + y$. Thus $\underline{\epsilon}(x) \oplus \underline{\epsilon}(y)$ is a singleton so we can write

$$\underline{\epsilon}(x) \oplus \underline{\epsilon}(y) = \underline{\epsilon}(z) \text{ for all } z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)$$

In a similar way, using the properties of γ^* in $\mathbf{R}$, one can prove that

$$\gamma^*(a) \otimes \underline{\epsilon}(x) = \underline{\epsilon}(z) \text{ for all } z \in \gamma^*(a) \cdot \underline{\epsilon}(x)$$

The WASS and the weak distributivity on $\mathbf{R}$ and $\mathbf{M}$ guarantee that the associativity and the distributivity are valid for the quotient $\mathbf{M}/\underline{\epsilon}$ over $\mathbf{R}/\gamma^*$. Therefore $\mathbf{M}/\underline{\epsilon}$ is a module over R/γ^*.

Now let σ be an equivalence relation in $\mathbf{M}$ such that $\mathbf{M}/\sigma$ is a module over $\mathbf{R}/\gamma^*$. Denote $\sigma(x)$ the class of x. Then $\sigma(x) \oplus \sigma(y)$ and $\gamma^*(a) \otimes \sigma(x)$ are singletons for all $a \in \mathbf{R}$ and $x, y \in \mathbf{M}$, i.e.

$$\sigma(x) \oplus \sigma(y) = \sigma(z) \text{ for all } z \in \sigma(x) + \sigma(y),$$

$$\gamma^*(a) \otimes \sigma(x) = \sigma(z) \text{ for all } z \in \gamma^*(a) \cdot \sigma(x).$$

Thus we can write, for every $a \in \mathbf{R}$, $x, y \in \mathbf{M}$ and $A \subset \gamma^*(a)$, $\mathbf{X} \subset \sigma(x)$,$\mathbf{Y} \subset \sigma(x)$

$$\sigma(x) \oplus \sigma(y) = \sigma(x+y) = \sigma(\mathbf{X}+\mathbf{Y}),\ \gamma^*(a) \otimes \sigma(x) = \sigma(ax) = \sigma(\mathbf{A} \cdot \mathbf{X})$$

By induction, we extend these relations on finite sums and products. Thus, $\forall u \in \mathbf{U}$, we have $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently

$$x' \in \epsilon(x) \text{ implies } x' \in \sigma(x) \text{ for every } x \in \mathbf{M}.$$

But σ is transitively closed, so we obtain:

$$x' \in \underline{\epsilon}(x) \text{ implies } x' \in \sigma(x).$$

That means that $\underline{\epsilon}$ is the smallest equivalence relation in $\mathbf{M}$ such that $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$, i.e. $\underline{\epsilon} = \epsilon^*$. □

The general definition of an H_v-Lie algebra was given in [15], [32],[32] as follows:

Definition 4.2. *Let $(\mathbf{L}, +)$ be an H_v-vector space over the $(\mathbf{F}, +, \cdot)$, $\phi : \mathbf{F} \to \mathbf{F}/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\phi' : \mathbf{L} \to \mathbf{L}/\epsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\epsilon^*$. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

then $\boldsymbol{L}$ is an H_v-Lie algebra over $\boldsymbol{F}$ if the following axioms are satisfied:

(L1) *The bracket hope is bilinear, i.e.*
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \varnothing$,
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences. We can see theta hopes in H_v-vector spaces and H_v-Lie algebras:

Theorem 4.3. *Let $(\mathbf{V}, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f : \mathbf{V} \to \mathbf{V}$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\mathbf{V}, +, \partial)$ is an H_v-algebra over F, where the related properties are weak. If, moreover f is linear then we have more strong properties.*

Theorem 4.4. *Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F. Take any map $f : \mathbf{A} \to \mathbf{A}$, then the ∂-hope on the Lie bracket $[x, y] = xy - yx$, is defined as follows*

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

then $(\mathbf{A}, +, \partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

5 The 'Judging from the results' proof and applications

The uniting elements method was introduced by Corsini-Vougiouklis [3] in 1989. This leads, through hyperstructures, to structures satisfying additional properties.

The **uniting elements method** is described as follows: Let $\mathbf{G}$ be an algebraic structure and d, a property which is not valid. Suppose that d is described by a set of equations; then, consider the partition in $\mathbf{G}$ for which it is put together, in the same class, every pair of elements that causes the non-validity of the property d. The quotient by this partition $\mathbf{G}/d$ is an H_v-structure. Then, quotient out of the H_v-structure $\mathbf{G}/d$ by the fundamental relation β^*, is a stricter structure $(\mathbf{G}/d)\beta^*$ for which d is valid, is obtained.

An interesting application of the uniting elements is when more than one property is desired. It is better to apply the straightforward classes followed by the others. We can do this because:

Theorem 5.1. *Let* $(\mathbf{G}, \cdot)$ *be a groupoid, and*

$$F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$$

be a system of equations on $\boldsymbol{G}$ *consisting of two subsystems*

$$F_m = \{f_1, ..., f_m\} \text{ and } F_n = \{f_{m+1}, ..., f_{m+n}\}.$$

Let σ, σ_m *be the equivalence relations defined by the uniting elements procedure using the systems* F *and* F_m *respectively, and let* σ_n *be the equivalence relation defined using the induced equations of* F_n *on the groupoid* $\mathbf{G}_m = (\mathbf{G}/\sigma_m)/\beta^*$. *Then*

$$(\mathbf{G}/\sigma)/\beta^* \cong (\mathbf{G}_m/\sigma_n)/\beta^*$$

Theorem 5.2. *Let* $(S, \cdot)$ *be a commutative semigroup which has at least one element* u *such that the set* uS *is finite. Consider the transitive closure* L^* *of the relation* $\mathfrak{R}^*$ *of the relation* $\mathfrak{R}$ *defined as follows:*

$$x\mathfrak{R}y \text{ iff } \exists x \in S \text{ such that } xs_1 = xs_2.$$

Then $< S/\mathfrak{R}^*, ^\circ > /\beta^*$ *is finite commutative group.*

Proof. The proof follows the one on the fundamental relation. It is the special proof that depends on the way that it can be in an analytical way but counting the fact that the classes depend on the results.

It is clear that, the fundamental structure it is very important, mainly if it is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures. Combining the uniting elements procedure with the enlarging theory we can obtain stricter structures. □

Theorem 5.3. *In the ring* $(\mathbf{Z}_n, +, \cdot)$, *with* $n = ms$ *we enlarge the multiplication only in the product of the special elements* $0 \cdot m$ *by setting* $0 \otimes m = \{0, m\}$ *and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot).$$

Proof. First, we remark that the only expressions of sums and products which contain more than one elements are the expressions which have at least one time the hyperproduct $0 \otimes m$. Adding to this special hyperproduct the element 1, several times we have the modm equivalence classes. On the other side, since m is zero divisor, adding or multiplying elements of the same class the results are remaining in one class, the class obtained by using only the representatives. Therefore, γ^*-classes form a ring isomorphic to $(\mathbf{Z}_m, +, \cdot)$. □

Corollary 5.4. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ps$ where p is prime, we enlarge only the product $0 \cdot p$ by $0 \otimes p = \{0, p\}$ and the rest remain the same. Then $(\mathbf{Z}_n, +, \otimes)$ is very thin H_v-field.*

In [28] the 'enlarged' hyperstructures were examined in the sense that a new element appears in one result. The enlargement or reduction is on Hv-structures with the same fundamental structure.

Theorem 5.5. *Attach Construction. Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. Then, we extend the hope $(\cdot)$ in the set $\underline{H} = H \cup \{v\}$ as follows: $x \cdot v = v \cdot x = v, \forall x \in H$, and $v \cdot v = H$. Since $v \cdot v = H$ we obtain that all elements of H are β-equivalent in $(\underline{H}, \cdot)$. Therefore, there are two β-equivalent classes: H and $\{v\}$. The products of those classes are scalars, so $(\underline{H}, \cdot)/\beta^* \cong Z_2$. Therefore $(\underline{H}, \cdot)$ is h/v-group and v is a single.*

The core of $(\underline{H}, \cdot)$ is obviously the set H. Moreover, all scalar elements of $(H, \cdot)$ are scalars in $(\underline{H}, \cdot)$ and any unit of $(H, \cdot)$ is unit of $(\underline{H}, \cdot)$.

Theorem 5.6. *Attached h/v field. Let $(H, \cdot)$ be H_v-semigroup, $v \notin H$ and $(\underline{H}, \cdot)$ be its attached h/v-group. Consider an element $0 \notin \underline{H}$ and define in $\underline{H}_o = H \cup \{v, 0\}$ two hopes as follows: hypersum (+): $\forall x, y \in H$*

$$0 + 0 = x + v = v + x = 0, 0 + v = v + 0 = x + y = v, 0 + x = x + 0 = v + v = H.$$

hyperproduct $(\cdot)$: the hope remains the same as in $\underline{H}$ and

$$0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \forall x \in \underline{H}$$

Then $(\underline{H}_o, +, \cdot)$ is h/v-field with $(\underline{H}_o, +, \cdot)/\gamma^* \cong \mathbf{Z}_3$. The hope (+) is associative, $(\cdot)$ is WASS and weak distributive with respect to (+). 0 is zero absorbing and single element but not scalar in (+).

$(\underline{H}_o, +, \cdot)$ is called the attached h/v-field of the H_v-semigroup $(H, \cdot)$.

Proof. See [28], [33] □

The magic single elements!

Recall that, an element is called single if its fundamental class is singleton. Thus, in an H_v-group if s is **single** then $\beta^*(s) = \{s\}$. Denote S_H the set of singles. If $S_H \neq \varnothing$, then we can answer to the very hard problem, that is to find the fundamental classes. The following theorems are proved [24], [25], [35]:

Theorem 5.7. *Let $(H, \cdot)$ be an H_v-group and $s \in S_H \neq \varnothing$. Let $a \in H$, take any element $v \in H$ such that $s \in av$, then $\beta^*(a) = \{h \in H : hv = s\}$, and the core of H is $\omega_H = \{u \in H : us = s\} = \{u \in H : su = s\}$.*

Theorem 5.8. *Let $(H, \cdot)$ be an H_v-group and $s \in S_H \neq \varnothing$. Then*

$$sx = \beta^*(sx) \text{ and } xs = \beta^*(xs) \text{ for all } x \in H.$$

6 Conclusions

Two elements a,b are in the fundamental relation β if there are two elements x,y who bring a,b in the relation β. That means that the fundamental relation β^* 'depends' on the results. This fact leads to a special proof where we need to discover the 'reason' to have the results. Every relation needs even the last one result to characterize its classes. However, if there are special elements, as the singles, which are strictly formed and carry inside them the relation, then these elements form the fundamental classes.

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